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DIRICHLET SPLINES AS FRACTIONAL INTEGRALS OF B-SPLINES

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ABSTRACT. Using *Dirichlet averages* we generalize the notion of a classical divided difference of a function by introducing a parameter $\mathbf{r}$ in $\mathbf{R}_+^{k+1}$. The case $\mathbf{r}$ in $\mathbf{N}^{k+1}$ is related to divided differences with multiple knots. We give an interpretation of these generalized differences in terms of fractional operators applied to classical divided differences considered as functions of their knots. The result is then applied to show that *Dirichlet splines* can be seen as fractional derivatives of B-splines.

1. Introduction. Splines are a well-established class of functions in many fields of applied analysis. Their properties, for example, allow good approximations and efficient algorithms for computation. In statistics, splines have an even older history. They occur as density functions of multivariate probability measures. Both fields seemed to have been unaware of their common interest in spline functions up into the 1980s. In 1986 papers by Dahmen and Micchelli [**7**] and Karlin, Micchelli and Rinott [**10**] appeared which began to point out these connections.

One of the key concepts in approximation with spline functions is a convenient basis for the underlying function space, the so-called *B-splines*. B-splines are nonnegative, compactly supported and are solutions to certain minimalization problems, just to mention a few of their central properties. From the statistical point of view, spline functions occur in connection with the uniform distribution over the standard simplex. To investigate more general classes of distributions over the simplex, density functions have been introduced in the defining equation for spline densities, e.g., the Dirichlet distribution and the Gamma distribution, respectively. This has led to so-called *Dirichlet*

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splines, which are probability densities with respect to the Dirichlet distribution.

In this paper, we give a new approach to Dirichlet splines. To be more precise, we will prove that Dirichlet splines can be interpreted as fractional derivatives of classical B-splines. What is surprising in this context is the fact that B-splines have to be treated as a multivariate function of their knots, while the one-dimensional variable serves as parameter.

Our method to define and investigate Dirichlet splines is based on the concept of *Dirichlet averages* introduced by Carlson [**4**]. Dirichlet averages are integral averages of functions with respect to the Dirichlet measure. In Section 2, we give the definition and summarize some of their properties. We further use the representation of divided differences as Dirichlet averages to introduce *generalized divided differences* which allow us to connect the averages to B-splines. In Section 3, we introduce operators of fractional integration and differentiation and formulate our main result in terms of divided differences. Section 4 finally states some known results concerning Dirichlet splines and gives a reformulation of our theorem in terms of spline functions.

2. Dirichlet averages. For $\mathbf{b} \in \mathbf{R}_+^{k+1}$ we denote by

$$B(\mathbf{b}) = \frac{\Gamma(b_0)\cdots\Gamma(b_k)}{\Gamma(b_0+\cdots+b_k)}, \quad \mathbf{b} = (b_0, \ldots, b_k) \in \mathbf{R}_+^{k+1},$$

the $(k+1)$-*dimensional beta function.* Let $\Delta^k = \{\mathbf{u} = (u_0, \ldots, u_k) \in \mathbf{R}^{k+1} \mid u_j \geq 0, u_0 + \cdots + u_k = 1\}$ be the standard simplex in $\mathbf{R}^{k+1}$. The *generalized Dirichlet measure* on the standard simplex Δ^k is then defined as[1]

$$d\mu_{\mathbf{b}}(\mathbf{u}) = \frac{1}{B(\mathbf{b})} u_1^{b_1-1} \cdots u_k^{b_k-1} (1-u_1-\cdots-u_k)^{b_0-1}\, du_1 \cdots du_k.$$

Setting $\mathbf{b} = \mathbf{e} = (1, \ldots, 1)$ the measure reduces to the classical Lebesgue measure $k!\, du$. Let Ω denote a convex set in $\mathbf{C}$ and let

[1] The measure formally is taken over the set $\{\mathbf{u} \in \mathbf{R}^k \mid 0 \leq u_j, j = 1, \ldots, k, \sum_{j=1}^k, u_j \leq 1\}$ which in this paper will be identified with Δ^k.

$z = (z_0, \dots, z_k) \in \Omega^{k+1}$, $k \geq 1$. The *Dirichlet average* of a function f, measurable on Ω, is then defined as the integral

$$F(\mathbf{b}; \mathbf{z}) = \int_{\Delta^k} f(\mathbf{z} \cdot \mathbf{u})\, d\mu_{\mathbf{b}}(\mathbf{u}). \tag{2.1}$$

Dirichlet averages have been introduced by Carlson [**4**] in 1969 to give a new approach to some classical special functions. They can be seen as a weighted average of a function f over a set of points $z_0, \dots, z_k$ in the complex plane. The special choice of the *Dirichlet distribution* as a weight function explains the name given to these averages. We will go on using a capital letter to denote the average of the function under consideration. The following useful properties can immediately be seen from the definition, (cf [**4**, Chapter 5]):

- *Dirichlet averages are symmetric in the arguments* $z_0, \dots, z_k$ *if the same permutation is applied to both, the components of* $\mathbf{z}$ *and of the parameter* $\mathbf{b}$.

- *If two of the components of* $\mathbf{z}$ *coalesce, they can be replaced by one of them and adding up the corresponding components of the parameter* b.

We will see below that there is a representation of Dirichlet averages which will allow us to extend the value of the parameter to arbitrary complex vectors with $b_0 + \cdots + b_k \notin -\mathbf{N}_0$. But let us first look at two examples which give rise to a wide class of special functions. Following Carlson's notation we denote the average of the power function by

$$R_\gamma(\mathbf{b}; \mathbf{z}) = \int_{\Delta^k} (\mathbf{z} \cdot \mathbf{u})^\gamma \, d\mu_{\mathbf{b}}(\mathbf{u}), \quad \mathbf{z} \in \Omega^{k+1},$$

where $\Omega = \mathbf{C}$, if $\gamma \in \mathbf{C} \setminus \mathbf{N}$, and Ω is some half plane in the domain $\mathbf{C} \setminus \{0\}$ otherwise. Note that the average $R_\gamma(\mathbf{b}; \mathbf{z})$ is equivalent to *Lauricella's function* $F_D(-\gamma, b_0, \dots, b_k; b_0 + \cdots + b_k; 1 - z_0, \dots, 1 - z_k)$, cf. [**8**, Chapters 2 and 3] for details.

For $k = 1$, we get *Gauss's hypergeometric* ${}_2F_1$*-function*:

$$R_{-\gamma}(\mathbf{b}; \mathbf{z}) = z_0^{-\gamma} {}_2F_1 \left[\begin{array}{c} \gamma, b_1 \\ b_0 + b_1 \end{array} \middle| 1 - \frac{z_1}{z_0} \right], \quad \mathbf{z} = (z_0, z_1) \in \mathbf{R}^2,$$
$$z_0 \neq 0, \quad \Re(b_0 + b_1) > \Re b_1 > 0.$$

The Dirichlet average of the exponential function is defined as

$$S(\mathbf{b};\mathbf{z}) = \int_{\Delta^k} e^{\mathbf{z}\cdot\mathbf{u}}\, d\mu_{\mathbf{b}}(\mathbf{u}), \quad \mathbf{z} \in \mathbf{C}^{k+1}.$$

Again setting $k = 1$, we get *Kummer's confluent hypergeometric* ${}_1F_1$-*function*:

$$S(\mathbf{b};\mathbf{z}) = e^{z_0}\, {}_1F_1\left[\begin{array}{c} b_1 \\ b_1 + b_0 \end{array}\middle|\, z_1 - z_0\right], \quad \mathbf{z} = (z_0, z_1) \in \mathbf{C}^2.$$

Using the symmetry property one immediately obtains Gauss's and Kummer's transformation formulae, respectively.

Carlson states two further representations of Dirichlet averages, (cf [**4**, Chapter 6]) which lead to a generalized definition for arbitrary complex parameters with nonnegative real part using analytic continuation.

If the Taylor expansion of f converges in an open disk $B_r(\lambda)$ of radius r with center $\lambda \in \mathbf{C}$, we have,

$$F(\mathbf{b};\mathbf{z}) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(\lambda) R_n(\mathbf{b}; \mathbf{z} - \lambda\mathbf{e}), \quad \mathbf{z} \in B_r^{k+1}(\lambda). \tag{2.2}$$

Using Cauchy's integral formula and assuming $b_0 + \cdots + b_k \notin -\mathbf{N}_0$, the following representation of Dirichlet averages is valid for a holomorphic function f under a suitable choice of the Jordan curve Γ:

$$F^{(n)}(\mathbf{b};\mathbf{z}) = \frac{n!}{2\pi i} \int_{\Gamma} f(t) R_{-n-1}(\mathbf{b}; \mathbf{z} - t\mathbf{e})\, dt, \quad n \in \mathbf{N}_0, \tag{2.3}$$

where $F^{(n)}(\mathbf{b};\mathbf{z})$ here and in the following denotes the Dirichlet average of the nth derivative $f^{(n)}$ of f. The righthand side of (2.2) is an analytic function in $\mathbf{b} \in \mathbf{C}^{k+1}$ with $\Re b_j > 0$, $j = 0, \ldots, k$. Representation (2.3) finally allows to further relax the conditions on $\mathbf{b}$. We can therefore assume $\mathbf{b} \in \mathbf{C}^{k+1}$ with $b_0 + \cdots + b_k \notin -\mathbf{N}_0$.

In this sense, it can be shown that a vanishing parameter b_j, $j = 0, \ldots, k$, can be omitted together with the corresponding component z_j of $\mathbf{z}$. Nevertheless, for the rest of the paper we want to assume the parameter vector $\mathbf{b}$ to be in $\mathbf{R}_+^{k+1}$, just for convenience.

Divided differences can be expressed in terms of Dirichlet averages with $\mathbf{b} = \mathbf{e}$. To see this, we have to calculate the average of a differentiable function f at two points x, y in its domain of differentiability

$$F^{(1)}(1,1;x,y) = \frac{1}{y-x}\int_x^y f'(t)\,dt = \frac{f(y)-f(x)}{y-x} = [x,y]f.$$

This simple observation can be inductively carried on to higher order differences to get a representation for $k+1$ pairwise different knots $x_0,\dots,x_k$ and a k-times differentiable function f. Indeed,

$$[x_0,\dots,x_k]f = \frac{1}{k!}F^{(k)}(1,\dots,1;x_0,\dots,x_k) = \frac{1}{k!}F^{(k)}(\mathbf{e};\mathbf{x}). \tag{2.4}$$

Recall that the kth *divided difference* of a function f at the pairwise different knots $x_0,\dots,x_k$ are recursively defined as

$$[x_0]f = f(x_0), \quad k = 0,$$

and

$$[x_0,\dots,x_k]f = \frac{[x_0,\dots,x_{k-1}]f - [x_1,\dots,x_k]f}{x_0 - x_k}, \quad k \geq 1.$$

Allowing knots to coalesce, we have to interpret the lefthand side of (2.4) in terms of derivatives, cf. [**9**, Chapter 6], i.e.,

$$[\underbrace{x_0,\dots,x_0}_{n_0+1},\dots,\underbrace{x_k,\dots,x_k}_{n_k+1}]f = \frac{1}{\mathbf{n}!}D^{\mathbf{n}}[x_0,\dots,x_k]f, \tag{2.5}$$

where $\mathbf{n}! = n_0!\cdots n_k!$ and $D^n = D_0^{n_0}\cdots D_k^{n_k}$. The notation $D_j^n[x_0,\dots,x_k]f$ is thereby and in the following understood to be an abbreviation for the exact term

$$(D_j^n[x_0,\dots,x_{j-1},(\cdot)_j,x_{j+1},\dots,x_k]f)(x_j).$$

Expressing the same fact in terms of Dirichlet averages, we get the following result, stated by Carlson, cf. [**5**].

Theorem (Carlson). *Let* $\mathbf{m}$ *be a multi-index in* $\mathbf{N}^{k+1}$*, and let* $f \in C^{|\mathbf{m}|_1}(\mathbf{R})$*. Then the function* $[\mathbf{x} \mapsto F(\mathbf{b};\mathbf{x})]$ *belongs to* $C^{|\mathbf{m}|_1}(\mathbf{R}^{k+1})$ *and its derivative is given by*

$$D^{\mathbf{m}} F(\mathbf{b};\mathbf{x}) = \frac{B(\mathbf{b}+\mathbf{m})}{B(\mathbf{b})} F^{(|\mathbf{m}|_1)}(\mathbf{b}+\mathbf{m};\mathbf{x}). \tag{2.6}$$

We use the representation (2.4) as our motivation to generalize the notion of a divided difference. For $\mathbf{r} \in \mathbf{R}_+^{k+1}$, we define

$$\begin{gathered}[\mathbf{r}; x_0, \dots, x_k] f = \frac{1}{k!} F^{(k)}(\mathbf{r}; x_0, \dots, x_k) = \frac{1}{k!} F^{(k)}(\mathbf{r};\mathbf{x}),\\ \mathbf{x} = (x_0, \dots, x_k) \in \mathbf{R}^{k+1}.\end{gathered} \tag{2.7}$$

We keep the constant $1/k!$ in the definition to include the classical divided differences by setting $\mathbf{r} = \mathbf{e}$. In the latter case we skip the parameter $\mathbf{e}$ to get the traditional notation. Setting $\mathbf{r} = \mathbf{n}$ for some $\mathbf{n} \in \mathbf{N}^{k+1}$, we just have the special case of classical divided differences with every knot x_j, $j = 0, \dots, k$, repeated exactly n_j-times, $\mathbf{n} = (n_0, \dots, n_k)$. Using Carlson's theorem we can therefore write

$$[\mathbf{n}; x_0, \dots, x_k] f = \frac{B(\mathbf{e})}{B(\mathbf{n})} D^{\mathbf{n}-\mathbf{e}} [x_0, \dots, x_k](I^{|\mathbf{n}-\mathbf{e}|_1} f), \tag{2.8}$$

where $|\mathbf{n}|_1 = n_0 + \dots + n_k$ and where the operator $I^n, n \in \mathbf{N_0}$, denotes ordinary n-fold integration.

To give the generalized divided differences a similar interpretation in the case of an arbitrary real parameter $\mathbf{r}$, we have to find an analogous formula for (2.8). To do so, we need to introduce fractional integrals and derivatives.

3. Fractional integrals and derivatives of Dirichlet averages. The following notations and basic facts on fractional operators can be found in the book of Samko et al. [**15**, Chapter 2]. For a suitable function f on $\mathbf{R}$ and some $\beta > 0$, we define the *Riemann-Liouville type fractional integral* as

$$(I_+^\beta f)(x) = \frac{1}{\Gamma(\beta)} \int_{-\infty}^x \frac{f(t)}{(x-t)^{1-\beta}}\, dt, \quad x \in \mathbf{R}, \tag{3.9}$$

and the *Weyl type fractional integral* as

$$(I_-^\beta f)(x) = \frac{1}{\Gamma(\beta)} \int_x^\infty \frac{f(t)}{(t-x)^{1-\beta}}\, dt, \quad x \in \mathbf{R}. \tag{3.10}$$

These operators can be seen as generalizations of the ordinary n-fold integration $(I^n f)(x) = 1/(n-1)! \int_{-\infty}^x (x-t)^{n-1} f(t)\, dt$. Their inverse operators are defined as the *fractional derivatives of Liouville type*. For $0 < \beta < 1$, we get the two operators

$$(D_+^\beta f)(x) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_{-\infty}^x (x-t)^{-\beta} f(t)\, dt, \quad x \in \mathbf{R}, \tag{3.11}$$

and

$$(D_-^\beta f)(x) = \frac{-1}{\Gamma(1-\beta)} \frac{d}{dx} \int_x^\infty (t-x)^{-\beta} f(t)\, dt, \quad x \in \mathbf{R}. \tag{3.12}$$

For $\beta \geq 1$ we set $n = \lfloor \beta \rfloor + 1$ and define

$$\begin{aligned} (D_\pm^\beta f)(x) &= \frac{(\pm 1)^n}{\Gamma(n-\beta)} \frac{d^n}{dx^n} \int_0^\infty t^{n-\beta-1} f(x \mp t)\, dt \\ &= (D^n I_\pm^{n-\beta} f)(x), \quad x \in \mathbf{R}. \end{aligned} \tag{3.13}$$

Next we state some useful properties. Let f, g be functions for which the following integrals are defined, and let $\beta_1, \beta_2 \in \mathbf{R}_+$.

$$I_\pm^{\beta_1} I_\pm^{\beta_2} f = I_\pm^{\beta_1+\beta_2} f \quad \text{semi-group property,} \tag{3.14}$$

$$\int_{-\infty}^\infty f(x)(I_+^\beta g)(x)\, dx = \int_{-\infty}^\infty (I_-^\beta f)(x) g(x)\, dx \quad \text{partial integration,} \tag{3.15}$$

$$\int_{-\infty}^\infty f(x)(D_+^\beta g)(x)\, dx = \int_{-\infty}^\infty (D_-^\beta f)(x) g(x)\, dx. \tag{3.16}$$

For the proof of the theorem below, we need fractional operators to be understood in the weak sense. This can be done due to formula

(3.15) which allows us to define the operators of fractional integration by their action on an appropriate space of test-functions. To explain the related theory, we follow the approach given by Lizorkin [**11**].

By Ψ we denote the subspace in *Schwartz's space* of rapidly decreasing functions $\mathcal{S} = \mathcal{S}(\mathbf{R})$, whose elements together with all their derivatives vanish at the origin:

$$\Psi = \{\psi \in \mathcal{S} \mid \psi^{(k)}(0) = 0, \forall k = 0, 1, 2, \dots\}.$$

The space of functions $\phi \in \mathcal{S}$, the Fourier transform of which is an element of Ψ, is called the *Lizorkin space* Φ.

A prominent example is the function

$$k_\lambda(x) = \frac{1}{2\pi} \int_{\mathbf{R}} e^{-\lambda^2[v^2+(1/v^2)]} \cos(vx)\, dv, \quad x \in \mathbf{R}, \lambda > 0.$$

Its Fourier transform $\hat{k}_\lambda(v) = e^{-\lambda^2[v^2+(1/v^2)]}$, $v \in \mathbf{R}$, $\lambda > 0$, is called the *completely balanced kernel*.

In contrast to the space $\mathcal{S}$, the Lizorkin space Φ has the property that it is closed under the action of the operators $I_\pm^\beta$. Furthermore, we have

Lemma (Lizorkin). *For all* $\phi \in \Phi$

$$\widehat{(I_\pm^\beta \phi)}(v) = (\mp iv)^{-\beta} \hat{\phi}(v), \quad v \in \mathbf{R}, \quad \Re\beta \geq 0. \tag{3.17}$$

Lizorkin then defines the *completely balanced averages*

$$(K_\lambda f)(x) = \frac{1}{2\pi} \int_{\mathbf{R}} k_\lambda(x-y) f(y)\, dy, \quad x \in \mathbf{R},$$

which converge to $f \in L_p(\mathbf{R})$, $1 \leq p < \infty$, in norm. Formula (3.17) is stated here for test-functions only. For arbitrary functions, the relation is valid just for the values $0 < \beta \leq 1$, as long as the integrals exist. To obtain a result in the weak sense for $\Re\beta \geq 0$, we use the partial fractional integration (3.15).

We are now ready to state our main result in terms of Dirichlet averages. In analogy to derivatives denoted by superscripts, we will use subscripts for fractional integrals, i.e., $f_{(\beta)} = I_+^\beta f$. Following the convention of the preceding section, we will denote the corresponding Dirichlet averages by $F^{(\beta)}(\mathbf{b};\cdot)$ and $F_{(\beta)}(\mathbf{b};\cdot)$, respectively.

Theorem 1. *Let $f \in C(\mathbf{R})$ and $\mathbf{r} = (r_0, \dots, r_k) \in \mathbf{R}_+^{k+1}$, $0 \le r_j < 1$, $j = 0, \dots, k$, and $\mathbf{b} = (b_0, \dots, b_k) \in \mathbf{R}_+^{k+1}$, satisfying $b_j > r_j$, $j = 0, \dots, k$. Let us further assume that the fractional integral*

$$f_{(|\mathbf{r}|_1)}(x) = (I_+^{r_0+\cdots+r_k} f)(x), \quad x \in \mathbf{R},$$

exists. The fractional integral of $F(\mathbf{b};\cdot)$ is then given by

$$(I_+^{\mathbf{r}} F(\mathbf{b};\cdot))(\mathbf{x}) = \frac{B(\mathbf{b}-\mathbf{r})}{B(\mathbf{b})} F_{(|\mathbf{r}|_1)}(\mathbf{b}-\mathbf{r};\mathbf{x}). \tag{3.18}$$

Formula (3.18) is the exact analogue of formula (2.6). Combining both formulae and incorporating the definition (3.13) of fractional derivatives, we immediately get the following result for an arbitrary nonnegative parameter.

Corollary. *Let $\mathbf{b}, \mathbf{r} \in \mathbf{R}_+^{k+1}$ with $b_j > r_j$, $j = 0, \dots, k$, and $f \in C(\mathbf{R})$. Further assume that the fractional derivative*

$$f^{(|\mathbf{r}|_1)}(x) = (D_+^{(|\mathbf{r}|_1)} f)(x), \quad x \in \mathbf{R},$$

exists. The fractional derivative of $F(\mathbf{b};\cdot)$ is then given by

$$(D_+^{\mathbf{r}} F(\mathbf{b};\cdot))(\mathbf{x}) = \frac{B(\mathbf{b}+\mathbf{r})}{B(\mathbf{b})} F^{(|\mathbf{r}|_1)}(\mathbf{b}+\mathbf{r};\mathbf{x}). \tag{3.19}$$

Set $\mathbf{n} = (n_0, \dots, n_k) = (\lfloor r_0 \rfloor + 1, \dots, \lfloor r_k \rfloor + 1)$. We then have

$$\begin{aligned}(D_+^{\mathbf{r}} F(\mathbf{b};\cdot))(\mathbf{x}) &= (D_0^{n_0} I_{+,0}^{n_0-r_0} \cdots D_k^{n_k} I_{+,k}^{n_k-r_k} F(\mathbf{b};\cdot))(x) \\ &= \frac{B(\mathbf{b}-\mathbf{n}+\mathbf{r})}{B(\mathbf{b})} D_0^{n_0} \cdots D_k^{n_k} F_{(|\mathbf{n}-\mathbf{r}|_1)}(\mathbf{b}-\mathbf{n}+\mathbf{r};\mathbf{x}) \\ &= \frac{B(\mathbf{b}-\mathbf{n}+\mathbf{r})}{B(\mathbf{b})} \frac{B(\mathbf{b}-\mathbf{n}+\mathbf{r}+\mathbf{n})}{B(\mathbf{b}-\mathbf{n}+\mathbf{r})} \\ &\quad \times F_{(|\mathbf{n}-\mathbf{r}|_1)}^{(|\mathbf{n}|_1)}(\mathbf{b}-\mathbf{n}+\mathbf{r}+\mathbf{n};\mathbf{x}),\end{aligned}$$

giving the righthand side of (3.19).

Proof of the Theorem. To simplify notation, we set $\mathbf{x} = (\mathbf{x}', x) \in \mathbf{R}^{k+1}$, $\mathbf{u} = (\mathbf{u}', u) \in \Delta^k$ and $\mathbf{v} = (\mathbf{v}', v) \in \mathbf{R}^{k+1}$. Let further $\mathbf{u}'^{\mathbf{b}'-1}$ denote the vector $(u_0^{b_0-1}, \dots, u_{k-1}^{b_{k-1}-1})$, where as usual $u_0 = (1 - u_1 - \dots - u_k)$.

Let's first look at the integer case $\mathbf{r} = \mathbf{n} = (n_0, \dots, n_k) \in \mathbf{N}^{k+1}$.

$$
\begin{aligned}
(I_+^1 F(\mathbf{b}; \mathbf{x}', (\cdot)))(x) &= \int_{-\infty}^{x} \frac{1}{B(\mathbf{b})} \int_{\Delta^k} f(ut + \mathbf{u}' \cdot \mathbf{x}') u^{b_k - 1} \mathbf{u}'^{b'-1} \, d\mathbf{u} \, dt \\
&\overset{\sigma = ut + \mathbf{u}' \cdot \mathbf{x}'}{=} \frac{1}{B(\mathbf{b})} \int_{\Delta^k} u^{b_k - 1} \mathbf{u}'^{\mathbf{b}'-1} \int_{-\infty}^{ux + \mathbf{u}' \cdot \mathbf{x}'} f(\sigma) \frac{d\sigma}{u} d\mathbf{u} \\
&= \frac{\Gamma(b_k - 1)\Gamma(b_0 + \dots + b_k)}{\Gamma(b_0 + \dots + b_k - 1)\Gamma(b_k)} F_{(1)}(\mathbf{b}', b_k - 1; \mathbf{x}', x).
\end{aligned}
$$

By symmetry we can do so for the other variables as well to obtain

$$
(I_{+,0}^{n_0} \cdots I_{+,k}^{n_k} F(\mathbf{b}; (\cdot)))(\mathbf{x}) = \frac{B(\mathbf{b} - \mathbf{n})}{B(\mathbf{b})} F_{|\mathbf{n}|}(\mathbf{b} - \mathbf{n}; \mathbf{x}), \tag{3.20}
$$

where $|n| = n_0 + \dots + n_k$.

To prove the general case, we will show that the Fourier transforms of both sides of equation (3.18) coincide.

As a first step, observe that for $g(x) = f(\mathbf{u}' \cdot \mathbf{x}' + ux)$, $x \in \mathbf{R}$, and $0 < u < 1$,

$$
\hat{g}(v) = u^{-1} e^{i\mathbf{u}' \cdot \mathbf{x}' (v/u)} \hat{f}\left(\frac{v}{u}\right), \tag{3.21}
$$

which can easily be seen by a change of variables. Since

$$
\begin{aligned}
(I_+^{\beta} F(\mathbf{b}; \mathbf{x}', (\cdot)))(x) &= \frac{1}{\Gamma(\beta)} \int_{-\infty}^{\infty} (x - t)_+^{\beta - 1} F(\mathbf{b}; \mathbf{x}', t) \, dt \\
&= \frac{2\pi}{\Gamma(\beta)} [(\cdot)_+^{\beta-1} * F(\mathbf{b}; \mathbf{x}', (\cdot))](x)
\end{aligned} \tag{3.22}
$$

is a convolution of the functions $F(\mathbf{b}; (\mathbf{x}', \cdot))$ and $2\pi(\cdot)_+^{\beta-1}$, its Fourier transform is therefore just the product of the Fourier transforms of its factors. The truncated power function $(\cdot)_+$ thereby is defined as $(t)_+ = t$, if $t > 0$, and 0 otherwise. Note that the constant 2π follows from the definition of the convolution according to the definition of the Fourier transform, cf., formula (3.24) below.

To evaluate the Fourier transform of the truncated power function, observe that it can be rewritten as a Laplace integral. From the tables, cf. [**14**, (1.3.3)], we get $\int_0^\infty t^{\beta-1}e^{-zt}\,dt = \Gamma(\beta)z^{-\beta}$, $z \in \mathbf{C}$ with $\Re z > 0$. Setting $z = \sigma + iv$, $\sigma > 0$, and letting σ tend to zero, we obtain

(3.23)
$$[2\pi(\cdot)_+^{\beta-1}]^\wedge(v) = \lim_{\sigma\to 0^+}\Gamma(\beta)(\sigma+iv)^{-\beta} = \Gamma(\beta)(iv)^{-\beta}, \quad 0<\beta<1,$$

which is valid as an Abel mean.

Using (3.21) we get the Fourier transform of the Dirichlet average:

(3.24)
$$\begin{aligned}\hat{F}(\mathbf{b};\mathbf{x}',(\cdot))(v) &= \frac{1}{B(\mathbf{b})}\int_{\Delta^k}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(ux+\mathbf{u}'\cdot\mathbf{x}')e^{-ivx}\,dx\right)u^{b_k-1}\mathbf{u}'^{b'-1}\,d\mathbf{u}\\ &\overset{(3.21)}{=} \frac{1}{B(\mathbf{b})}\int_{\Delta^k} e^{\mathbf{u}'\cdot\mathbf{x}'(v/u)}\hat{f}\left(\frac{v}{u}\right)u^{b_k-2}\mathbf{u}'^{\mathbf{b}'-1}\,d\mathbf{u}.\end{aligned}$$

For the righthand side of (3.18), we have

(3.25)
$$\begin{aligned}\hat{F}_{(\beta)}(\mathbf{b};\mathbf{x}',(\cdot))(v) &= \frac{1}{B(\mathbf{b})}\int_{\Delta^k}\frac{1}{2\pi}\int_{-\infty}^{\infty} f_{(\beta)}(ux+\mathbf{u}'\cdot\mathbf{x}')e^{-ivx}\,dx u^{b_k-1}\mathbf{u}'^{\mathbf{b}'-1}\,d\mathbf{u}\\ &\overset{(3.21)}{=} \frac{1}{B(\mathbf{b})}\int_{\Delta^k} e^{i\mathbf{u}'\cdot\mathbf{x}'(v/u)}f_{(\beta)}\left(\frac{v}{u}\right)u^{b_k-2}\mathbf{u}'^{\mathbf{b}'-1}\,d\mathbf{u}\\ &\overset{(3.17)}{=} \frac{(iv)^{-\beta}}{B(\mathbf{b})}\int_{\Delta^k} e^{i\mathbf{u}'\cdot\mathbf{x}'(v/u)}\hat{f}\left(\frac{v}{u}\right)u^{b_k+\beta-2}\mathbf{u}'^{\mathbf{b}'-1}\,d\mathbf{u}.\end{aligned}$$

Putting the pieces together, (3.22), (3.23) and (3.24) add up to the Fourier transform of the lefthand side of (3.18), while (3.25) gives the righthand side. Therefore, for $0 < \beta < 1$:

$$(I_+^\beta F(\mathbf{b};\mathbf{x}',(\cdot)))^\wedge(v) = \frac{B(b_k-\beta,\mathbf{b}')}{B(\mathbf{b})}\hat{F}_{(\beta)}(\mathbf{b}',b_k-\beta;\mathbf{x}',(\cdot))(v).$$

Because of the symmetry of Dirichlet averages, we can deduce the equivalent relation for the other variables, which completes the proof of the theorem. □

4. Dirichlet splines. A motivation to define generalized splines is given by the Peano representation of the classical B-splines. Before doing so, let us first recall the definition of B-splines.

The *B-spline of order k* at the pairwise different knots $x_0, \dots, x_k$ is given by the function

$$M_k(u \mid x_0, \dots, x_k) = [x_0, \dots, x_k]\{k(\cdot - u)_+^{k-1}\}, \quad u \in \mathbf{R}.$$

Multiplicities are allowed, as long as not all of the knots coalesce. In the latter case, the spline has to be interpreted in the distributional sense, i.e., as point evaluation at $x_0 = \cdots = x_k$. B-splines are piecewise polynomial functions over the partition spanned by their knots and have their support in the convex hull of the knots; on this interval they are nonnegative.

An alternative way to define B-splines is to assume the following *Peano representation* to hold true for all $f \in C^k(\mathbf{R})$:

$$[x_0, \dots, x_k]f = \frac{1}{k!} \int_{\mathbf{R}} f^{(k)}(u) M_k(u \mid x_0, \dots, x_k)\, du. \tag{4.26}$$

In terms of Dirichlet averages, the representation reads

$$\int_{\mathbf{R}} f(u) M_k(u \mid x_0, \dots, x_k)\, du = \int_{\Delta^k} f(\mathbf{x} \cdot \mathbf{u})\, d\mu_{\mathbf{e}}(\mathbf{u}) = F(\mathbf{e}; \mathbf{x}),$$
$$\mathbf{x} = (x_0, \dots, x_k) \in \mathbf{R}^{k+1}.$$

Note that the Peano representation is used in the multi-dimensional setting to define the so-called *simplex splines*, cf. [**12**]. In the same manner, we define *Dirichlet splines* as the function $M_{\mathbf{b}}(u \mid \mathbf{x})$, $u \in \mathbf{R}$, $\mathbf{x} \in \mathbf{R}^{k+1}$ and $\mathbf{b} \in \mathbf{R}_+^{k+1}$ to be the density function for which the following relation holds true for all $f \in C(\mathbf{R})$:

$$\int_{\mathbf{R}} f(u) M_{\mathbf{b}}(u \mid \mathbf{x})\, du = \int_{\Delta^k} f(\mathbf{x} \cdot \mathbf{u})\, d\mu_{\mathbf{b}}(\mathbf{u}) = F(\mathbf{b}; \mathbf{x}), \quad x \in \mathbf{R}^{k+1}. \tag{4.27}$$

As in the classical case, we have to interpret the definition in the distributional sense, if all of the components of $\mathbf{x}$ coincide.

Dirichlet splines have been known in the literature for a long time, although they have not been denoted as spline. They occur as density functions in multivariate statistics, cf. [**10**] and the references therein. Dahmen and Micchelli gave a representation via a contour integral, cf. [**7**], i.e.,

(4.28)
$$M_{\mathbf{b}}(u \mid \mathbf{x}) = \frac{c-1}{2\pi i} \int_{\Gamma_{l+1}} (z-u)^{c-2} \prod_{j=0}^{k} (z-x_j)^{-b_j} \, dz, \quad x_l < u < x_{l+1},$$

where $c = b_0 + \cdots + b_k$ and Γ_{l+1} denotes a path, surrounding the knots $x_{l+1}, \ldots, x_k$ but not $x_0, \ldots, x_l$.

From the representation (4.28) it is easily seen that, for $c \geq b_j + 1$, $j = 0, \ldots, k$, the Dirichlet spline has its support in the convex hull of its knots. Furthermore, as a density function, the spline is nonnegative on its support.

Carlson [**6**] used the connection between Dirichlet averages and Dirichlet splines to reprove basic properties of B-splines and to deduce the system of *Euler-Poisson equations*, to which Dirichlet splines are solutions:

$$\begin{gathered} [(x_\mu - x_\nu) D_\mu D_\nu + b_\mu D_\nu - b_\nu D_\mu] u(\mathbf{x}) = 0, \\ \mathbf{x} \in \mathbf{R}^{k+1}, \ \mu, \nu \in \{0, \ldots, k\}. \end{gathered} \tag{4.29}$$

Neuman [**13**] investigated Dirichlet splines of higher dimensions which are defined the same way as simplex splines. He derived recurrence relations for their moments and an algorithm to compute them.

Looking at the system of differential equations (4.29), the spline is treated as a multivariate function of its knots. B-splines as functions of their knots play an important role in the investigation of Fourier transforms of functions which are radial with respect to the ℓ_1-norm. Cambanis, Keener and Simons [**2**] gave a characterization of these functions using divided differences. Another proof was given independently by Berens and Xu [**1**]. They used the Peano representation to introduce the B-spline and calculated its multi-dimensional Fourier transform with respect to the knots. In a forthcoming paper, we will investigate an integral transform where Dirichlet splines take over the role of the B-splines.

The connection of the generalized divided differences to splines is the analogue of the Peano representation for B-splines; indeed,

$$[\mathbf{r}; x_0, \dots, x_k]f = \frac{1}{k!} \int_{\mathbf{R}} f^{(k)}(u) M_{\mathbf{r}}(u \mid \mathbf{x})\, du, \quad \mathbf{x} \in \mathbf{R}^{k+1}. \tag{4.30}$$

Finally, we state the representation of Dirichlet splines as fractional derivatives of B-splines considered as a function of the knots:

Theorem 2. *Let* $\mathbf{b}$ *be a parameter in* $\mathbf{R}_+^{k+1}$*, and let* f *be a test-function in* $C(\mathbf{R})$*. Then the kth* B*-spline at the knots* $\mathbf{x} = (x_0, \dots, x_k) \in \mathbf{R}^{k+1}$*, and the Dirichlet spline* $M_{\mathbf{b}}(\cdot \mid \mathbf{x})$ *satisfy*
(4.31)

$$B(\mathbf{b}) \int_{\mathbf{R}} f(u) M_{\mathbf{b}}(u \mid \mathbf{x})\, du = B(\mathbf{e}) \int_{\mathbf{R}} f_{(|\mathbf{b}-\mathbf{e}|_1)}(u) D_+^{\mathbf{b}-\mathbf{e}} M_{\mathbf{e}}(u \mid \mathbf{x})\, du.$$

Equation (4.31) follows directly by applying the Corollary to the definition of the Dirichlet spline.

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