ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 32, Number 1, Spring 2002

CONGRUENCES FOR THE COEFFICIENTS WITHIN A GENERALIZED FACTORIAL POLYNOMIAL

M.A. NYBLOM

1. Introduction. The Stirling numbers of first order, denoted by $s(n,k)$ can be defined for $n > 0$ as the coefficient of x^k in the expansion of the rising factorial polynomial

$$x(x+1)\cdots(x+n-1) = \sum_{k=1}^{n} s(n,k)x^k.$$

The many varied properties of this class of numbers have been extensively studied, see, for example, [**2**]. Yet, in spite of this, congruences for the Stirling numbers $s(n,k)$ are apparently not well known. A few congruences for prime moduli can be found in [**2**] and other texts, but it has only been in recent times that certain papers have appeared dealing specifically with the problem of Stirling number congruences (see [**1**], [**3**]). Of these papers, the one of most interest to us here is due to Howard, who found congruences (mod p) for $s(n,k)$ and the associated Stirling numbers. We briefly list some of the main congruences as follows:

$$s(p,k) \equiv 0 \pmod{p} \quad \text{for } 2 \le k \le p-1 \tag{1}$$

$$s(p-1,k) \equiv 1 \pmod{p} \quad \text{for } 1 \le k \le p-1 \tag{2}$$

$$s(p-2,k) \equiv (2^{p-k-1}-1) \pmod{p} \quad \text{for } 0 \le k \le p-2 \tag{3}$$

$$s(hp+m,k) \equiv \sum_{i=0}^{h} \binom{h}{i} (-1)^{h-i} s(m, k-h-i(p-1)) \pmod{p} \quad \text{for } 0 \le m < p. \tag{4}$$

As an application of the above results, a complete examination of the congruences (mod p) for $s(n,k)$, where $n \ge p$, was given in [**3**] for the special cases $p = 2, 3$ and 5. In this paper we propose to extend

Received by the editors on August 31, 1999, and in revised form on May 3, 2000.

the results of Howard to the coefficients $\phi(n,k)$ within the polynomial expansion of a generalized factorial polynomial given by

$$(x)_{a_n} = (x+a_1)(x+a_2)\cdots(x+a_n),$$

where $a_n = a_1 + (n-1)d$ with $a_1, d \in \mathbf{Z}$. These coefficients, which are essentially the symmetric function on n objects, will be shown for all arithmetic progressions $a_n \in \mathbf{Z}$ to satisfy congruences similar to those in (1) and (2) with the residue in the later result given by $(d-a_1)^{p-k-1}$. The congruences in (3) and (4) will be generalized to the case of those coefficients generated within the expansion of $(x)_{a_n}$, where $a_n = (n-1)d$ and with $d \in \mathbf{Z} \setminus \{0\}$. For these results we shall see that the residues differ from those in (3) and (4) by a suitable multiplicative factor. In this latter case, we shall find as in [**3**] all the congruences (mod p) of $\phi(n,k)$ for $n \geq p$ and illustrate our method by finding the residues when $p = 2, 3$ and 5. To conclude the paper an alternate notion of the associated Stirling number is introduced and a divisibility result is proved for these numbers which mirrors that which was found by Howard in [**3**].

2. Elementary symmetric functions. For an arbitrary sequence of reals $\{a_i\}_{i=1}^n$, one can define the elementary symmetric function $\phi(n,k)$ in the n variables $a_1, \ldots, a_n$ as the coefficient of x^k in the polynomial expansion

$$(x)_{a_n} = (x+a_1)(x+a_2)\cdots(x+a_n) = \sum_{k=0}^{n} \phi(n,k)x^k. \tag{5}$$

It follows from (5) that

$$\phi(n,k) = a_n\phi(n-1,k) + \phi(n-1,k-1) \tag{6}$$

with

$$\phi(n,0) = \prod_{i=1}^{n} a_i \quad \text{if } n > 0, \tag{7}$$

$$\phi(n,n) = 1, \tag{8}$$

$$\phi(n,k) = 0 \quad \text{if } k > n \text{ or } k < 0. \tag{9}$$

Clearly $\phi(n,k) = s(n,k)$ in the case when $a_n = n-1$. For later convenience, we shall define $s(0,k) = 0$ for all $k \neq 0$ and $s(0,0) = 1$. In what follows, we shall study the congruences for prime moduli p, of those symmetric functions $\phi(n,k)$ generated with respect to an arbitrary sequence of integers in arithmetic progression, that is, for $a_n = a_1 + (n-1)d$ with $a_1,\ d \in \mathbf{Z}$ and $d \neq 0$. We begin by generalizing the well-known result (see [**2**], [**3**]) that $s(p,k) \equiv 0 \pmod{p}$ for $k = 2, \ldots, p-1$.

Theorem 2.1. *If p is a prime number, then $\phi(p,k) \equiv 0 \pmod{p}$ for $k = 2, \ldots, p-1$.*

Proof. By making the substitution $dX = (x + a_1)$, observe from the definition of $(x)_{a_n}$ that

$$(x)_{a_n} = dX(dX+d)\cdots(dX+d(n-1)) = d^n (X)_{n-1}.$$

Expanding out the polynomial $(X)_{n-1}$, one finds

$$\begin{aligned}
(x)_{a_n} &= d^n \sum_{i=0}^{n} s(n,n-i) X^{n-i} \\
&= d^n \sum_{i=0}^{n} s(n,n-i) d^{i-n} \sum_{j=0}^{n-i} \binom{n-i}{j} x^{n-i-j} a_1^j \\
&= \sum_{i=0}^{n} s(n,n-i) d^{i} \sum_{j=0}^{n-i} \binom{n-i}{j} x^{n-i-j} a_1^j.
\end{aligned} \tag{10}$$

Consequently, by collecting powers of x^k in (10), we obtain a relation between the Stirling numbers $s(n,k)$ and $\phi(n,k)$ as follows

$$\phi(n,k) = \sum_{i=0}^{n-k} \binom{n-i}{k} s(n,n-i) a_1^{n-i-k} d^i. \tag{11}$$

Now, after setting $n = p$ in (11), observe that $p \mid s(p,p-i)$ for $i = 1, 2, \ldots, p-k$, provided $k = 2, \ldots, p-1$ while $p \mid \binom{p}{k}$ for $k = 1, 2, \ldots, p-1$. Thus, all terms in the above summation will be divisible by p if we restrict $k = 2, \ldots, p-1$. $\square$

By making use of Theorem 2.1 and (6), we can now show that $\phi(p-1,k) \equiv (d-a_1)^{p-k-1} \pmod{p}$, which extends the congruence result $s(p-1,k) \equiv 1 \pmod{p}$ also obtained in [**3**].

Corollary 2.1. *If $a \neq d$, then $\phi(p-1,k) \equiv (d-a_1)^{p-k-1} \pmod{p}$ for $k = 1, \dots, p-1$. While $\phi(p-1,k) \equiv 0 \pmod{p}$ for $k = 1, \dots, p-2$ when $a_1 = d$.*

Proof. We first establish the result in the case $a_1 \neq d$ via backward induction on k. Clearly, the congruence holds for $k = p-1$ as $\phi(p-1,p-1) = 1 \equiv (d-a_1)^0 \pmod{p}$. Thus assume the result holds for $k = p-r$ where $1 \leq r < p-1$; then, by (6) and Theorem 2.1, $p \,|\, \{\phi(p-1,p-r-1) + (a_1-d)\phi(p-1,p-r)\}$. However, by the inductive assumption $(a_1-d)\phi(p-1,p-r) \equiv -(d-a_1)^r \pmod{p}$. Consequently, we deduce that

$$p \mid \{\phi((p-1,p-r-1) - (d-a_1)^r)\}.$$

That is, $\phi(p-1,p-r-1) \equiv (d-a_1)^r \pmod{p}$ and so the result holds for $k = p-r-1$. In the case when $a_1 = d$, we have from (6) that $\phi(p-1,l-1) = \phi(p,l) - pd\phi(p-1,l)$. Now the righthand side of this equation is, by Theorem 2.1, clearly divisible by p for $l = 2, \dots, p-1$ and so $p \,|\, \phi(p-1,k)$ for $k = 1, \dots, p-2$. □

We see, in particular, from Corollary 2.1 for $a_1 = 0$, $d = 1$, that $\phi(p-1,1) = s(p-1,1) = (p-2)! \equiv 1 \pmod{p}$. Thus, by multiplying both sides of this congruence by $p-1$, one finds $(p-1)! \equiv (p-1) \pmod{p}$ which leads to the statement of Wilson's theorem that is $(p-1)! \equiv -1 \pmod{p}$. In order to generalize the result $s(p-2,k) \equiv (2^{p-k-1}-1) \pmod{p}$, we shall restrict attention in the final corollary to those symmetric functions $\phi(n,k)$ generated with respect to the sequence $a_n = (n-1)d$.

Corollary 2.2. *If $a_1 = 0$ and $p > 2$, then $\phi(p-2,k) \equiv d^{p-k-2}(2^{p-k-1}-1) \pmod{p}$, for $k = 1, \dots, p-2$.*

Proof. We establish the result again via backward induction on k. As before, the congruence holds trivially for $k = p-2$ as $\phi(p-2,p-2) = 1$.

Thus assume the result holds for $k = p - r$ where $2 \le r < p - 1$, then by (6) and Corollary 2.1,

$$p \mid (\phi(p-2, p-r-1) - 2d\phi(p-2, p-r) - d^{r-1}).$$

However, by the inductive assumption $-2d\phi(p-2,p-r) \equiv -2d^{r-1}(2^{r-1}-1) \pmod p$. Consequently, we deduce that $p \mid \{\phi(p-2, p-r-1) - 2d^{r-1}(2^{r-1}-1) - d^{r-1}\}$. That is, $\phi(p-2, p-r-1) \equiv d^{r-1}(2^r - 1) \pmod p$ and so the result holds for $k = p - r - 1$. □

We now establish a more general result in connection with those symmetric functions $\phi(n,k)$ which are again generated with respect to $a_n = (n-1)d$.

Theorem 2.2. *If p is a prime number, $h > 0$ and $0 \le m < p$, then*

$$\begin{aligned}(12)\quad & \phi(hp+m,k) \\ & \equiv d^{hp+m-k} \sum_{i=0}^{h} \binom{h}{i} (-1)^{h-i} s(m, k-h-i(p-1)) \pmod p.\end{aligned}$$

Proof. Consider first the case $h = 1$. That is, we wish to establish that for $m \ge 0$,

$$(13)\qquad \phi(p+m,k) \equiv d^{p+m-k}(-s(m,k-1) + s(m,k-p)) \pmod p.$$

The proof of (13) is by induction on m. Recalling $s(0,k) = 0$ for $k \ne 0$ and $s(0,0) = 1$ together with Wilson's theorem, we clearly have (13) holding for $m = 0$. Assume it is true for $m = 0, 1, \ldots, j-1$. Then, again by (13),

$$\begin{aligned}\phi(p+j,k) &= d(p+j-1)\,\phi(p+j-1,k) + \phi(p+j-1,k-1) \\ &\equiv (j-1)\,d^{p+j-k}[-s(j-1,k-1) + s(j-1,k-p)] \\ &\quad + d^{p+j-k}[-s(j-1,k-2) + s(j-1,k-p-1)] \pmod p \\ &= -d^{p+j-k}[(j-1)\,s(j-1,k-1) + s(j-1,k-2)] \\ &\quad + d^{p+j-k}[(j-1)\,s(j-1,k-p) + s(j-1,k-p-1)] \\ &\equiv d^{p+j-k}(-s(j,k-1) + s(j-1,k-p)) \pmod p,\end{aligned}$$

and so (13) holds for $m = j$. Let $h \geq 2$ and suppose for $r = 1, 2, \dots, h-1$ and all m that

$$\phi(rp+m,k) \equiv d^{rp+m-k} \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} s(m, k-r-i(p-1)) \pmod p.$$

Thus, by writing $\phi(hp+m,k) = \phi((h-1)p+(m+p),k)$, observe from the inductive assumption

$$\begin{aligned}\phi(hp+m,k) \equiv d^{hp+m-k} \sum_{i=0}^{h-1} \binom{h-1}{i} (-1)^{h-1-i} \\ \cdot s(p+m, k-(h-1)-i(p-1)) \pmod p.\end{aligned} \tag{14}$$

Now, for each i, in the above summation we have, after setting $d = 1$ in (13), that

$$\begin{aligned}&s(m+p, k-(h-1)-i(p-1)) \\ &\quad \equiv -s(m, k-h-i(p-1)) + s(m, k-(h-1)-i(p-1)-p) \pmod p.\end{aligned}$$

Consequently, by breaking up the summation in (14) and relabeling the index variable in the resulting second summation from i to $i-1$, one has

$$\begin{aligned}&\phi(hp+m,k) \\ &\quad \equiv d^{hp+m-k} \sum_{i=0}^{h-1} \binom{h-1}{i} (-1)^{h-i} s(m, k-h-i(p-1)) \\ &\qquad + d^{hp+m-k} \sum_{i=1}^{h} \binom{h-1}{i-1} (-1)^{h-i} s(m, k-h-i(p-1)) \pmod p \\ &\quad = d^{hp+m-k} \Big[(-1)^h s(m, k-h) + \sum_{i=1}^{h-1} \Big\{ \binom{h-1}{i} + \binom{h-1}{i-1} \Big\} (-1)^{h-i} \\ &\qquad\qquad \cdot s(m, k-h-i(p-1)) + s(m, k-hp) \Big] \\ &\quad \equiv d^{hp+m-k} \sum_{i=0}^{h} \binom{h}{i} (-1)^{h-i} s(m, k-h-i(p-1)) \pmod p.\end{aligned}$$

Hence, (14) holds for $r = h$. □

Recalling from (9) that $s(n,k) = 0$ for $k > n$ or $k < 0$, one can determine using Theorem 2.2 the congruences for prime moduli of $\phi(hp+m,k)$ for all k in the special cases $m = 0,1,2,p-1$ and $p-2$ as follows. Considering firstly $m = 0$, we have $s(0,k-h-i(p-1)) = 1$ if $k = h+i(p-1)$ and $s(0,k-h-i(p-1)) = 0$ for all other k. Similarly, in the case $m = 1$, we have $s(1,k-h-i(p-1)) = 0$ for $h+1+i(p-1) < k$ and $k < h+1+i(p-1)$ and $s(1,k-h-i(p-1)) = 1$ for $k = h+1+i(p-1)$. However, in the case $m = 2$, one again has $s(2,k-h-i(p-1)) = 0$ for $k < h+i(p-1)$ or $k > 2+h+i(p-1)$, while $s(2,k-h-i(p-1)) = 1$ either for $k = 2+h+i(p-1)$ or $k = 1+h+i(p-1)$ as $s(2,1) = 1$. Thus we have from (12) that for $m = 0,1,2$ and all $h \geq 1$, $\phi(hp+m,k) \equiv 0 \pmod{p}$ except for the following: For $i = 0,\ldots,h$

$$\begin{aligned}&\phi(hp+m,h+m+(p-1)i)\\ &\qquad\equiv d^{(p-1)(h-i)}\binom{h}{i}(-1)^{h-i} \pmod{p} \quad \text{for } m = 0,1\end{aligned}$$

and

$$\begin{aligned}&\phi(hp+2,h+r+(p-1)i)\\ &\qquad\equiv d^{(h-i)(p-1)+2-r}\binom{h}{i}(-1)^{h-i} \pmod{p}, \quad r = 1,2.\end{aligned}$$

In the case $m=p-1$ again $s(p-1,k-h-i(p-1))=0$ for $k \leq h+i(p-1)$ or $k > (p-1)(i+1)+h$, and so for all $i = 0,\ldots,h$, we have $s(p-1,k-h-i(p-1)) \neq 0$ when $h+i(p-1) < k \leq (p-1)(i+1)+h$. Thus, using the fact that $s(p-1,r) \equiv 1 \pmod{p}$ for $r = 1,\ldots,p-1$ and (12), we deduce for all $h \geq 1$ that $\phi(hp+p-1,k) \equiv 0 \pmod{p}$ except for the following. For $i = 0,\ldots,h$ and $t = 1,\ldots,p-1$,

$$\phi(hp+p-1,h+t+i(p-1)) \equiv d^{(p-1)(h-i+1)-t}\binom{h}{i}(-1)^{h-i} \pmod{p}.$$

Finally, in the case $m = p-2 > 0$, we similarly have $s(p-2,k-h-i(p-1)) \neq 0$ for $h+i(p-1) < k \leq (p-2)+h+i(p-1)$, and so using the fact that $s(p-2,r) \equiv (2^{p-r-1}-1) \pmod{p}$ for $r = 1,\ldots,p-2$ and

(12), we find for all $h \geq 1$ that $\phi(hp+p-2,k) \equiv 0 \pmod p$, except for the following: For $i=0,\dots,h$ and $t=1,\dots,p-2$,

$$\begin{aligned}\phi(hp+p-2),h+t+i(p-1))\\ \equiv d^{(p-1)(h-i+1)-t-1}\binom{h}{i}(-1)^{h-i}(2^{p-t-1}-1) \pmod p.\end{aligned}$$

As a consequence, we note the following special cases. For $p=2,3$ or 5 and $n \geq p$,

$$\phi(n,k) \equiv 0 \pmod p$$

except for the following: For $h \geq 1$ and $i=0,\dots,h$,

$$\phi(2h+r,h+r+i) \equiv d^{h-i}\binom{h}{i}(-1)^{h-i} \pmod 2, \quad r=0,1$$

together with

$$\begin{aligned}\phi(3h,h+2i) &\equiv d^{2(h-i)}\binom{h}{i}(-1)^{h-i} \pmod 3\\ \phi(3h+1,h+1+2i) &\equiv d^{2(h-i)}\binom{h}{i}(-1)^{h-i} \pmod 3\\ \phi(3h+2,h+t+2i) &\equiv d^{2(h-i)-t}\binom{h}{i}(-1)^{h-i} \pmod 3, \quad t=1,2\end{aligned}$$

and finally

$$\begin{aligned}\phi(5h,h+4i) &\equiv d^{4(h-i)}\binom{h}{i}(-1)^{h-i} \pmod 5\\ \phi(5h+1,h+1+4i) &\equiv d^{4(h-i)}\binom{h}{i}(-1)^{h-i} \pmod 5\\ \phi(5h+2,h+t+4i) &\equiv d^{4(h-i)+2-t}\binom{h}{i}(-1)^{h-i} \pmod 5, \quad t=1,2\\ \phi(5h+3,h+t+4i) &\equiv d^{4(h-i+1)-t-1}(2^{4-t}-1)\binom{h}{i} \pmod 5,\\ &\qquad t=1,\dots,3,\end{aligned}$$

$$\phi(5h+4, h+t+4i) \equiv d^{4(h-i+1)-t}\binom{h}{i}(-1)^{h-i} \pmod 5,$$
$$t = 1, \dots, 4.$$

3. Generalized associated Stirling numbers of the first kind. It is well known that $s(n, n-k)$ is a polynomial in n of degree $2k$. Indeed,

$$s(n, n-k) = \sum_{i=0}^{k} d(2k-i, k-i)\binom{n}{2k-i},$$

where $d(n,k)$ is the associated Stirling number of the first kind (see [**2**]). In a recent paper [**4**], the author showed that in the polynomial expansion of $(x)_{a_n}$, where $a_n = a_1 + (n-1)d$ with a_1, $d \in \mathbf{R}$, the coefficients were given by

$$\phi(n, n-k) = \sum_{i=1}^{k+1} \theta_i^{(k)}\binom{n+k+1-i}{2k+1-i} \tag{15}$$

where $\theta_i^{(k)}$ satisfy for $k = 2, \dots, n$ and $i = 2, \dots, k$ the recurrence

$$\theta_i^{(k)} = \theta_{i-1}^{(k-1)}(d(i-k-2)+a_1) + \theta_i^{(k-1)}(2k-i)d \tag{16}$$

with $\theta_1^{(k)} = 1 \cdot 3 \dots (2k-1)d$ and $\theta_{k+1}^{(k)} = (a_1 - d)^k$. Clearly, when $a_1, d \in \mathbf{Z}$, one has $\theta_i^{(k)} \in \mathbf{Z}$ and in this instance we refer to the numbers $\theta_i^{(k)}$ as the generalized associated Stirling numbers of the first kind. In [**3**] a number of congruences (mod p) for $d(n,k)$ were found, in particular, $d(p,k) \equiv 0 \pmod p$ for $2 \le k \le p$. With this result in mind, it is natural to question whether there exists an arithmetic progression $\{a_n\} \subseteq \mathbf{Z}$ such that in general, $p \mid \theta_i^{(k)}$ for some k and i. To help answer this question, it will be convenient to arrange, for $n \ge 1$, the set of numbers $\theta_i^{(k)}$ for $k = 1, \dots, n$ and $i = 1, \dots, k$ as the elements of the following lower triangular matrix

$$A_n = \begin{bmatrix} \theta_1^{(0)} & 0 & 0 & \cdots & 0 \\ \theta_1^{(1)} & \theta_2^{(1)} & 0 & \cdots & 0 \\ \theta_1^{(2)} & \theta_2^{(2)} & \theta_3^{(2)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \theta_1^{(n)} & \theta_2^{(n)} & \theta_3^{(n)} & \cdots & \theta_{n+1}^{(n)} \end{bmatrix}.$$

Note, in A_n we set $\theta_1^{(0)} \equiv 1$ which by (15) is consistent with the equality $\phi(n,n) = 1$. We now establish, using the recurrence in (16) and Theorem 2.1, the following divisibility result for the generalized Stirling numbers of the first kind.

Theorem 3.1. *Let $p > 3$ be a prime. Then, for any nonconstant arithmetic progression $\{a_n\} \subseteq \mathbf{Z}$, one has $\theta_j^{[((p+1)/2)+i]} \equiv 0 \pmod p$ for $i = 0, \dots, (p-1)/2$ and $j = 1, \dots, i+1$.*

Proof. Note, when $(p+1)/2$ is even, the numbers in question can equivalently be found as a subset of the components within the two adjacent truncated column vectors of A_{p+1} given here by

$$\mathbf{C}_{2s-1}^T = (\theta_{2s-1}^{[((p+1)/2)+s-1]}, \theta_{2s-1}^{[((p+1)/2)+s]}, \dots, \theta_{2s-1}^{(p+1)})$$

and

$$\mathbf{C}_{2s}^T = (\theta_{2s}^{[((p+1)/2)+s-1]}, \theta_{2s}^{[((p+1)/2)+s]}, \dots, \theta_{2s}^{(p+1)}),$$

for $s = 1, \dots, (p+1)/4$. While if $(p+1)/2$ is odd, then the numbers are again found in the above column vectors, for $s = 1, \dots, (p-1)/4$, together with a subset of the array of components in $\mathbf{C}_{(p+1)/2}^T$. We shall prove here a stronger result than required, namely, that all the components of $\mathbf{C}_{2s-1}^T, \mathbf{C}_{2s}^T$ are divisible by p for the values of s indicated. Considering first the case when $(p+1)/2$ is even, it will suffice to demonstrate via the following inductive argument on the index, $s \in [1, (p+1)/4]$, that the components of $\mathbf{C}_{2s-1}^T$ and $\mathbf{C}_{2s}^T$ are divisible by p. When $s = 1$, it is clear from the definition of $\theta_1^{(i)}$ that $p \,|\, \theta_1^{[((p+1)/2)+r]}$ for $r = 0, 1, \dots, (p+1)/2$. However, for the components of $\mathbf{C}_2^T$ it will first be convenient to show $p \,|\, \theta_2^{[(p+1)/2]}$ as one can then easily deduce via repeated application of

$$\begin{aligned}\theta_2^{[((p+1)/2)+r]} &= \theta_1^{[((p+1)/2)+r-1]} \Big(a_1 - d\Big(\frac{p+1}{2} + r\Big)\Big) \\ &\quad + \theta_2^{[((p+1)/2)+r-1]}\,(p+2r-1)d,\end{aligned}$$

that $p \,|\, \theta_2^{[((p+1)/2)+r]}$ for $r = 1, 2, \dots, (p+1)/2$. To this end, consider the coefficient $\phi(p, p-k)$ in the polynomial expansion of $(x)_{a_p}$ given in

(15). Setting $k = (p-1)/2$, observe, after some simplification of the binomial coefficients, that

$$\begin{aligned}\phi(p, p-k) = \theta_1^{[(p+1)/2]} \binom{p+k-1}{p+1} + \theta_2^{[(p+1)/2]} \binom{p+k}{p} \\ + \sum_{l=3}^{p-k+1} \theta_l^{[(p+1)/2]} \binom{2p-k+1-l}{2p-2k+1-l}.\end{aligned}$$

Now the $p-k-1$ binomial coefficients in the above are equal to $(k!)^{-1} \prod_{i=0}^{k-1} (2p-k+1-l-i)$, with $2p-k+1-l-i = p$ when $i = p-k+1-l$ with $0 \le i \le p-k-2 = k-1$. Moreover, as $(p, k!) = 1$, one can deduce that

$$p \Big| \sum_{l=3}^{p-k+1} \theta_l^{[(p+1)/2]} \binom{2p-k+1-l}{2p-2k+1-l}.$$

However, by Theorem 2.1, $p \,|\, \phi(p, p-(p+1)/2)$, but as $p \,|\, \theta_1^{[(p+1)/2]}$ and $p \nmid \binom{p+k}{p}$, we can conclude that $p \,|\, \theta_2^{[(p+1)/2]}$ as required. Suppose now that the components of $\mathbf{C}_{2s-1}^T$ and $\mathbf{C}_{2s}^T$ are divisible by p for all $s = 1, \ldots, m$, where $1 \le m \le [(p+1)/4] - 1 = (p-3)/4$. Considering the column vector $\mathbf{C}_{2m+1}^T$, we see again from (16) that

$$\begin{aligned}\theta_{2m+1}^{[((p+1)/2)+r]} = \theta_{2m}^{[((p+1)/2)+r-1]} \Big(d\Big(2m - \frac{p+1}{2} - r - 1\Big) + a_1 \Big) \\ + \theta_{2m+1}^{[((p+1)/2)+r-1]} (p + 2r - 2m) d,\end{aligned} \tag{17}$$

for $r = m, \ldots, (p+1)/2$. Observe, after setting $r = m$ in (17), that the coefficient of $\theta_{2m+1}^{[((p+1)/2)+m-1]}$ is a multiple of p, thus, via the inductive assumption, $p \,|\, \theta_{2m+1}^{[((p+1)/2)+m]}$. Now again, by repeated application of (17) together with the inductive assumption, we further deduce that $p \,|\, \theta_{2m+1}^{[((p+1)/2)+r]}$, for $r = m+1, \ldots, (p+1)/2$. For the components of $\mathbf{C}_{2m+2}^T$, it will be necessary to first show that $p \,|\, \theta_{2m+2}^{[((p+1)/2)+m]}$ as one can then deduce via the inductive assumption and repeated application of

$$\begin{aligned}\theta_{2m+2}^{[((p+1)/2)+r]} = \theta_{2m+1}^{[((p+1)/2)+r-1]} \Big(d\Big(2m - \frac{p+1}{2} - r\Big) + a_1 \Big) \\ + \theta_{2m+2}^{[((p+1)/2)+r-1]} (p + 2r - 2m - 1),\end{aligned}$$

that $p\,|\,\theta_{2m+2}^{[((p+1)/2)+r]}$ for $r = m+1, \ldots, (p+1)/2$. To this end, consider now the coefficient of $\phi(p, p-((p+1)/2)-m)$ in the polynomial expansion of $(x)_{a_p}$, which is equal to

$$\sum_{l=1}^{2m+1} \theta_l^{[((p+1)/2)+m]} \binom{p+k+m+2-l}{p+2m+2-l} + \theta_{2m+2}^{[((p+1)/2)+m]} \binom{p+k-m}{p} + \sum_{l=2m+3}^{k+m+2} \theta_l^{[((p+1)/2)+m]} \binom{p+k+m+2-l}{p+2m+2-l},$$

where again $k = (p-1)/2$. Now $((p+1)/2)+m \leq (3p-1)/4$ and $(3p-1)/4 \leq p-2$ for $p \geq 7$; however, as the smallest prime $p > 3$ for which $(p+1)/2$ is even is 7, we can conclude from Theorem 2.1 that $p\,|\,\phi(p, p-((p+1)/2)-m)$. Furthermore, as previously, the $k-m$ binomial coefficients in the above are equal to $((k-m)!)^{-1}\prod_{i=0}^{k-m-1}(p+k+m+2-l-i)$ with $p+k+m+2-l-i = p$ when $i = k+m+2-l$ with $0 \leq i \leq k-m-1$. Consequently, as $(p, (k-m)!) = 1$, one can deduce that

$$p \,\Big|\, \sum_{l=2m+3}^{k+m+2} \theta_l^{[((p+1)/2)+m]} \binom{p+k+m+2-l}{p+2m+2-l}.$$

In addition, as $\theta_r^{[((p+1)/2)+m]}$ is a component of $\mathbf{C}_r^T$ for $r = 1, 2, \ldots, 2m+1$, we also have

$$p \,\Big|\, \sum_{l=1}^{2m+1} \theta_l^{[((p+1)/2)+m]} \binom{p+k+m+2-l}{p+2m+2-l}.$$

Combining the above with the fact that p does not divide $\binom{p+k-m}{p}$ yields the required result, that $p\,|\,\theta_{2m+2}^{[((p+1)/2)+m]}$. Hence, the components of C_{2m+1}^T and C_{2m+2}^T are all divisible by p.

Considering now the case when $(p+1)/2$ is odd, we first establish via induction on $s = 1, \ldots, (p-1)/4$ that the components of $\mathbf{C}_{2s-1}^T$ and $\mathbf{C}_{2s}^T$ are divisible by p. This argument, however, follows precisely in the same manner as above with the exception of one detail within the inductive step which we attend to as follows. If one assumes that the components of $\mathbf{C}^T_{2s-1}$ and $\mathbf{C}^T_{2s}$ are divisible by p for al $s = 1, \dots, m$, where $1 \le m \le ((p-1)/4) - 1 = (p-5)/4$, then when considering the coefficient $\phi(p, ((p-1)/2) - m)$, to show that $p \mid \theta^{[((p+1)/2)+m]}_{2m+2}$, we must have that $p \mid \phi(p, ((p-1)/2) - m)$. To establish this, first note that $((p+1)/2) + m \le (3p-3)/4$ and $(3p-3)/4 \le p-2$ for $p \ge 5$; however, the smallest prime $p > 3$ for which $(p+1)/2$ is odd is 5, thus, as before, one need now only invoke Theorem 2.1 to deduce the required divisibility condition. Considering finally the column vector $\mathbf{C}^T_{(p+1)/2}$, we have from (16)

$$\begin{aligned}\theta^{[((p+1)/2)+r]}_{(p+1)/2} &= \theta^{[((p+1)/2)+r-1]}_{[((p+1)/2)-1]}(a_1 - d(r+2)) \\ &\quad + \theta^{[((p+1)/2)+r-1]}_{(p+1)/2}\Big(\frac{p+1}{2} + 2r\Big)d,\end{aligned} \tag{18}$$

for $r = (p-1)/4, \dots, (p+1)/2$. Observe, after setting $r = (p-1)/4$ in (18), that the coefficient of $\theta^{[((p+1)/2)+r-1]}_{(p+1)/2}$ is a multiple of p. Moreover, as $\theta^{[(3p-3)/4]}_{[((p+1)/2)-1]} \in \mathbf{C}^T_{[((p+1)/2)-1]}$, one can deduce that $p \mid \theta^{[((p+1)/2)+((p-1)/4)]}_{[(p+1)/2]}$. Now by repeated application of (18), we can further conclude that $p \mid \theta^{[((p+1)/2)+r]}_{(p+1)/2}$ for $r = ((p-1)/4) + 1, \dots, (p+1)/2$. Hence the components of $\mathbf{C}^T_{(p+1)/2}$ are all divisible by p. $\square$

REFERENCES

1. L. Carlitz, *A theorem of Glaisher*, Canad. J. Math. **5** (1953), 306–316.

2. L. Comtet, *Advanced combinatorics*, Reidel, Dordrecht, 1974.

3. F.T. Howard, *Congruences for the Stirling numbers and associated Stirling numbers*, Acta Arith. **55** (1990), 29–41.

4. M.A. Nyblom, *On a generalisation of the binomial theorem*, Fibonacci Quart. **37** (1999), 3–13.

Mathematics Department, RMIT University, GPO Box 2476V, Melbourne 3001, Victoria, Australia
E-mail address: `michael.nyblom@rmit.edu.au`