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DIVISION PROBLEM OF MOMENT FUNCTIONALS

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ABSTRACT. For a quasi-definite moment functional σ and nonzero polynomials $A(x)$ and $D(x)$, we define another moment functional τ by the relation

$$D(x)\tau = A(x)\sigma.$$

In other words, τ is obtained from σ by a linear spectral transform. We find necessary and sufficient conditions for τ to be quasi-definite when $D(x)$ and $A(x)$ have no nontrivial common factor. When τ is also quasi-definite, we also find a simple representation of orthogonal polynomials relative to τ in terms of orthogonal polynomials relative to σ. We also give two illustrative examples when σ is the Laguerre or Jacobi moment functional.

1. Introduction. Let σ be a quasi-definite moment functional, i.e., a linear function on $\mathbf{P}$, the space of polynomials in one variable, satisfying the Hamburger condition: $\Delta_n := |[\sigma_{i+j}]_{i,j=0}^n| \neq 0$, $n \geq 0$, where $\sigma_n := \langle \sigma, x^n \rangle$, $n \geq 0$, are the moments of σ. Then the monic orthogonal polynomial system (MOPS) $\{P_n(x)\}_{n=0}^\infty$, relative to σ, is given by
(1.1)

$$P_0(x) = 1 \quad \text{and } P_n(x) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} \sigma_0 & \sigma_1 & \cdots & \sigma_n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_{n+1} \\ \vdots & \vdots & & \vdots \\ \sigma_{n-1} & \sigma_n & \cdots & \sigma_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}, \quad n \geq 1.$$

However, in the computational viewpoint, the formula (1.1) is of little practical value for large n. Instead we might use the three-term recurrence relation satisfied by any MOPS

$$P_{n+1}(x) = (x-b_n)P_n(x) - c_n P_{n-1}(x), \quad n \geq 0, (P_{-1}(x) = 0, P_0(x) = 1)$$

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if the coefficients b_n and c_n are easily computable.

On the other hand, if τ is another quasi-definite moment functional which is obtained from σ by a simple modification, then it is natural and useful to represent the MOPS $\{Q_n(x)\}_{n=0}^{\infty}$ relative to τ in terms of $\{P_n(x)\}_{n=0}^{\infty}$. For example, when σ and τ are given by positive weights as

$$\langle \sigma, \pi(x)\rangle := \int_a^b \pi(x)w(x)\,dx$$
$$\langle \tau, \pi(x)\rangle := \int_a^b \pi(x)\tilde{w}(x)\,dx$$

and $\tilde{w}(x) = R(x)w(x)$ where $R(x) = (A(x)/D(x))$ is a suitable rational function, representation of $\{Q_n(x)\}_{n=0}^{\infty}$ in terms of $\{P_n(x)\}_{n=0}^{\infty}$ was already considered by Uvarov [**16**], (see also [**12**]).

We now consider a more general situation for any two generic moment functionals σ and τ satisfying

$$D(x)\tau = A(x)\sigma, \tag{1.2}$$

where $A(x)$ and $D(x)$ are nonzero polynomials. In terms of Stieltjes functions of moment functionals, τ is obtained from σ by a linear spectral transform, (see [**16**]). Assuming that σ is quasi-definite, we may ask: When is the other moment functional τ also quasi-definite? If so, what is the relation between their corresponding orthogonal polynomials $\{P_n(x)\}_{n=0}^{\infty}$ relative to σ and $\{Q_n(x)\}_{n=0}^{\infty}$ relative to τ?

When $D(x) = 1$, σ is the Legendre moment functional defined by

$$\langle \sigma, \pi(x)\rangle := \int_{-1}^1 \pi(x)\,dx, \quad \pi \in \mathbf{P},$$

and $A(x)$ is nonnegative on $[-1, 1]$ so that $\tau = A(x)\sigma$ is also positive-definite, Christoffel [**6**] found representation of $\{Q_n(x)\}_{n=0}^{\infty}$ in terms of the Legendre polynomials $\{P_n(x)\}_{n=0}^{\infty}$. More generally, when $D(x) = 1$ and $A(x)$ is any nonzero polynomial, Belmehdi [**2**] found necessary and sufficient conditions for τ to be quasi-definite and a representation of $\{Q_n(x)\}_{n=0}^{\infty}$ in terms of $\{P_n(x)\}_{n=0}^{\infty}$,(for some special cases see also Ronveaux [**14**] and Szegö [**15**].

When $D(x)$ is of degree ≥ 1, the equation (1.2) gives rise to a division problem of moment functions, in which we are interested. When

$A(x) = 1$ and $D(x)$ is of degree 1 and 2, respectively, Maroni [**10**] and Branquinho and Marcellán [**3**], respectively, found necessary and sufficient conditions for τ to be quasi-definite. When $A(x) = D(x)$, τ is obtained from σ by a generalized Uvarov transform, i.e., by adding finitely many mass points and their derivatives. In this case, the quasi-definiteness of τ was handled in [**7**] and [**8**].

In this work we consider the case when $A(x)$ and $D(x)$ have no non-trivial common factor. In this case, we find necessary and sufficient conditions for τ to be quasi-definite and give representations of $\{Q_n(x)\}_{n=0}^{\infty}$ in terms of $\{P_n(x)\}_{n=0}^{\infty}$.

2. Preliminaries. For a polynomial $\pi(x)$ we let $\partial(\pi)$ be the degree of $\pi(x)$ with the convention $\partial(0) = -1$. For a moment functional σ and a polynomial $\phi(x) = \sum_{k=0}^{n} a_k x^k$, define [**11**]

$$\langle \phi\sigma, \pi\rangle := \langle \sigma, \phi\pi\rangle; \langle \sigma', \phi\rangle = -\langle \sigma, \phi'\rangle;$$
$$\langle (x-c)^{-1}\sigma, \phi\rangle := \langle \sigma, \theta_c\phi\rangle, \quad \pi \in \mathbf{P},$$

where $\theta_c\pi = (\pi(x) - \pi(x))/(x-c)$, $c \in \mathbf{C}$,

$$(\sigma\phi)(x) := \sum_{k=0}^{n}\left(\sum_{j=k}^{n} a_j\sigma_{j-k}\right)x^k;$$
$$F(\sigma)(x) := \sum_{n=0}^{\infty}\frac{\sigma_n}{x^{n+1}}.$$

We call the formal series $F(\sigma)(x)$ the Stieltjes function of σ. When σ is quasi-definite, we let $\{P_n(x)\}_{n=0}^{\infty}$ be the MOPS relative to σ and

$$P_{n+1}(x) = (x-b_n)P_n(x) - c_nP_{n-1}(x), \quad n \geq 0, (P_{-1}(x) = 0) \tag{2.1}$$

the three-term recurrence relation of $\{P_n(x)\}_{n=0}^{\infty}$. In this case, we also let $\{P_n^{(1)}(x)\}_{n=0}^{\infty}$ be the numerator MOPS for $\{P_n(x)\}_{n=0}^{\infty}$ satisfying the three-term recurrence relation

$$P_{n+1}^{(1)}(x) = (x-b_{n+1})P_n^{(1)}(x) - c_{n+1}P_{n-1}^{(1)}(x), \quad n \geq 0, (P_{-1}^{(1)}(x) = 0)$$

and $\{P_n(x;c)\}_{n=0}^{\infty}$ the co-recursive MOPS, [**5**], for $\{P_n(x)\}_{n=0}^{\infty}$ satisfying

$$P_n(x;c) = P_n(x) - cP_{n-1}^{(1)}(x), \quad n \geq 0, c \in \mathbf{C}.$$

Let

$$K_n(x,y) = \sum_{k=0}^{n} \frac{P_k(x)P_k(y)}{\langle \sigma, P_k^2 \rangle}$$

be the nth kernel polynomial of $\{P_n(x)\}_{n=0}^{\infty}$.

3. Division problem. From now on let σ be a quasi-definite moment functional and $\{P_n(x)\}_{n=0}^{\infty}$ the MOPS relative to σ satisfying the three-term recurrence relation (2.1). Define another moment functional τ by the relation

$$D(x)\tau = A(x)\sigma, \tag{3.1}$$

where $A(x)$ and $D(x)$ are nonzero polynomials of degree s and t, respectively. When τ is also quasi-definite, we denote the MOPS relative to τ by $\{Q_n(x)\}_{n=0}^{\infty}$. We may and shall assume that $A(x)$ and $D(x)$ are monic. In terms of the corresponding Stieltjes functions, the relation (3.1) can be written as

$$F(\tau)(x) = \frac{A(x)F(\sigma)(x) + B(x)}{D(x)},$$

where $B(x) = (\tau\theta_0 D)(x) - (\sigma\theta_0 A)(x)$. In other words, (see [**17**]), $F(\tau)(x)$ is a linear spectral transform of $F(\sigma)(x)$.

For any complex numbers λ and β, let

$$C(\lambda)F(\sigma) := (x-\lambda)F(\sigma) - \sigma_0$$

and

$$G(\lambda;\beta)F(\sigma)(x) := \frac{\beta + F(\sigma)}{x-\lambda}$$

be the Christoffel transform and the Geronimus transform, [**17**], of $F(\sigma)$, respectively. In terms of moment functionals, we have

(i) $C(\lambda)F(\sigma) = F(\tau)$ if and only if $\tau := C(\lambda)\sigma = (x-\lambda)\sigma$;

(ii) $G(\lambda;\beta)F(\sigma) = F(\tau)$ if and only if $\tau := G(\lambda;\beta)\sigma = (x-\lambda)^{-1}\sigma + \beta\delta(x-\lambda)(\beta = \tau_0)$.

Proposition 3.1. *For any complex numbers λ and β,*

(i) $\tau = C(\lambda)\sigma$ *is quasi-definite if and only if* $P_n(\lambda) \neq 0$, $n \geq 0$. *When* τ *is also quasi-definite,*

$$Q_n(x) = P_n^*(\lambda; x) = \frac{\langle \sigma, P_n^2 \rangle}{P_n(\lambda)} K_n(x, \lambda), \quad n \geq 0$$

is the monic kernel polynomials for $\{P_n(x)\}_{n=0}^\infty$ *with* K*-parameter* λ.

(ii) $\tau = G(\lambda, \beta)\sigma$ *is quasi-definite if and only if*

$$\beta P_n(\lambda) + \sigma_0 P_{n-1}^{(1)}(\lambda) = \beta P_n\left(\lambda; \frac{-\sigma_0}{\beta}\right) \neq 0, \quad n \geq 0. \tag{3.2}$$

When τ *is also quasi-definite,*

$$Q_0(x) = 1 \text{ and } Q_n(x) = P_n(x) - \frac{P_n(\lambda; -(\sigma_0/\beta))}{P_{n-1}(\lambda; (-\sigma_0/\beta))} P_{n-1}(x), \quad n \geq 1.$$

Proof. For (i), (see Theorem 7.1 in [**4**, Chapter 1]) and for (ii), (see Theorem 4.2 in [**9**]) and Theorem 1.1 in [**10**]. □

Note that we may rewrite the condition (3.2) as

$$\langle \tau, P_n \rangle \neq 0, \quad n \geq 0.$$

The division problem (3.1) can be solved for τ as

$$\tau = D(x)^{-1} A(x)\sigma + \sum_{i=1}^{k} \sum_{j=0}^{m_i - 1} c_{i,j} \delta^{(j)}(x - \nu_i) \tag{3.3}$$

where $D(x) = (x - \nu_1)^{m_1} \cdots (x - \nu_k)^{m_k}$, $\nu_i \neq \nu_j$ for $i \neq j$ and $c_{i,j}$ are constants which depend on the first t moments $\{\tau_i\}_{i=0}^{t-1}$ of τ.

Lemma 3.2. *For any two* MOPS's $\{P_n(x)\}_{n=0}^\infty$ *and* $\{Q_n(x)\}_{n=0}^\infty$ *relative to* σ *and* τ*, respectively, the following are equivalent*:

(i) σ *and* τ *satisfy the relation* (3.1) *for some nonzero monic polynomials* $A(x)$ *and* $D(x)$;

(ii) *there are nonnegative integers* s *and* t *such that*

$$A(x)Q_n(x) = P_{n+s}(x) + \sum_{k=n-t}^{n+s-1} a_{n,k}P_k(x), \quad n \geq t, \tag{3.4}$$

where $a_{n,k}$ *are constants with* $a_{n,n-t} \neq 0$.

Proof. Assume that (3.1) holds. Expand $A(x)Q_n(x)$ as

$$A(x)Q_n(x) = \sum_{k=0}^{n+s} a_{n,k}P_k(x).$$

Then

$$a_{n,k}\langle\sigma, P_k^2\rangle = \langle\sigma, AQ_nP_k\rangle = \langle\tau, Q_nDP_k\rangle = \begin{cases} 0 & \text{if } k+t<n, \\ \text{nonzero} & \text{if } k+t=n \end{cases}$$

so that $a_{n,n-t} \neq 0$ and $a_{n,k} = 0$ for $0 \leq k < n-t$. Hence (3.4) holds.

Conversely, assume that (3.4) holds. Then

$$\begin{aligned} \langle A\sigma, Q_n\rangle = \langle\sigma, AQ_n\rangle &= \langle\sigma, P_n\rangle + \sum_{k=n-t}^{n+s-1} a_{n,k}\langle\sigma, P_k\rangle \\ &= \begin{cases} 0 & \text{if } n \geq t+1 \\ \text{nonzero} & \text{if } n=t. \end{cases} \end{aligned}$$

Hence, $A\sigma = D\tau$ for some polynomial $D(x)$ of degree t. □

Lemma 3.3. *Let* $A(x) = (x-a_1)\cdots(x-a_s)$ *and* $D(x) = (x-d_1)\cdots(x-d_t)$ *be monic polynomials of degree* s *and* t*, respectively. If* $a_i \neq d_j$ *for* $1 \leq i \leq s$ *and* $1 \leq j \leq t$*, then* $\{x^iA(x), x^jD(x) \mid 0 \leq i \leq t-1, 0 \leq j \leq s-1\}$ *are linearly independent.*

Proof. Let

$$A_j(x) = \begin{cases} 1 & j=0 \\ (x-a_1)\cdots(x-a_j) & 1 \leq j \leq s, \end{cases}$$

and

$$D_i(x) = \begin{cases} 1 & i = 0 \\ (x-d_1)\cdots(x-d_i) & 1 \le i \le t. \end{cases}$$

Assume that

$$\sum_{i=0}^{t-1} \alpha_i D_i(x)A(x) + \sum_{j=1}^{s-1} \beta_j A_j D(x) \equiv 0, \tag{3.5}$$

where α_i and β_j are constants. Set $x = d_1$. Then $\alpha_0 A(d_1) = 0$ so that $\alpha_0 = 0$. Then

$$\sum_{i=1}^{t-1} \alpha_i \frac{D_i(x)}{x-d_1} A(x) + \sum_{j=0}^{s-1} \beta_j A_j(x) \frac{D(x)}{x-d_1} \equiv 0$$

in which, if we set $x = d_2$, then $\alpha_1 A(d_2) = 0$ so that $\alpha_1 = 0$. Continuing the same process, we obtain $\alpha_0 = \alpha_1 = \cdots = \alpha_{t-1} = 0$, and so $\beta_0 = \beta_1 = \cdots = \beta_{s-1} = 0$ from (3.5). Hence $\{D_i(x)A(x), A_j(x)D(x) \mid 0 \le i \le t-1, 0 \le j \le s-1\}$ are linearly independent, and so they span $\mathbf{P}_{s+t-1}$, the space of polynomials of degree $\le s+t-1$. Let H be the span of $\{x^i A(x), x^j D(x) \mid 0 \le i \le t-1, 0 \le j \le s-1\}$. Then $\{D_i(x)A(x), A_j(x)D(x) \mid 0 \le i \le t-1, 0 \le j \le s-1\} \subseteq H \subseteq \mathbf{P}_{s+t-1}$ so that $H = \mathbf{P}_{s+t-1}$, that is, $\{x^i A(x), x^j D(x) \mid 0 \le i \le t-1, 0 \le j \le s-1\}$ are linearly independent. □

Now we are ready to state and prove our main result.

Theorem 3.4. *Assume that $A(x)$ and $D(x)$ are nonzero monic polynomials of degree s and t, respectively, with $s + t \ge 1$. Let*

$$A(x) = (x-\alpha_1)^{s_1} \cdots (x-\alpha_m)^{s_m}$$

where $\alpha_i \ne \alpha_j$ for $i \ne j$ if $s := s_1 + \cdots + s_m \ge 1$ and

$$D(x) = (x-d_1)(x-d_2)\cdots(x-d_t) \text{ if } t \ge 1.$$

We also assume that $\alpha_i \ne d_j$ for $1 \le i \le m$ and $1 \le j \le t$. Then the moment functional τ defined by the relation (3.1) is quasi-definite if and only if

$$|M_k| \ne 0 \text{ for } 1 \le k \le s+t \text{ and } |N_n| \ne 0 \text{ for } n \ge s+t, \tag{3.6}$$

where

$$M_k := \begin{bmatrix} \langle \mu_0, P_0 \rangle & \langle \mu_0, P_1 \rangle & \cdots & \langle \mu_0, P_{k-1} \rangle \\ \vdots & \vdots & & \vdots \\ \langle \mu_{k-1}, P_0 \rangle & \langle \mu_{k-1}, P_1 \rangle & \cdots & \langle \mu_{k-1}, P_{k-1} \rangle \end{bmatrix}, \quad 1 \le k \le s+t$$

$$N_n := \begin{bmatrix} \langle \mu_0, P_{n-t} \rangle & \langle \mu_0, P_{n-t+1} \rangle & \cdots & \langle \mu_0, P_{n+s-1} \rangle \\ \vdots & \vdots & & \vdots \\ \langle \mu_{t-1}, P_{n-t} \rangle & \langle \mu_{t-1}, P_{n-t+1} \rangle & \cdots & \langle \mu_{t-1}, P_{n+s-1} \rangle \\ P_{n-t}(\alpha_1) & P_{n-t+1}(\alpha_1) & \cdots & P_{n+s-1}(\alpha_1) \\ \vdots & \vdots & & \vdots \\ P_{n-t}^{(s_1-1)}(\alpha_1) & P_{n-t+1}^{(s_1-1)}(\alpha_1) & \cdots & P_{n+s-1}^{(s_1-1)}(\alpha_1) \\ \vdots & \vdots & & \vdots \\ P_{n-t}^{(s_m-1)}(\alpha_m) & P_{-t+1}^{(s_m-1)}(\alpha_m) & \cdots & P_{n+s-1}^{(s_m-1)}(\alpha_m) \end{bmatrix}, \quad n \ge s+t,$$

where

$$\mu_i = \begin{cases} D_i(x)\tau & 0 \le i \le t, \\ A_{i-t}(x)D(x)\tau & t \le i \le s+t, \end{cases}$$

$D_0(x) = 1$, $D_i(x) = (x-d_1)\cdots(x-d_i)$ *for* $1 \le i \le t$, $A_0(x) = 1$, $A_i(x) = (x-\alpha_1)^{s_1}\cdots(x-\alpha_{k-1})^{s_{k-1}}(x-\alpha_k)^l$ *for* $1 \le i = s_1 + \cdots + s_{k-1} + l \le s$. *When* τ *is quasi-definite,* $Q_0(x) = 1$,

$$(3.7) \qquad Q_k(x) = \frac{(-1)^k}{|M_k|} \begin{vmatrix} P_0(x) & P_1(x) & \cdots & P_k(x) \\ \langle \mu_0, P_0 \rangle & \langle \mu_0, P_1 \rangle & \cdots & \langle \mu_0, P_k \rangle \\ \vdots & \vdots & & \vdots \\ \langle \mu_{k-1}, P_0 \rangle & \langle \mu_{k-1}, P_1 \rangle & \cdots & \langle \mu_{k-1}, P_k \rangle \end{vmatrix}, \quad 1 \le k \le s+t-1,$$

(3.8)
$$A(x)Q_n(x) = \frac{(-1)^{s+t}}{|N_n|} \begin{vmatrix} P_{n-t}(x) & P_{n-t+1}(x) & \cdots & P_{n+s}(x) \\ \langle \mu_0, P_{n-t} \rangle & \langle \mu_0, P_{n-t+1} \rangle & \cdots & \langle \mu_0, P_{n+s} \rangle \\ \vdots & \vdots & & \vdots \\ \langle \mu_{t-1}, P_{n-t} \rangle & \langle \mu_{t-1}, P_{n-t+1} \rangle & \cdots & \langle \mu_{t-1}, P_{n+s} \rangle \\ P_{n-t}(\alpha_1) & P_{n-t+1}(\alpha_1) & \cdots & P_{n+s}(\alpha_1) \\ \vdots & \vdots & & \vdots \\ P_{n-t}^{(s_m-1)}(\alpha_m) & P_{n-t+1}^{(s_m-1)}(\alpha_m) & \cdots & P_{n+s}^{(s_m-1)}(\alpha_m) \end{vmatrix},$$
$$n \geq s+t.$$

Proof. Assume that τ is quasi-definite. Then we have (3.4) so that

$$\sum_{k=n-t}^{n+s-1} a_{n,k} \langle \mu_i, P_k \rangle = -\langle \mu_i, P_{n+s} \rangle, \quad 0 \leq i \leq t-1$$
$$\sum_{k=n-t}^{n+s-1} a_{n,k} P_k^{(j)}(\alpha_i) = -P_{n+s}^{(j)}(\alpha_i),$$
$$1 \leq i \leq m \quad \text{and} \quad 0 \leq j \leq s_i - 1$$

or equivalently in matrix form

$$N_n [a_{n,k}]_{k=n-t}^{n+s-1} = -[\langle \mu_0, P_{n+s} \rangle, \dots, P_{n+s}^{(s_m-1)}(\alpha_m)]^T \tag{3.9}$$

if $n \geq s+t$. Assume $|N_n| = 0$ for some $n \geq s+t$. Then the system of equation (3.9) has another solution $[b_k]_{k=n-t}^{n+s-1} \neq [a_{n,k}]_{k=n-t}^{n+s-1}$. That is, if we set

$$R_{n+s}(x) := P_{n+s}(x) + \sum_{k=n-t}^{n+s-1} b_k P_k(x),$$

then

$$R_{n+s}(x) = A(x) S_n(x) (\partial(S_n) = n) \quad \text{and} \quad \langle \mu_i, R_{n+s} \rangle = 0, \quad 0 \leq i \leq t-1.$$

Hence

$$R_{n+s}(x) - A(x)Q_n(x) = \sum_{k=n-t}^{n+s-1} (b_k - a_{n,k}) P_k(x) = A(x) T_{n-1}(x)$$

where $T_{n-1}(x) = S_n(x) - Q_n(x)$ is of degree $\leq n-1$. Then

$$\langle \mu_i, A(x)T_{n-1}(x)\rangle = 0, \quad 0 \leq i \leq t-1 \tag{3.10}$$

and

$$\langle x^i D(x)\tau, T_{n-1}(x)\rangle = 0, \quad 0 \leq i \leq s-1. \tag{3.11}$$

Since (3.10) implies $\langle \tau, x^i A(x)T_{n-1}(x)\rangle = 0$, $0 \leq i \leq t-1$ and $\{x^i A(x), x^j D(x) \mid 0 \leq i \leq t-1, 0 \leq j \leq s-1\}$ are linearly independent by Lemma 3.3, (3.10) and (3.11) imply

$$\langle \tau, x^i T_{n-1}(x)\rangle = 0, \quad 0 \leq i \leq s+t-1. \tag{3.12}$$

Set $T_{n-1}(x) = \sum_{k=0}^{n-1} e_k Q_k(x)$. Then

$$\begin{aligned} A(x)T_{n-1}(x) &= \sum_{k=0}^{n-1} e_k \sum_{j=k-t}^{k+s} a_{k,j} P_j(x) \\ &= \sum_{j=0}^{s-1} \left(\sum_{k=0}^{j+t} e_k a_{k,j} \right) P_j(x) \\ &\quad + \sum_{j=s}^{n-t-1} \left(\sum_{k=j-s}^{j+t} e_k a_{k,j} \right) P_j(x) \\ &\quad + \sum_{j=n-t}^{n+s-1} \left(\sum_{k=j-s}^{n-1} e_k a_{k,j} \right) P_j(x). \end{aligned}$$

$(a_{k,k+s} = 1 \text{ and } P_j(x) = 0 \text{ for } j < 0)$

Since $A(x)T_{n-1}(x) = \sum_{k=n-t}^{n+s-1} (b_k - a_{n,k}) P_k(x)$,

$$\sum_{k=0}^{j+t} e_k a_{k,j} = 0, \quad 0 \leq j \leq s-1 \tag{3.13}$$

$$\sum_{k=j-s}^{j+t} e_k a_{n,k} = 0, \quad s \leq j \leq n-t-1. \tag{3.14}$$

By (3.12), $e_0 = e_1 = \cdots = e_{s+t-1} = 0$ so that (3.13) holds and then by induction on (3.14) $e_k = 0$ for $0 \le k \le n-1$. Hence $T_{n-1}(x) = 0$ and so $b_k = a_{n,k}$ for $n-t \le k \le n+s-1$, which is a contradiction. Hence $|N_n| \ne 0$, $n \ge s+t$.

For $1 \le k \le s+t$, write $Q_k(x)$ as

$$Q_k(x) = P_k(x) + \sum_{j=0}^{k-1} a_{k,j} P_j(x).$$

Then

$$\sum_{j=0}^{k-1} \langle \mu_i, P_j \rangle a_{k,j} = -\langle \mu_i, P_k \rangle, \quad 0 \le i \le k-1,$$

that is,

$$M_k [a_{k,j}]_{j=0}^{k-1} = -[\langle \mu_i, P_k \rangle]_{i=0}^{k-1}. \tag{3.15}$$

Assume $|M_k| = 0$ for some k with $1 \le k \le s+t$. Then the system of equation (3.15) has another solution $[b_j]_{j=0}^{k-1} \ne [a_{k,j}]_{j=0}^{k-1}$. That is, if we set

$$S_k(x) = P_k(x) + \sum_{j=0}^{k-1} b_j P_j(x),$$

then $\langle \mu_i, S_k \rangle = 0$, $0 \le i \le k-1$. Then $\langle \tau, x^i S_k \rangle = 0$, $0 \le i \le k-1$ so that $S_k(x) = Q_k(x)$, which is a contradiction. Hence $|M_k| \ne 0$, $1 \le k \le s+t$.

Conversely, assume that the conditions (3.6) hold. Define polynomials $Q_0(x) = 1$ and $Q_n(x)$, $n \ge 1$, by (3.7) and (3.8). Then $Q_n(x)$ are monic polynomials of degree n. Now $\langle \tau, Q_0 \rangle = \langle \tau, P_0 \rangle = M_1 \ne 0$. For $1 \le k \le s+t-1$,

$$\langle \mu_i, Q_k \rangle = \begin{cases} 0 & \text{if } 0 \le i \le k-1 \\ |M_{k+1}|/|M_k| & \text{if } i = k \end{cases}$$

so that

$$\langle \tau, x^i Q_k \rangle = \begin{cases} 0 & \text{if } 0 \le i \le k-1 \\ \text{nonzero} & \text{if } i = k. \end{cases}$$

For $n \geq s+t$, $\langle \mu_i, AQ_n(x)\rangle = 0$, $0 \leq i \leq t-1$, so that

$$\begin{cases} \langle \tau, x^i AQ_n\rangle = 0 & 0 \leq i \leq t-1 \\ \langle \tau, x^j DQ_n\rangle = 0 & 0 \leq j \leq n = t = 1. \end{cases} \tag{3.16}$$

Since $\{x^i A(x), x^j D(x) \mid 0 \leq i \leq t-1, 0 \leq j \leq n-t-1\}$ are linearly independent for $n \geq s+t$, (3.16) implies $\langle \tau, x^m Q_n\rangle = 0$, $0 \leq m \leq n-1$ for $n \geq s+t$. Finally,

$$\langle \tau, x^n Q_n(x)\rangle = \langle \sigma, x^{n-t} A(x) Q_n(x)\rangle = (-1)^{s+t}\langle \sigma, P_{n-t}^2\rangle \frac{|N_{n+1}|}{|N_n|} \neq 0, \quad n \geq s+t.$$

Hence, $\{Q_n(x)\}_{n=0}^{\infty}$ is the MOPS relative to τ. □

We now consider two special cases.

Corollary 3.5. *Assume that $D(x) = 1$ so that $\tau = A(x)\sigma$. Then τ is quasi-definite if and only if $|M_k| \neq 0$, $1 \leq k \leq s$ and $|N_n| \neq 0$, $n \geq s$, where*

$$N_n = \begin{bmatrix} P_n(\alpha_1) & P_{n+1}(\alpha_1) & \cdots & P_{n+s-1}(\alpha_1) \\ \vdots & \vdots & & \vdots \\ P_n^{(s_1-1)}(\alpha_1) & P_{n+1}^{(s_1-1)}(\alpha_1) & \cdots & P_{n+s-1}^{(s_1-1)}(\alpha_1) \\ \vdots & \vdots & & \vdots \\ P_n^{(s_m-1)}(\alpha_m) & P_{n+1}^{(s_m-1)}(\alpha_m) & \cdots & P_{n+s-1}^{(s_m-1)}(\alpha_m) \end{bmatrix}, \quad n \geq s.$$

Corollary 3.5 was first proved by Belmehdi [**2**] as: $\tau = A(x)\sigma$ is quasi-definite if and only $|N_n| \neq 0$, $n \geq 0$, which are equivalent to the conditions in Corollary 3.5.

Corollary 3.6. *Assume that $A(x) = 1$ so that $D(x)\tau = \sigma$. Then τ is quasi-definite if and only if $|M_k| \neq 0$, $1 \leq k \leq t-1$ and $|N_n| \neq 0$, $n \geq t$, where*

(3.17)

$$N_n = \begin{bmatrix} \langle \mu_0, P_{n-t}\rangle & \langle \mu_0, P_{n-t+1}\rangle & \cdots & \langle \mu_0, P_{n-1}\rangle \\ \vdots & \vdots & & \vdots \\ \langle \mu_{t-1}, P_{n-t}\rangle & \langle \mu_{t-1}, P_{n-t+1}\rangle & \cdots & \langle \mu_{t-1}, P_{n-1}\rangle \end{bmatrix}, \quad n \geq t.$$

Corollary 3.6 for $t = 2$ was proved by Branquinho and Marcellán [**3**], (see also [**1**]), by a different method in which they must handle the two cases separately when roots of $D(x)$ are simple or not. When $D(x)$ has simple roots, the condition found in [**3**] is different from ours. To see the connection between them, let's reformulate the condition (3.17) when $D(x)$ has only simple roots, i.e., $d_i \neq d_j$ for $i \neq j$. In this case the polynomials $\prod_{\substack{i=1\\ i\neq j}}^{k}(x-d_i)$, $1 \leq j \leq t$, are linearly independent so that they can span all polynomials of degree $\leq t-1$. Then we may replace the matrix N_n in (3.17) by

$$\tilde{N}_n := [\langle \tilde{\mu}_i, P_{n-t+j-1}\rangle]_{i,j=1}^{t},$$

where $\tilde{\mu}_j := \prod_{\substack{i=1\\ i\neq j}}^{k}(x-d_i)\tau$, $1 \leq j \leq t$.

For example, if $t = 2$, then

$$\langle \tilde{\mu}_1, P_n\rangle = \langle (x-d_2)\tau, P_n\rangle = \sigma_0 P_{n-1}^{(1)}(d_1) + (\tau_1 - d_2\tau_0)P_n(d_1)$$

since $(x-d_2)\tau = (x-d_1)^{-1}\sigma + (\tau_1 - d_2\tau_0)\delta(x-d_1)$ and $\langle (x-d_1)^{-1}\sigma, P_n\rangle = \sigma_0 P_{n-1}^{(1)}(d_1)$. Now the conditions $|\tilde{N}_n| \neq 0$, $n \geq 2$, and $|M_1| \neq 0$ coincide with the one in [**3**, Theorem 6].

Finally, we give two examples illustrating Theorem 3.4.

Example 3.1. Let $x^3\tau = \sigma$, where σ is the Laguerre moment functional:

$$\langle \sigma, \pi(x)\rangle = \int_0^\infty \pi(x)x^3 e^{-x}\,dx, \quad \pi(x) \in \mathbf{P}.$$

Then

$$\begin{aligned}\langle \tau, \pi(x)\rangle = \int_0^\infty & \pi(x)e^{-x}\,dx + a\langle \delta(x), \pi\rangle \\ & + b\langle \delta'(x), \pi\rangle + c\langle \delta''(x), \pi\rangle, \quad \pi(x) \in \mathbf{P},\end{aligned}$$

where a, b, c are constants. In fact we have $a = \tau_0 - 1$, $b = 1 - \tau_1$, $c = (1/2)\tau_2 - 1$. Let $\{P_n(x)\}_{n=0}^\infty$ be the MOPS relative to σ.

In order to compute $\langle x^k\tau, P_n\rangle$ for $k = 0, 1, 2$, we need the following for the Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$, $\alpha > -1$, [**4, 13**],

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}, \quad n \geq 0,$$

$$\frac{d}{dx} L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x), \quad n \geq 0, \tag{3.18}$$

$$L_n^{(\alpha+1)}(x) = \sum_{k=0}^{n} L_k^{(\alpha)}(x), \quad n \geq 0. \tag{3.19}$$

Then $P_n(x) = (-1)^n n! L_n^{(3)}(x)$, $n \geq 0$, so that by (3.18)

$$\begin{aligned} P_n(0) &= (-1)^n \frac{(n+3)!}{6}, \quad n \geq 0, \\ P_n'(0) &= (-1)^{n+1} \frac{n(n+3)!}{24}, \quad n \geq 0, \\ P_n''(0) &= (-1)^n \frac{n(n-1)(n+3)!}{120}, \quad n \geq 0. \end{aligned}$$

Using (3.19) repeatedly, we obtain

$$\begin{aligned} L_n^{(3)}(x) &= \sum_{k=0}^{n} L_k^{(2)}(x) = \sum_{k=0}^{n} (n-k+1) L_k^{(1)}(x) \\ &= \sum_{k=0}^{n} \frac{1}{2}(n-k+1)(n-k+2) L_k^{(0)}(x), \quad n \geq 0, \end{aligned}$$

so that

$$\begin{aligned} \int_0^\infty x^2 e^{-x} P_n(x)\,dx &= (-1)^n n! \int_0^\infty x^2 e^{-x} L_0^{(2)}(x)\,dx \\ &= 2(-1)^n n!, \quad n \geq 0, \\ \int_0^\infty x e^{-x} P_n(x)\,dx &= (-1)^n n!(n+1) \int_0^\infty x e^{-x} L_0^{(1)}(x)\,dx \\ &= (-1)^n (n+1)!, \quad n \geq 0, \\ \int_0^\infty e^{-x} P_n(x)\,dx &= (-1)^n n! \frac{(n+1)(n+2)}{2} \int_0^\infty e^{-x} L_0^{(0)}(x)\,dx \\ &= (-1)^n \frac{(n+2)!}{2}, \quad n \geq 0. \end{aligned}$$

Hence

$$\begin{aligned}\langle \tau, P_n\rangle &= \int_0^\infty e^{-x} P_n(x)\,dx + aP_n(0) - bP_n'(0) + cP_n''(0)\\ &= (-1)^n(n+2)!\left[\frac{1}{2} + \frac{a}{6}(n+3) + \frac{b}{24}n(n+3)\right.\\ &\qquad\left. + \frac{c}{120}(n-1)n(n+3)\right], \quad n \geq 0,\\ \langle x\tau, P_n\rangle &= \int_0^\infty xe^{-x} P_n(x)(x)\,dx - bP_n(0) + 2cP_n'(0)\\ &= (-1)^n(n+1)!\left[1 - \frac{b}{6}(n+2)(n+3)\right.\\ &\qquad\left. - \frac{c}{12}n(n+2)(n+3)\right], \quad n \geq 0,\\ \langle x^2\tau, P_n\rangle &= \int_0^\infty x^2e^{-x} P_n(x)\,dx + 2cP_n(0)\\ &= (-1)^n n!\left[2 + \frac{c}{3}(n+1)(n+2)(n+3)\right], \quad n \geq 0.\end{aligned}$$

Therefore,

$$\begin{aligned}|M_1| &= \langle \tau, P_0\rangle = 1 + a,\\ |M_2| &= \begin{vmatrix} \langle \tau, P_0\rangle & \langle \tau, P_1\rangle \\ \langle x\tau, P_0\rangle, \langle x\tau, P_1\rangle & \end{vmatrix}\\ &= 1 + 2a + 2b + 2c + 2ac - b^2,\\ |N_n| &= \begin{vmatrix} \langle \tau, P_{n-3}\rangle & \langle \tau, P_{n-2}\rangle & \langle \tau, P_{n-1}\rangle \\ \langle x\tau, P_{n-3}\rangle & \langle x\tau, P_{n-2}\rangle & \langle x\tau, P_{n-1}\rangle \\ \langle x^2\tau, P_{n-3}\rangle & \langle x^2\tau, P_{n-2}\rangle & \langle x^2\tau, P_{n-1}\rangle \end{vmatrix}\\ &= (-1)^n(n-3)!(n-2)!(n-1)!D_n, \quad n \geq 3,\end{aligned}$$

where

$$\begin{aligned}D_n = \frac{1}{2160}[&c^3n^9 - 6c^3n^7 + 216c^2n^6 + (9c^3 - 720c^2 + 360bc)n^5\\ &+ 360(b^2 + c^2 - 2ac - 2bc)n^4 - (4c^3 - 720c^2 + 4320c + 360bc)n^3\\ &- (360b^2 + 4320b - 8640c - 720ac - 720bc + 576c^2)n^2\\ &- 4320(a - b + c)n - 4320].\end{aligned}$$

Note that $D_1 = -2|M_1|$ and $D_2 = -2|M_2|$.

Hence, by Corollary 3.6, τ is quasi-definite if and only if $D_n \neq 0$, $n \geq 1$.

In particular, when $c = 0$, i.e., $\tau_2 = 2$, we have

$$\begin{aligned}
|M_1| &= 1 + a,\\
|M_2| &= 1 + 2a + 2b - b^2,\\
|N_n| &= (-1)^n (n-3)!(n-2)!(n-1)! D_n, \quad n \geq 3,
\end{aligned}$$

where

$$D_n = \frac{1}{6}[b^2 n^4 - (b^2 + 12b)n^2 + 12(b-a) - 12].$$

Hence τ is quasi-definite, if and only if

$$b^2 n^4 - (b^2 + 12b)n^2 + 12(b-a) - 12 \neq 0, \quad n \geq 1.$$

In this case, the MOPS $\{Q_n(x)\}_{n=0}^{\infty}$ relative to τ is

$$Q_1(x) = \frac{-1}{|M_1|}\begin{vmatrix} P_0(x) & P_1(x) \\ \langle \tau, P_0 \rangle & \langle \tau, P_1 \rangle \end{vmatrix} = P_1(x) + \frac{4a + b + 3}{|M_1|} P_0(x),$$

$$\begin{aligned}
Q_2(x) &= \frac{1}{|M_2|}\begin{vmatrix} P_0(x) & P_1(x) & P_2(x) \\ \langle \tau, P_0 \rangle & \langle \tau, P_1 \rangle & \langle \tau, P_2 \rangle \\ \langle x\tau, P_0 \rangle & \langle x\tau, P_1 \rangle & \langle x\tau, P_2 \rangle \end{vmatrix}\\
&= P_2(x) + \frac{2(3 + 7a + 9b - 5b^2)}{|M_2|} P_1(x)\\
&\quad + \frac{2(3 + 8a + 13b - 10b^2)}{|M_2|} P_0(x),
\end{aligned}$$

$$Q_n(x) = \frac{-1}{|N_n|}\begin{vmatrix} P_{n-3}(x) & P_{n-2}(x) & P_{n-1}(x) & P_n(x) \\ \langle \tau, P_{n-3}\rangle & \langle \tau, P_{n-2}\rangle & \langle \tau, P_{n-1}\rangle & \langle \tau, P_n\rangle \\ \langle x\tau, P_{n-3}\rangle & \langle x\tau, P_{n-2}\rangle & \langle x\tau, P_{n-1}\rangle & \langle x\tau, P_n\rangle \\ \langle x^2\tau, P_{n-3}\rangle & \langle x^2\tau, P_{n-2}\rangle & \langle x^2\tau, P_{n-1}\rangle & \langle x^2\tau, P_n\rangle \end{vmatrix}$$

$$\begin{aligned} &= P_n(x) + \frac{n}{6D_n}[3b^2n^4 + 4b^2n^3 - 3b(b+12)n^2 \\ &\qquad - 4(9a - 3b + b^2)n - 12a + 12b - 36]P_{n-1}(x) \\ &\quad + \frac{n(n-1)}{6D_n}[3b^2n^4 + 8b^2n^3 + (3b^2 - 36b)n^2 \\ &\qquad - 2(18a + 6b + b^2)n - 24a + 12b - 36]P_{n-2}(x) \\ &\quad + \frac{n(n-1)(n-2)}{6D_n}[b^2n^4 + 4b^2n^3 + b(5b-12)n^2 \\ &\qquad - (12a + 12b - 2b^2)n - 12(a+1)]P_{n-3}(x), \\ &\qquad n \geq 3. \end{aligned}$$

Example 3.2. Let $(1+x)\tau = (1-x)\sigma$, where σ is the Jacobi moment functional:

$$\langle \sigma, \pi(x)\rangle = \int_{-1}^{1} \pi(x)(1+x)\,dx, \quad \pi(x) \in \mathbf{P}.$$

Then

$$\langle \tau, \pi(x)\rangle = \int_{-1}^{1} \pi(x)(1-x)\,dx + a\langle \delta(1+x), \pi\rangle, \quad \pi(x) \in \mathbf{P},$$

where $a = \tau_0 - 2$ is a constant. Let $\{P_n(x)\}_{n=0}^{\infty}$ be the MOPS relative to σ.

In order to compute $\langle \tau, P_n\rangle$ and $\langle (1+x)\tau, P_n\rangle$, we need the following for the Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$, $\alpha, \beta > -1$, [**4, 13**]:

$$P_n^{(\alpha,\beta)}(x) = 2^{-n}\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\binom{n+\beta}{k}(x-1)^k(x+1)^{n-k}, \quad n \geq 0,$$

$$\begin{aligned} (1-x)P_n^{(0,1)}(x) &= -A(n)P_{n+1}^{(0,1)}(x) + (1-B(n))P_n^{(0,1)}(x) \\ &\quad - C(n)P_{n-1}^{(0,1)}(x), \quad n \geq 1, \end{aligned} \tag{3.20}$$

where

$$A(n) = \frac{n+2}{2n+3}, \quad B(n) = \frac{1}{(2n+1)(2n+3)}, \quad C(n) = \frac{n}{2n+1},$$

$$(3.21) \qquad (2n+1)P_n^{(0,0)}(x) = (n+1)P_n^{(0,1)}(x) + nP_{n-1}^{(0,1)}(x), \quad n \geq 1.$$

Then $P_n(x) = (2^n n!(n+1)!/(2n+1)!)P_n^{(0,1)}(x)$, $n \geq 0$, so that

$$P_n(1) = \frac{2^n n!(n+1)!}{(2n+1)!}, \quad n \geq 0,$$
$$P_n(-1) = \frac{(-1)^n 2^n ((n+1)!)^2}{(2n+1)!}, \quad n \geq 0.$$

Using (3.21) inductively, we have

$$P_n^{(0,1)}(x) = \frac{2n+1}{n+1}P_n^{(0,0)}(x) - \frac{2n-1}{n+1}P_{n-1}^{(0,0)}(x) + \frac{2n-3}{n+1}P_{n-2}^{(0,0)}(x) + \cdots + \frac{(-1)^n}{n+1}P_0^{(0,0)}(x).$$

Then by (3.20) the coefficient of $P_0^{(0,0)}(x)$ in the expansion of $(1-x)P_n^{(0,1)}(x)$ in terms of $\{P_k^{(0,0)}(x)\}_{k=0}^{n+1}$ is

$$-A(n)\frac{(-1)^{n+1}}{n+2} + (1-B(n))\frac{(-1)^n}{n+1} - C(n)\frac{(-1)^{n-1}}{n} = 2\frac{(-1)^n}{n+1}.$$

Hence

$$\langle \tau, P_n \rangle = \int_{-1}^{1} (1-x)P_n \, dx + aP_n(-1)$$
$$= \frac{(-1)^n 2^n (n!)^2}{(2n+1)!}[4 + a(n+1)^2], \quad n \geq 1,$$
$$\langle \tau, P_0 \rangle = \int_{-1}^{1} (1-x) \, dx + a = 2 + a.$$

Using (3.20) we also have

$$\langle (1+x)\tau, P_0 \rangle = \int_{-1}^{1} (1+x)(1-x) \, dx = \frac{4}{3},$$
$$\langle (1+x)\tau, P_1 \rangle = \int_{-1}^{1} (1+x)(1-x)P_1(x) \, dx = -\frac{4}{9}.$$

Therefore,

$$|M_1| = \langle \tau, P_0 \rangle = 2 + a,$$

$$|M_2| = \begin{vmatrix} \langle \tau, P_0 \rangle & \langle \tau, P_1 \rangle \\ \langle (1+x), \tau, P_0 \rangle & \langle (1+x)\tau, P_1 \rangle \end{vmatrix} = \frac{4}{9}(2+3a),$$

$$|N_n| = \begin{vmatrix} \langle \tau, P_{n-1} \rangle & \langle \tau, P_n \rangle \\ P_{n-1} & P_n(1) \end{vmatrix} = \frac{n((n-1)!)^4 4^n}{2((2n-1)!)^2} D_n, \quad n \geq 2,$$

where

$$D_n = 4 + n(n+1)a.$$

Note that $D_1 = 2|M_1|$, $D_2 = (9/2)|M_2|$.

Hence, by Theorem 3.4, τ is quasi-definite if and only if $D_n \neq 0$, $n \geq 1$, i.e.,

$$a \neq \frac{-4}{n(n+1)}, \quad n \geq 1.$$

In this case the MOPS $\{Q_n(x)\}_{n=0}^{\infty}$ relative to τ is

$$Q_1(x) = \frac{-1}{|M_1|} \begin{vmatrix} P_0(x) & P_1(x) \\ \langle \tau, P_0 \rangle & \langle \tau, P_1 \rangle \end{vmatrix} = P_1(x) + \frac{4(a+1)}{3(a+2)} P_0(x),$$

$$\begin{aligned}(1-x)Q_n(x) &= \frac{-1}{|N_n|} \begin{vmatrix} P_{n-1}(x) & P_n(x) & P_{n+1}(x) \\ \langle \tau, P_{n-1} & \langle \tau, P_n \rangle & \langle \tau, P_{n+1} \rangle \\ P_{n-1}(1) & P_n(1) & P_{n+1}(1) \end{vmatrix} \\ &= -P_{n+1}(x) - \frac{n+1}{(2n+3)(2n+1)D_n} \\ &\quad \times (an^3 + 5an^2 + 4(a+1)n - 4a - 12)P_n(x) \\ &\quad + \frac{n(n+1)}{(2n+1)^2 D_n}(an^2 + 3an + 2a + 4)P_{n-1}(x), \quad n \geq 2.\end{aligned}$$

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