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THE ISOTOPY CLASSIFICATION OF AFFINE QUARTIC CURVES

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To the memory of D.A. Gudkov

ABSTRACT. In this paper we obtain the isotopy classification of affine quartic curves, which contains 647 classes, and the topological classification of pairs ($\mathbf{R}^2$, *quartic curve*), which contains 516 classes (see Theorem 7). We also present the isotopy classification of real projective quartic curves, which contains 66 classes.

We prove that each of these classifications is equivalent to the classification of all real (affine or projective) quartic curves having only singular points, if any, of types A_1, A_1^*, D_4 or X_9 (see Theorems 5 and 6 and Corollaries 6.1–6.4).

1. Introduction. One of the most important problems in the topology of real algebraic curves and surfaces is the problem of their classification. Any such classification is based on an equivalence relation for the set of varieties being considered. The quotient set with respect to the equivalence relation is called a *classification*. There are a number of different approaches to the subject. One can consider classification of varieties either of fixed degree or of fixed dimension or of both. One can also consider only nonsingular varieties, or varieties with other prescribed algebraic or topological properties. Historically, the first and most basic approach is to classify real algebraic varieties up to affine equivalences in $\mathbf{R}^n$, respectively, projective equivalence in $\mathbf{R}P^n$. Two affine (projective) varieties are called *affine* (*projective*) *equivalent* provided there exists a nondegenerate affine (projective) linear transformation that carries one variety to the other. The affine (projective) linear transformation does not change the degree of a variety. Thus

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one can consider the problem of classification of varieties of fixed degree. The affine (projective) classification of hyperplanes in $\mathbf{R}^n$ ($\mathbf{R}P^n$) is trivial and consists of one class.

The affine classification of hypersurfaces of degree 2 in $\mathbf{R}^n$ leads to the classification of expressions $x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 + D$ where D is equal to 0 or 1 or x_{p+q+1}, up to the rank $p+q$ and signature $p-q \geq 0$ of the quadratic form, where $1 \leq p+q \leq n$ if $D = 0$ or 1; and $1 \leq p+q \leq n-1$ if $D = x_{p+q+1}$. The number of affine classes of such hypersurfaces is equal to $n^2 + 3n - 1$.

The projective classification of hypersurfaces of degree 2 in $\mathbf{R}P^n$ leads to the classification, up to the rank $p+q$ and signature $p-q \geq 0$, of the quadratic form $x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$, where $1 \leq p+q \leq n$. The number of projective classes is equal to $(n^2+4n)/4$ if n is even, and equal to $(n^2+4n-1)/4$ if n is odd.

The situation is changed for degree $d \geq 3$ in all dimensions: there is an infinite number of affine (projective) classes of the varieties. In order to obtain a finite classification of affine (projective) varieties of a fixed degree, one should apply a new equivalence relation to the varieties. One natural approach is isotopy classification. Let X be $\mathbf{R}^n$ or $\mathbf{R}P^n$. Recall that two topological subspaces T_1 and T_2 in X are called *isotopy equivalent* if there exists a homeomorphism of X isotopic to the identity map that carries the pair (X, T_1) onto the pair (X, T_2) [1].

The isotopy classification approach has been successfully applied to nonsingular projective curves in $\mathbf{R}P^2$ and nonsingular projective surfaces in $\mathbf{R}P^3$. Isotopy classifications of nonsingular curves of degree $d \leq 7$ in $\mathbf{R}P^2$ and of nonsingular surfaces of degree $d \leq 4$ in $\mathbf{R}P^3$ have been obtained. For degree $d \leq 5$ the isotopy classification of nonsingular projective curves was known in the 19th century [**32**]. In 1969 Gudkov [**6**] (see also [**7**] and [**30**]) completed the isotopy classification for degree 6, and in 1980 Viro [**45**] (see also [**44**]) completed the classification for $d = 7$. The classification of nonsingular projective surfaces of degree $d = 1$ and 2 is trivial, of degree $d = 3$ was known in the 19th century [**40**], [**34**], and of degree $d = 4$ was completed by Kharlamov [**33**] in 1978. We exhibit Tables 1 and 2, which show the numbers of isotopy classes in these classifications.

TABLE 1.

Degree of curves	1	2	3	4	5	6	7
Number of isotopy classes	1	2	2	6	7	56	121

TABLE 2.

Degree of surfaces	1	2	3	4
Number of isotopy classes	1	3	5	112

Starting from this point we will consider only algebraic curves in $\mathbf{R}^2$ and in $\mathbf{R}P^2$.

The isotopy classification of algebraic curves in $\mathbf{R}^2$ or in $\mathbf{R}P^2$ has a topological character. This is the reason why the isotopy classification of algebraic curves reveals only topological properties and does not reveal the algebraic structure of the curves. This phenomenon is inherent especially in the case of singular curves. If a curve C_1 of degree d has a real k-fold singular point with $r \geq 1$ real branches, then this curve is isotopic to a curve C_2 that has instead of the previous k-fold singular point just an r-fold ordinary point. Or if a curve C_1 of degree d has a real k-fold singular point without real branches (*isolated point*), then this curve is isotopic to a curve C_2 that has instead of the previous k-fold singular point just an ordinary double isolated point, i.e., a point with two complex conjugate transversal branches. These examples suggest that we consider real singular points of algebraic curves more precisely. From a topological point of view a curve with multiple components looks like a curve that has the same, but 1-fold components, and therefore represents an isotopy class of curves of lesser degree. This means that one should study curves with and without multiple components separately.

Note that an open neighborhood U (in $\mathbf{R}^2$ or $\mathbf{R}P^2$) of a point P of a curve C is called *regular* [**8**] if 1) the sets U and $C \cap \operatorname{Cl} U$ are 1-connected, 2) the set $C \cap \operatorname{Fr} U$ consists of a finite number of points (equal to twice the number of real branches with center at P), and 3) there are no singular or inflection points of the curve C in $\operatorname{Cl} U$ except perhaps at P. Each point of an algebraic curve without multiple components has a regular neighborhood. Let P_1 and P_2 be points of the curves C_1 and C_2, and let U_1 and U_2 be regular neighborhoods of P_1 and P_2, respectively. The points P_1 and P_2 are called *isotopy equivalent* if the pairs $(U_1, C_1 \cap U_1)$ and $(U_2, C_2 \cap U_2)$ are topologically equivalent. An n-fold point is called an *ordinary n-fold point* if it has n branches, the tangent lines of which are distinct. A point of an algebraic curve is called a *singular-simple* point 1) if it is an n-fold ordinary point with $n \geq 1$ real branches, or 2) if it is an ordinary isolated double point. It is easy to see that if $r \geq 1$, then an n-fold point with r real branches is isotopy equivalent to an r-fold singular-simple point, and if $r = 0$, then an n-fold point (without real branches) is isotopy equivalent to an ordinary isolated double point. In the latter case, n is even. In particular, an n-fold point with 1 real branch is isotopy equivalent to a nonsingular point. Thus, the number of real branches of a point is an isotopy invariant; in other words, each isotopy class of points of an algebraic curve contains points with the same number of real branches and, in particular, contains a singular-simple point with the same number of branches. We consider nonsingular points of a curve as ordinary 1-fold points. A nonsingular point can be an inflection point of a curve.

A projective algebraic curve that has only real singular-simple points is called a *singular-simple* curve. In particular, a nonsingular curve is a singular-simple curve. If each isotopy class of a given set of algebraic curves contains a singular-simple curve, then to obtain the isotopy classification of such curves it is sufficient to classify the singular-simple curves.

By virtue of its topology, the set of complex points of a real algebraic curve can contain additional information on the arrangement of the real components. Studying the topology of the complex part has made it possible to get new regularities and restrictions on the arrangement of the real components. As this subject would take us too far afield, we refer the reader to the surveys [**8**] and [**39**].

The complex singular points of a real algebraic curve appear as pairs of complex conjugate points. Every real deformation of one of them comes together with an analogous deformation of its complex conjugate. Such a deformation leads to one of two possibilities: either the deformation changes the isotopy type of the real part of the curve or it does not. If the deformation does not change the isotopy type of any

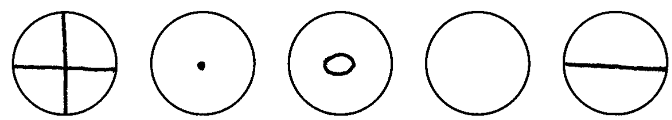


FIGURE 1. Isotopy and singular-isotopy classifications of projective conics coincide.

curve from a given set of curves, then each isotopy class contains a curve without complex singular points. Thus, to get the isotopy classification, one can take into consideration only curves without complex singular points. We will show that every isotopy class that contains a quartic curve with a set of complex singular points contains a singular-simple quartic curve.

Conjecture 1. *Each isotopy class of plane projective algebraic curves of degree d without multiple components contains a singular-simple curve of degree d.*

This conjecture is obvious for isotopy classes of projective curves of degree 1, 2 and 3. In Figures 1 and 2 (see Remark about pictures below) one can see the isotopy classification of projective conics and cubic curves, respectively. We show the isotopy classification of the conics for completeness. The isotopy classification of cubic curves is well known and contains eight classes.

Remark about pictures. The model of the affine plane that we use is an open disk D^2 that can be obtained as an image under the composition $\mathbf{R}^2 \to S^2_- \to D^2$ of the homeomorphism $g : \mathbf{R}^2 \to S^2_-$, where $S^2_- = \{(x,y,z) \in \mathbf{R}^3 \mid x^2+y^2+(z-1)^2 = 1, z < 1\}$ is the open unit hemisphere and

$$g(x,y) = \left(\frac{x}{\sqrt{x^2+y^2+1}}, \frac{y}{\sqrt{x^2+y^2+1}}, 1 - \frac{1}{\sqrt{x^2+y^2+1}}\right)$$

is the central projection from the center $(0,0,1)$ of the hemisphere S^2_-, and the homeomorphism $h : S^2_- \to D^2$, where $D^2 = \{(x,y) \in \mathbf{R}^2 \mid$

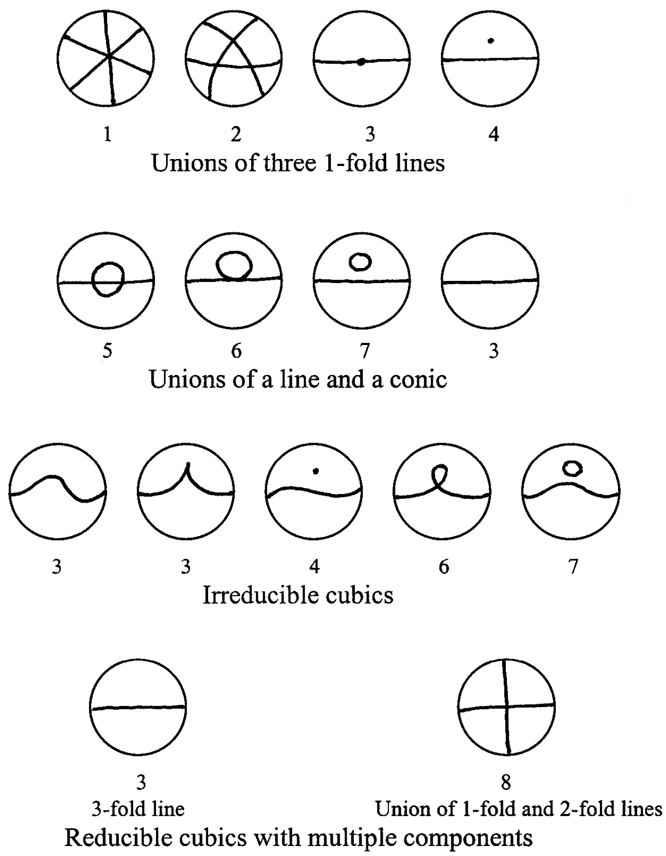


FIGURE 2. Isotopy classification (8 classes) and singular-isotopy classification (15 classes) of projective cubics.

$x^2 + y^2 < 1\}$ and $h(x, y, z) = (x, y, 0)$ is a projection parallel to the z-axis. The boundary ∂D^2 of the disk D^2 represents the line at infinity L_∞. The union $D^2 \cup \partial D^2$ is a model of the projective plane, which is usually called a *projective disk*. To visualize the affine plane it is enough to remove the line at infinity $L_\infty = \partial D^2$ from the projective disk. We have selected this model of the affine plane to exhibit the behavior of a quartic curve at infinity.

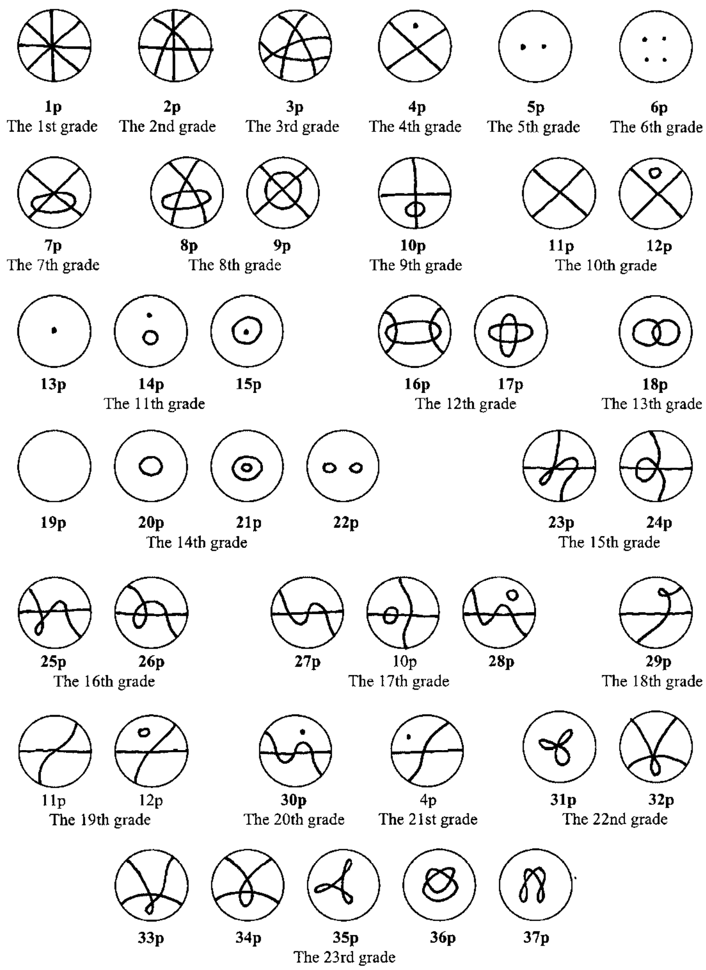


FIGURE 3 (beginning). Isotopy classification of projective quartic curves.

The isotopy classification of projective quartic curves is shown in Figure 3. It contains 66 classes and can be easily obtained by applying the projective version of Theorem 5(a), Shustin's Theorem 3 and Corollary 6.1, all from Section 2, to the union of the classification of irreducible quartic curves due to Gudkov, Utkin and Tai [**31**] and the isotopy classification of reducible quartic curves. One property

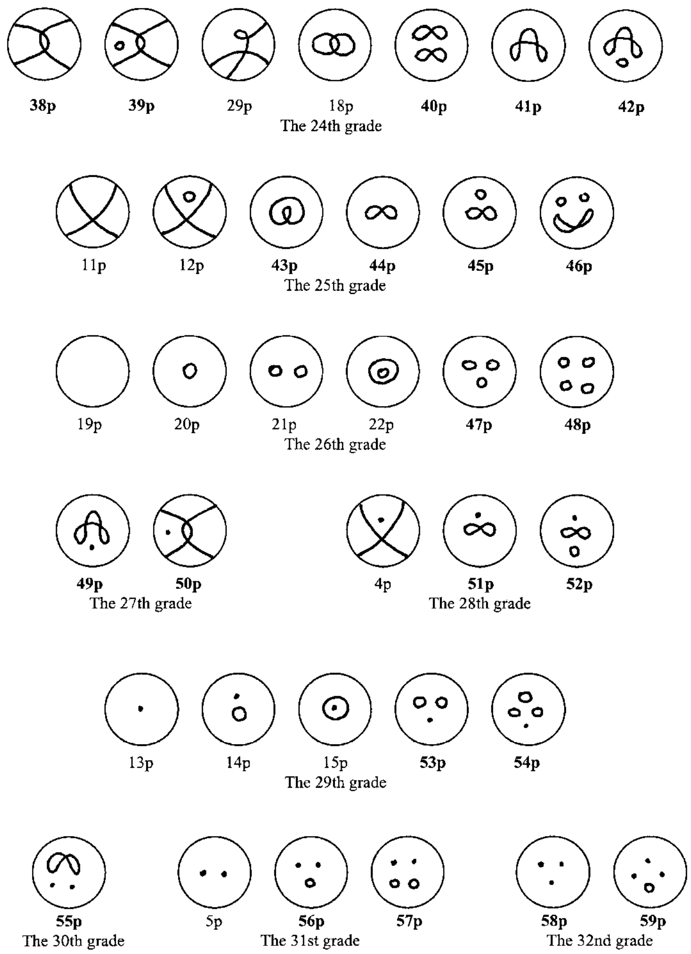


FIGURE 3 (continued). Isotopy classification of projective quartic curves.

of quartic curves that our enumeration is based on is the kind of decomposition into irreducible algebraic components. There are 11 kinds of such decompositions of projective quartic curves with respect to the kinds of its irreducible algebraic components, where for all cases l, c, cb and q denote a line, irreducible conic, cubic and quartic curve, respectively:

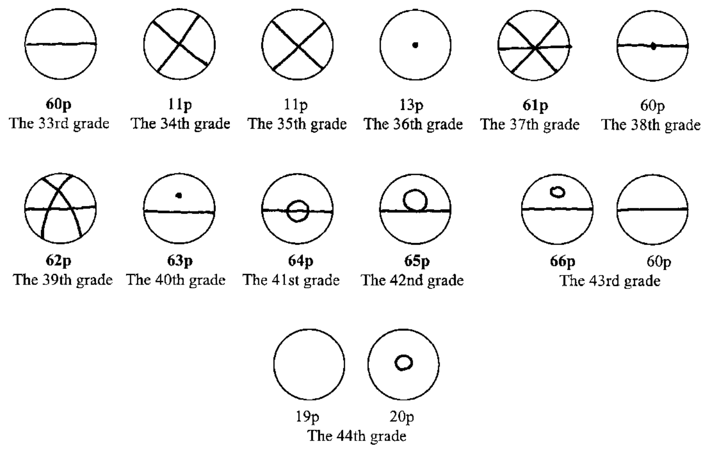


FIGURE 3 (conclusion). Isotopy classification of projective quartic curves.

1) four lines $l_1 l_2 l_3 l_4$,

2) two lines and a conic $l_1 l_2 c$,

3) a line and a cubic curve lcb,

4) two conics $c_1 c_2$,

5) a quartic curve q,

6) a 4-fold line l^4,

7) a 1- and 3-fold line $l_1 l_2^3$,

8) two 1-fold lines and a 2-fold line $l_1 l_2 l_3^2$

9) two 2-fold lines $l_1^2 l_2^2$,

10) a 2-fold line and a conic $l^2 c$, and

11) a 2-fold conic curve c^2.

We make use of the topological classification of triples $(\mathbf{R}P^2,\ \mathbf{R}f,\ \mathrm{Sing}\,(f))$ as a kind of foundation for the isotopy classification of projective quartic curves. This turns out to be a most convenient bookkeeping device. Here $\mathbf{R}f$ is the set of zeros of a polynomial f in $\mathbf{R}P^2$ and 1) if f has no multiple components, then $\mathrm{Sing}\,(f)$ is the set of real singular points of a singular-simple representative of the isotopy class of the curve f and 2) if f has multiple components, then $\mathrm{Sing}\,(f)$ is the set of real singular points of a polynomial that has the same algebraic components as f, but all to the first degree. We take into consideration the topological classification of the triples because the addition of the third member $\mathrm{Sing}\,(f)$ simplifies the enumeration of isotopy classes. The set of isotopy classes of curves without multiple components and with the same kind of irreducible algebraic components and with the same set of singular-simple points is called a *grade* of quartic curves. Each grade of curves without multiple components can be described by a pair $(\mathbf{R}f, \mathrm{Sing}\,(f))$. One can see that there are 32 grades of quartic curves without multiple components (see Section 2 for an explanation of the notation for singular points):

The 1st grade $(\mathbf{R}(l_1l_2l_3l_4), \{X_9\})$.

The 2nd grade $(\mathbf{R}(l_1l_2l_3l_4), \{D_4, 3A_1\})$.

The 3rd grade $(\mathbf{R}(l_1l_2l_3l_4), \{6A_1\})$.

The 4th grade $(\mathbf{R}(l_1l_2l_3l_4), \{A_1, A_1^*\})$.

The 5th grade $(\mathbf{R}(l_1l_2l_3l_4), \{2A_1^*\})$.

The 6th grade $(\mathbf{R}(l_1l_2l_3l_4), \{4A_1^*\})$.

The 7th grade $(\mathbf{R}(l_1l_2c), \{D_4, 2A_1\})$.

The 8th grade $(\mathbf{R}(l_1l_2c), \{5A_1\})$.

The 9th grade $(\mathbf{R}(l_1l_2c), \{3A_1\})$.

The 10th grade $(\mathbf{R}(l_1l_2c), \{A_1\})$.

The 11th grade $(\mathbf{R}(l_1l_2c), \{A_1^*\})$.

The 12th grade $(\mathbf{R}(c_1c_2), \{4A_1\})$.

The 13th grade $(\mathbf{R}(c_1c_2), \{2A_1\})$.

The 14th grade $(\mathbf{R}(c_1c_2), \varnothing)$.

The 15th grade $(\mathbf{R}(lcb), \{D4, A_1\})$.

The 16th grade $(\mathbf{R}(lcb), \{4A_1\})$.

The 17th grade $(\mathbf{R}(lcb), \{3A_1\})$.

The 18th grade $(\mathbf{R}(lcb), \{2A_1\})$.

The 19th grade $(\mathbf{R}(lcb), \{A_1\})$.

The 20th grade $(\mathbf{R}(lcb), \{3A_1, A_1^*\})$.

The 21st grade $(\mathbf{R}(lcb), \{A_1, A_1^*\})$.

The 22nd grade $(\mathbf{R}q, \{D_4\})$.

The 23rd grade $(\mathbf{R}q, \{3A_1\})$.

The 24th grade $(\mathbf{R}q, \{2A_1\})$.

The 25th grade $(\mathbf{R}q, \{A_1\})$.

The 26th grade $(\mathbf{R}q, \varnothing)$.

The 27th grade $(\mathbf{R}q, \{2A_1, A_1^*\})$.

The 28th grade $(\mathbf{R}q, \{A_1, A_1^*\})$.

The 29th grade $(\mathbf{R}q, \{A_1^*\})$.

The 30th grade $(\mathbf{R}q, \{A_1, 2A_1^*\})$.

The 31st grade $(\mathbf{R}q, \{2A_1^*\})$.

The 32nd grade $(\mathbf{R}q, \{3A_1^*\})$.

For completeness we consider the 12 grades of projective quartic curves with multiple algebraic components. Each grade of curves with multiple components can be described by a pair $(\mathbf{R}f, \mathrm{Sing}\,(g))$, where g is the product of all the distinct irreducible algebraic components of f.

The 33rd grade $(\mathbf{R}(l^4), \mathrm{Sing}\,(l) = \varnothing)$.

The 34th grade $(\mathbf{R}(l_1 l_2^3), \mathrm{Sing}\,(l_1 l_2) = \{A_1\})$.

The 35th grade $(\mathbf{R}(l_1^2 l_2^2), \mathrm{Sing}\,(l_1 l_2) = \{A_1\})$.

The 36th grade $(\mathbf{R}(l_1^2 l_2^2), \mathrm{Sing}\,(l_1 l_2) = \{A_1^*\})$.

The 37th grade $(\mathbf{R}(l_1 l_2 l_3^2), \mathrm{Sing}\,(l_1 l_2 l_3) = \{D_4\})$.

The 38th grade $(\mathbf{R}(l_1 l_2 l_3^2), \mathrm{Sing}\,(l_1 l_2 l_3) = \{D_4^*\})$.

The 39th grade $(\mathbf{R}(l_1 l_2 l_3^2), \mathrm{Sing}\,(l_1 l_2 l_3) = \{3A_1\})$.

The 40th grade $(\mathbf{R}(l_1 l_2 l_3^2), \mathrm{Sing}\,(l_1 l_2 l_3) = \{A_1^*\})$.

The 41st grade $(\mathbf{R}(l^2 c), \mathrm{Sing}\,(lc) = \{2A_1\})$.

The 42nd grade $(\mathbf{R}(l^2 c), \mathrm{Sing}\,(lc) = \{A_3\})$.

The 43rd grade $(\mathbf{R}(l^2 c), \mathrm{Sing}\,(lc) = \{\varnothing\})$.

The 44th grade $(\mathbf{R}(c^2, \mathrm{Sing}\,(c) = \varnothing))$.

An affine curve is the restriction of a projective curve to $\mathbf{R}^2 = RP^2 \setminus L_\infty$, where L_∞ is the line at infinity. We say that the affine and

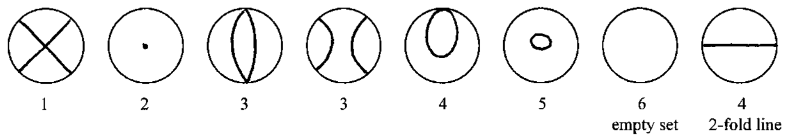


FIGURE 4. Isotopy classification of affine conics: 6 classes.

projective curves correspond to each other. The isotopy classification of affine algebraic curves in $\mathbf{R}^2$ has some distinctive features in comparison with the isotopy classification of projective curves in $\mathbf{R}P^2$. The distinctive features are related to the behavior of affine curves at infinity, namely, with the fact that an affine curve can have singular points at infinity.[2] If a projective curve has a nonsingular point on the line L_∞, which is not an inflection point, then either L_∞ is not a tangent line to the projective curve or it is. In the first case a tangent line to the projective curve at the point of intersection with L_∞ corresponds to an asymptote of the affine curve. The point of intersection of the curve and the line L_∞ in this case is called a point of *simple* intersection. The asymptote is real if the point of simple intersection is real, and is complex otherwise. In the second case the corresponding affine curve has two parabolic branches that look asymptotically like two common branches of a standard parabola. The point of quadratic tangency of the curve and the line L_∞ in this case is called a point of *simple* tangency. These branches are real if the point of simple tangency is real, and are complex otherwise. If a projective curve has points of simple intersection or points of simple tangency with the line L_∞, then such an intersection of the curve and the line L_∞ is called *quasi-simple*. If a projective curve is singular-simple and has quasi-simple intersection with the line L_∞, then the corresponding affine curve is called an (*affine*) *singular-simple* curve.

Conjecture 2. *Each isotopy class of plane affine algebraic curves of degree d without multiple components contains a singular-simple curve of degree d.*

This conjecture is obvious for isotopy classes of curves of degrees 1, 2 and 3. In Figures 4 and 5 one can see the isotopy classification of affine conics and cubics. The isotopy classification of the conics is

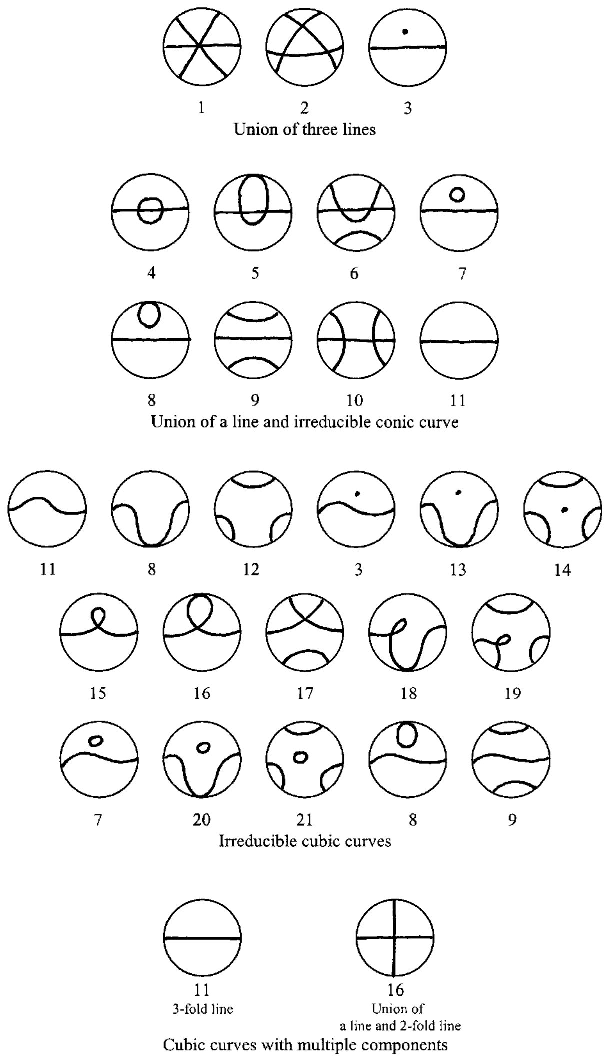


FIGURE 5. Isotopy classification of affine cubic curves.

trivial and contains 6 isotopy classes. We show it for completeness. The isotopy classification of affine cubic curves contains 21 classes. It was described for the first time by Weinberg [**47**]. He obtained this isotopy classification by means of simple algebraic manipulations of the general cubic equation. But it can also be obtained from Newton's classification of irreducible cubic curves [**38**], which contains 78 species, together with a consideration of reducible cubic curves.

To take into account the algebraic behavior of a curve at singular points, we formulate the following definition.

Two projective curves C_1 and C_2 are called *singular-isotopy equivalent* if 1) they are isotopy equivalent and 2) the isotopy connecting C_1 and C_2 preserves their singularlities.[3]

The singular-isotopy classification of curves of degrees 2 and 3 contains a finite number of classes, because the curves have only zero-modal singular points. The singular-isotopy classification of curves of degree 4 has an infinite number of classes: there are a finite number of classes of curves with zero-modal singular points, and there are three families of curves with unimodal singular points of types X_9, X_0^*, and X_9^{**} (see the notation in Section 2). Applying the previous definition to curves of degree 4, we make use of the following convention. In this paper we consider the three families of curves with unimodal singular points of types X_9, X_9^* and X_9^{**} as three distinct "singular-isotopy" classes.

It is clear that isotopy equivalence follows from singular isotopy equivalence, and that isotopy and singular-isotopy classifications of non-singular curves coincide. In this sense the singular-isotopy classification is situated between the affine and isotopy classifications. The singular-isotopy classification of plane projective curves of degree 3 is well known, contains 15 classes and probably was described for the first time in [**29**].[4] In [**31**] it was mentioned that the singular-isotopy classification of projective irreducible conics contains 2 classes (real and imaginary ellipses), and that the singular-isotopy classification of projective irreducible cubic curves contains 5 classes (Figure 21 in [**31**]). This singular-isotopy classification of projective irreducible cubic curves was described by Newton [**38**]. The singular-isotopy classification of irreducible projective quartic curves without complex singular points contains 99 classes [**31**], but the isotopy classification of the same curves contains 42 classes (Figure 3). A more detailed classification of irreducible projective quartic curves that takes into account complex singular points, contains 117 classes (the additional 18 classes of irreducible quartic curves with complex conjugate singular points were studied in [**24**], [**25**]. As is shown in [**29**], the singular-isotopy classification of reducible projective quartic curves contains 95 classes.[5] Thus, the singular-isotopy classification of all projective quartic curves consists of 212 classes.

Gudkov and his students obtained the so-called *classification of forms* of irreducible projective quartic curves. Following Gudkov, we call a nonsingular point of a curve an *n-flat* point if the intersection multiplicity of the tangent line and the curve at this point is equal to n. For $n = 2$, this is an ordinary point of the curve having quadratic tangency with the tangent line. For $n = 3$, this is an ordinary point of inflection. For $n = 4$, this is a *planar* point, etc. Two projective curves C_1 and C_2 are said to have the same *form* if they are 1) singular-isotopy equivalent and 2) the isotopy connecting C_1 and C_2 preserves their n-flat points for all $n \geq 2$. (Gudkov [**8**] gave another definition of the equivalence of the forms, equivalent to our definition.) It is clear that singular-isotopy equivalence follows from equivalence of forms, and that the classification of forms is situated between the affine and singular-isotopy classifications. The idea and spirit of this research is due to Gudkov. This research was done in 1981–1990 and naturally falls into three big parts.

First [**16**], [**17**], [**18**]–[**24**], they regarded the arrangement of the real inflection points of an irreducible quartic curve when the inflection points are in general position, i.e., these points do not coincide with each other (no planar points) and do not coincide with singular points (no inflection points at a center of a real branch at a singular point of the curve). Later [**10**]–[**14**], Gudkov called the forms of such curves *coarse* (or *rough*) *forms*. They proves that the 117 singular-isotopy classes of irreducible quartic curves with inflection points in general position have 384 coarse forms[6] (= 349 forms with only real singular points +35 forms with complex ones [**26**]). According to Gudkov's remark [**8**], Zeuthen [**48**] constructed all 42 existing coarse forms of nonsingular quartic curves but didn't prove that one of the admissible curves cannot be realized by a quartic curve. A topological curve T is called *admissible for degree d* if for any algebraic curve f of degree $n = 1$ or 2 the set $\mathbf{R}f \cap T$ consists of no more than nd points[7] and the number of components of the curve T formally satisfies Harnack's theorem for degree d. In this sense one says that the topological curve T *satisfies Bezout's theorem with respect to intersection with a line or a conic* and *satisfies Harnack's theorem*. The isotopy class that contains an admissible curve T for degree d is called an *admissible class* (*for the degree* d). The complete classification of coarse forms of nonsingular quartic curves was given in [**18**], [**16**], [**18**]–[**20**].

Second, Gudkov [**10**]–[**14**], [**26**] completed the classification of the so-called *special forms* of irreducible quartic curves. This involves the case when the inflection points of a quartic curve are not in general position, i.e., either 1) two inflection points coincide by forming a planar point, or 2) one real branch of an ordinary double point has an inflection point at its center, or 3) each of two real branches at an ordinary double point has an inflection point at its center. This classification of special forms of irreducible quartic curves contains 653 classes. As an auxiliary result of his method, Gudkov rederived the classification of coarse forms. Thus, the total number of all forms of irreducible projective quartic curves is 1037 forms (= 653 special +349 coarse +35 forms with complex singular points).

To extend the classification of forms to the case of reducible projective quartic curves, one should adopt the definition (Number 2 in [**9**]) for both coarse and special forms of reducible quartic curves. The singular-isotopy classification of plane reducible projective quartic curves contains 95 classes [**29**], each of which represents only one coarse form. Thus, the number of coarse forms of reducible quartic curves is equal to 95. Notice that reducible quartic curves that consist of either 4 lines, or 2 lines and a conic, or 2 conics, don't have inflection points at all and so can't represent any special forms. To obtain the missing classification of special forms of reducible quartic curves, it is sufficient to consider only the reducible quartic curves that consist of a line and a cubic curve that has inflection points at points of intersection with the line. It is not difficult to check that there are 26 such special forms of reducible quartic curves. Now one can see that the classification of coarse and special forms of irreducible and reducible projective quartic curves contains 1158 distinct forms ($= 1037 + 95 + 26$).

By the same method, Gudkov also obtained the classification of mutual arrangements of coarse and special forms of quartic curves with so-called singular lines A real line is called a *singular line with respect to* *a quartic curve* or simply a *singular line* if either 1) the line is tangent to the quartic curve at two distinct real points lying on the same complete real branch of the quartic curve (double tangent line), or 2) the line is tangent to the quartic curve at two complex points (double isolated tangent line), or 3) the line is tangent to the quartic curve at a planar point, or 4) the line is tangent to a real quadratic branch at a singular point, or 5) the line is tangent to a real cubic branch at a singular point, or 6) the line passes through two singular points of the quartic curve, or 7) the line is a double tangent at a singular point of the quartic curve.

The classification of these forms is based on applying the Klein-Viro formula [**45**], [**36**] and Shustin's inequality [**41**]. The Klein-Viro formula together with the famous formulas of Plücker allow one to calculate the possible numbers of inflection points and double tangent lines of a quartic curve with a fixed set of singular points. Gudkov realized the tremendous value of the Klein-Viro formula together with Plücker's formulas for vastly simplifying the enumeration of the admissible classes[8] for these forms of projective quartic curves. The set of admissible classes consists of isotopy classes that contain projective quartic curves and classes that do not. Gudkov studied both cases. Shustin's inequality describes the condition that allows one to perturb independently the singular points of an irreducible algebraic curve. Gudkov used Shustin's inequality for the construction of quartic curves. Without this inequality, the construction would be much more complicated.

And third, Gudkov and Polotovskiĭ [**27**]–[**29**] proved that the set of projective quartic curves of the same singular-isotopy class represents one stratum (connected component) in the space $\mathbf{R}P^{14}$ of all quartic curves, both irreducible and reducible. We already calculate that the number of such strata is 212. They proved the same statements for conics and cubics [**29**]. There are 5 and 15 strata of them, respectively. According to Rokhlin's remark [**39**, item 4.1], the fact that each singular-isotopy class of projective nonsingular quartic curves represents one stratum was known to Klein [**35**, p. 112]. Rokhlin found an example of two projective nonsingular curves of degree 5 that represent the same isotopy class, but belong to distinct strata in the space of curves of degree 5, (see [**39**, item 3]).

The works of Gudkov and his students on real projective quartic curves represent an important step and significant contribution to the classical theory of real algebraic curves. Gudkov and his mathematical school exerted a powerful influence on research in real algebraic geometry in general.

The subject of affine quartic curves was considered by many mathematicians. Ball [**2**] indicated that Waring applied the Newtonian method to classify affine quartic curves, indicated 12 characteristic forms of quartic equations and showed that the classification of affine quartic curves, treated in this way, contains no more than 84551 species; Cramer divided quartic curves into 9 classes but did not continue the subdivision into genera and species; Euler divided them into 8 classes and 146 genera; and Salmon gave ten classes.

Bruce and Giblin [**5**] gave a complete isotopy classification of complex projective quartic curves.

The main results of our paper are the isotopy classification of affine quartic curves which contains 647 classes, and the topological classification of pairs ($\mathbf{R}^2$, *quartic curve*) which contains 516 classes. There are at least two ways to obtain these two classifications. The first way goes as follows: one can

1) take all projective quartic curves that are provided by the Gudkov classification of coarse forms,

2) consider all unions of these curves with transversal and non-transversal lines,

3) designate these lines as lines at infinity, obtaining in each case an affine quartic curve,

4) apply the notion of isotopy equivalence to the resulting set of affine quartic curves, thus obtaining the required isotopy classification.

Our calculation shows that in this way one must compare about 9000 pictures of affine quartic curves.[9] We select the second way: in Section 2 we prove Conjecture 2 for degree 4, which establishes that it is sufficient to consider only singular-simple affine quartic curves. In Section 3, first we describe the most efficient way of enumerating the affine admissible isotopy classes of quartic curves; second, we produce a kind of recursive method of construction of representatives of the affine isotopy classes; and third, in Lemma 8 we prove that some of the admissible classes have no quartic representatives.

Notice that after one proves Conjecture 1 for degree 4, there arises a third way that is a combination of the two previous ones. Having proved Conjecture 1 for degree 4, one can reduce to the consideration of unions of the line L_∞ with projective quartic curves that are provided by the Gudkov classification of coarse forms, to unions that satisfy the following two additional properties: 1) the unions have arbitrary types of points of intersection of the quartic curve with the line L_∞, and 2) the unions have singular-simple points in $\mathbf{R}P^2 \setminus L_\infty$. The third way is simpler than the first one, but still more complicated than the second one.

The isotopy classification presented in this paper can be regarded as a first step toward obtaining the singular-isotopy classification of affine quartic curves.

2. Singular points of affine quartic curves and of some reducible projective quintic curves. A homogeneous real polynomial of degree d in three variables considered up to a constant factor is called a *real plane projective algebraic curve of degree d*, or simply a *curve*. If $F(x, y, z)$ is such a polynomial, then the set $\mathbf{R}F = \{x \in \mathbf{R}P^2 \mid F = 0\}$ is called the *set of real points* of the projective curve. We follow the classic tradition and call this set a projective curve when it does not lead to confusion. If $z = 0$ is the equation of the line at infinity, then the polynomial $f(x, y) \equiv F(x, y, 1)$ considered up to a constant factor is called a *real plane affine algebraic curve of degree* $d - k$, where k is the multiplicity of the factor z in the polynomial F. The set $\mathbf{R}f = \{x \in \mathbf{R}^2 \mid f = 0\}$ is called the *set of real points* of the affine curve. We follow the same tradition and call this set an affine curve when it does not lead to confusion. The curves $\mathbf{R}F$ and $\mathbf{R}f$ are said to correspond to each other.

The main aim of this section is to prove Theorem 6, which coincides with Conjecture 2 for quartic curves. Our proof of this conjecture is based on the following theorem of Shustin, which allows us to deform singular points of a curve independently.

Theorem 3 [**42**]. *Let a* (*real*) *curve F have singular points $z_1, \dots, z_s$ with Milnor numbers $\mu(z_1), \dots, \mu(z_s)$, respectively. If the* (*real*) *curve F satisfies the inequality*

$$\mu(z_1) + \cdots + \mu(z_s) \leq 4d - 5,$$

then there exists a family $F_t \in \mathbf{C}P^n$ in the space $\mathbf{C}P^n$, $n = d(d+3)/2$

of all complex curves of degree d, $t \in [0,1]$, with $F_0 = F$, that provides a (real) deformation of the singular points $z_1, \dots, z_s$ that is equivalent to any possible prescribed (real) deformation of these singular points.

To reach our aim we consider all quintic curves that decompose into a line and a projective quartic curve. We select this line as the line at infinity and enumerate all prescribed deformations of singular points of the quartic component of the quintic curve. After that we enumerate all sets of singular points of these quintic curves and show that these sets, with one exception, satisfy inequality (1). The exceptional case is treated separately.

It is known [**31**], [**16**], [**17**] that irreducible quartic curves can have only the following singular points:

$$A_1,\ A_1^*,\ A_2,\ A_3,\ A_3^*,\ A_4,\ A_5,\ A_5^*,\ A_6,\ D_4,\ D_4^*,\ D_5,\ E_6,\ 2A_1^i,\ 2A_2^i.$$

It is also known [**29**] that reducible quartic curves can have only the following singular points:

$$A_1,\ A_1^*,\ A_2,\ A_3,\ A_3^*,\ A_5,\ A_5^*,\ A_7,\ A_7^*,$$
$$D_4,\ D_4^*,\ D_5,\ D_6,\ E_7,\ X_9,\ X_9^*,\ X_9^{**},\ 2A_1^i,\ 2A_3^i.$$

The names of these singular points follow Arnold's notation for singularities [**1**] and Gudkov's special convention [**8**]: 1) if there is no asterisk in the notation of a point, then the point is real and all branches centered in it are real, 2) if there is one asterisk, then the point is real and two branches centered in this point are complex conjugate, 3) if there is an upper index i, then the point is complex, 4) an integer factor before a letter denotes the number of such points, 5) X_9, X_9^* and X_9^{**} are ordinary 4-fold points with four real branches, with two real and two complex branches, and with two pairs of complex conjugate branches, respectively.

Affine quartic curves have the same singular points and the same sets of singular points as projective curves. The difference from projective quartic curves, as we already mentioned in the Introduction, is in the fact that affine quartic curves can have singular points at infinity. Our primary aim now is to prove Conjecture 2 for affine quartic curves. We prove that for any affine quartic curve $\mathbf{R}f$, there exists an isotopy equivalent singular-simple affine quartic curve $\mathbf{R}f_0$. We obtain the

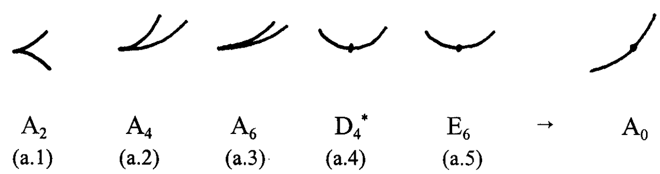


FIGURE 6.1.

curve $\mathbf{R}f_0$ by deformation of singular points of the curve $\mathbf{R}f$ such that all singular points of $\mathbf{R}f$ in $\mathbf{R}^2$ become singular-simple points of the curve $\mathbf{R}f_0$ and the intersection of the projective quartic curve $\mathbf{R}F$ with the line at infinity L_∞ becomes a quasi-simple intersection of the projective quartic curve $\mathbf{R}F_0$ with L_∞.

The prescribed deformations of real singular points in $\mathbf{R}^2$ and of complex singular points in $\mathbf{C}^2 \setminus \mathbf{R}^2$ that we apply are listed in Figures 6.1–6.4. The singular points A_2, A_4, A_6, D_4^* and E_6 are smoothed to the singular-simple point A_0. The singular points A_3, A_5, A_7, D_5, E_7 and X_9^* are deformed to the singular-simple point A_1. The singular point D_6 is deformed to D_4. The singular points A_3^*, A_5^*, A_7^* and X_9^{**} are deformed to the singular-simple point A_1^*. The complex singular points of a real quartic curve appear in $\mathbf{C}^2 \setminus \mathbf{R}^2$ as complex conjugate pairs $2A_1^i, 2A_2^i$ and $2A_3^i$ and are smoothed to the nonsingular complex point A_0^i. We do not deform the singular points A_1, D_4, X_9 and A_1^* since they are already singular-simple.

The prescribed deformations of singular points at infinity are listed in Figures 7.1–7.9. We add to Gudkov's convention a new one: A_{-1} means that a real point on the line at infinity does not belong to the quartic curve; $A_0^1 (A_0^{i,1})$ means a 1-fold real (complex) point of the curve

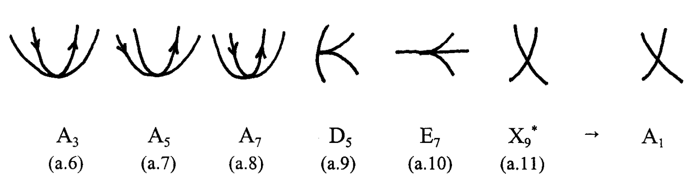
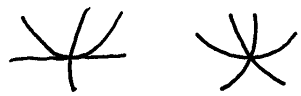


FIGURE 6.2.



$D_6 \rightarrow D_4$

(a.12)

FIGURE 6.3.

that has a 1-fold intersection with the line at infinity; $A_0^2(A_0^{i,2})$ means a 1-fold real (complex) point of the curve that has a 2-fold intersection with the line at infinity, i.e., of quadratic tangency; A_0^3 means a 1-fold real point of the curve that has a 3-fold intersection with the line at infinity, i.e., tangency at an inflection point; A_0^4 means a 1-fold real point of the curve that has a 4-fold intersection with the line at infinity, i.e., tangency at a planar point. In each picture the horizontal line represents the line at infinity. The double codes under the pictures mean, first, the type of the singular point of the quartic curve, and second, in parentheses, the type of the singular point of the union of the quartic curve and the line at infinity. The arrangement of a singular point at infinity can vary: the line at infinity can be a tangent line to some branches of the projective quartic curve at the (singular) point or not. The arrow in each string indicates the result of deforming the (singular) points in the string. We don't indicate the complex points of simple intersection of the quartic curve and the line at infinity that can arise under deformation of real points in their small complex neighborhoods. The sets of complex singular points $\{2A_0^{i,2}\}(\{2A_3^i\})$, $\{2A_1^i\}(\{2D_4^i\})$, $\{2A_2^i\}(\{2D_5^i\})$ and $\{2A_3^i\}(\{2D_6^i\})$ are deformed to $\{4A_0^{i,1}\}(\{4A_1^i\})$.

$X \rightarrow A_1^*$

FIGURE 6.4. $X = A_3^*, A_5^*, A_7^*, X_9^{**}$ from (a.16), (a.13), (a.14), (a.15) of Theorem 5, respectively.

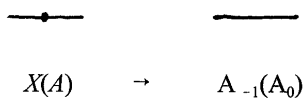


FIGURE 7.1. $X(A) = A_1^*(D_4^*)$, $A_3^*(D_6^*)$, $A_3^*(J_{10}^*)$, $A_5^*(D_8^*)$, $A_5^*(J_{12}^*)$, $A_7^*(D_{10}^*)$, $A_7^*(J_{14}^*)$, $X_9^{**}(N_{16}^{**})$ from (b.1)–(b.8) of Theorem 5, respectively.

In some cases of Theorem 5 we apply conditions when a quartic curve has singular points of the types A_μ, $\mu = 4, 5, 6$. These conditions follow from our Lemma 4. To prove this lemma we consider some special summations of monomials in the polynomial that represents a given curve.

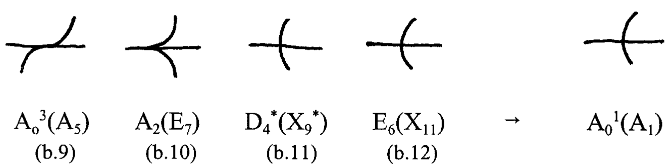


FIGURE 7.2.

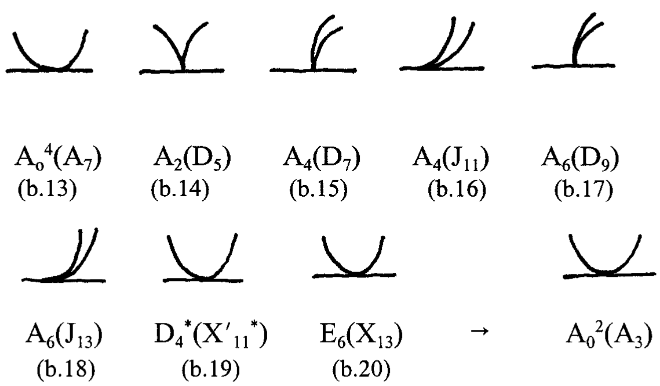


FIGURE 7.3.

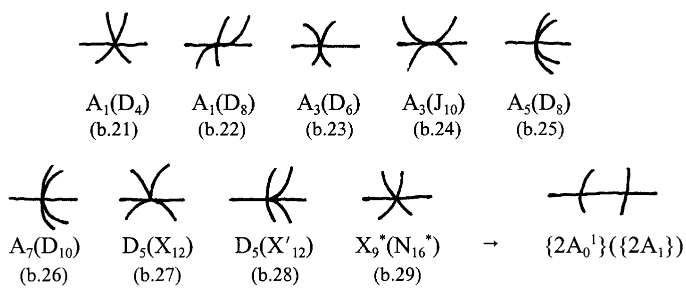


FIGURE 7.4.

Let

$$f(x,y) = \sum_{i+j\le n} f_{ij}x^i y^j$$

be a polynomial of a given curve. The set $\operatorname{supp}(f) = \{(i,j) \in \mathbf{Z}^2 \mid f_{i,j} \neq 0\}$ is called a *carrier* set for the curve f. The natural embedding $\mathbf{Z}^2 \to \mathbf{R}^2$ allows us to consider the carrier set as a subset of $\mathbf{R}^2$. The plane $\mathbf{R}^2$ with coordinates (i,j) is called the carrier plane. The convex hull of $\operatorname{supp}(f)$ in $\mathbf{R}^2$ is called *Newton's polygon* of the curve f and is denoted $N(f)$. Thus we can write

$$f(x,y) = \sum_{(i,j)\in N(f)} f_{ij}x^i y^j.$$

Let $\Gamma \subset \mathbf{R}^2$ be a subset. The polynomial

$$f^{\Gamma}(x,y) = \sum_{(i,j)\in N(f)\cap\Gamma} f_{ij}x^i y^j$$

is called the restriction of the polynomial f to Γ. For each fixed integer $r \ge 1$ let $L_r = \{l_{pqr}\}_{p=0,1,\dots;q=0,1,\dots,r-1}$ be a family of parallel lines in the carrier plane, where $l_{pqr}(i,j) = i + rj - rp - q$. The union of all lines of the family $L_r = \{l_{pqr}\}$ contains the carrier set of a given curve, and the carrier set of the curve intersects a finite set of the lines of the family. It is easy to check that if we denote $\Gamma_{pqr} = \mathbf{R}l_{pqr} \cap \operatorname{supp}(f)$, then $f^{\Gamma_{pqr}}(x,y) = x^q U_{pq}(x^r,y)$ where $U_{pq}(x,y)$ is a homogeneous polynomial of degree p. Thus the polynomial $f(x,y)$ can be written in the form

$$\sum_p \sum_q x^q U_{pq}(x^r,y).$$

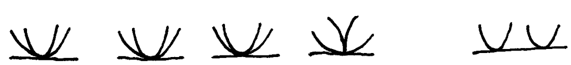
$A_1(D_6)$ (b.30) $D_5(X_{10})$ (b.31) $E_7(X'_{12})$ (b.32) $\to$ $\{A_0^1, A_0^2\}(\{A_1, A_3\})$

FIGURE 7.5.

This sum runs over the homogeneous polynomials $U_{pq}(x^r, y)$, the monomials of which correspond to the points of $\operatorname{supp}(f)$ lying on the lines l_{pqr}, respectively. The sum starts from the line $i + rj$ and runs over the lines in the direction of their common normal vector $(1, r)$.

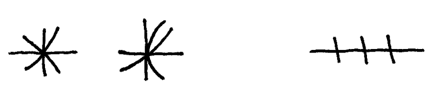
Lemma 4. *Let a curve be represented by the polynomial* $f(x, y) = U_{2,0}(x^n, y) + xU_{2,1}(x^n, y) + x^2U_{2,2}(x^n, y) + \cdots + x^{n-1}U_{2,n-1}(x^n, y) + U_{3,0}(x^n, y) + \cdots$ *of degree* $d \geq 2n$*, where* $n \geq 1$*. Then*

1) *if the polynomial* $U_{2,0}(x, y)$ *has two distinct roots (i.e., the discriminant* Δ *of* $U_{2,0}(x, y)$ *is not equal to zero), then the curve* f *has a singular point of the type* A_{2n-1} *at the origin when* $\Delta > 0$*, and a singular point of the type* A^*_{2n-1} *at the origin when* $\Delta < 0$*;*



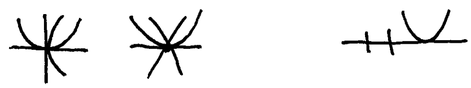
$A_3(J_{10})$ (b.33) $A_5(J_{12})$ (b.34) $A_7(J_{14})$ (b.35) $D_5(X_{12})$ (b.36) $\to$ $\{2A_0^2\}(\{2A_3\})$

FIGURE 7.6.



$D_4(X_9)$ (b.37) $D_6(X'_{11})$ (b.38) $\to$ $\{3A_0^1\}(\{3A_1\})$

FIGURE 7.7.



$D_6(X'_{13})$ (b.39) $D_4(X'_{11})$ (b.40) $\rightarrow$ $\{2A_0{}^1, A_0{}^2\}(\{2A_1, A_3\})$

FIGURE 7.8.

2) *if the polynomial* $U_{2,0}(x,y)$ *has a double root* $(\alpha : \beta)$, $U_{2,0}(0,1) \neq 0$ *and* $U_{2,1}(\alpha,\beta) \neq 0$, *then the curve* f *has a singular point of the type* A_{2n} *at the origin*;

3) *if the polynomial* $U_{2,0}(x,y)$ *has a double root* $(\alpha : \beta)$, $U_{2,0}(0,1) \neq 0$, $U_{2,1}(\alpha,\beta) = 0$, *and the polynomial* $f(x, y + \beta x^n/\alpha) = V_{2,0}(x^{n+1}, y) + xV_{2,1}(x^{n+1}, y) + x^2V_{2,2}(x^{n+1}, y) + \cdots + x^nV_{2,n-1}(x^{n+1}, y) + U_{3,0}(x^{n+1}, y) + \cdots$ *is such that the homogeneous polynomial* $V_{2,0}(x,y)$ *has two distinct roots* (*i.e., the discriminant* Δ *of* $V_{2,0}(x,y)$ *is not equal to zero*), *then the curve* $f(x,y)$ *has a singular point of the type* A_{2n+1} *at the origin when* $\Delta > 0$, *and a singular point of the type* A^*_{2n+1} *at the origin when* $\Delta < 0$.

Proof. 1) The statement of this item is obvious.

2) The condition $U_{2,0}(0,1) \neq 0$ implies $\alpha \neq 0$. The transformation $(x,y) \mapsto (x, y - \beta x^n/\alpha)$ is a diffeomorphism of the real plane and preserves the Milnor number of any singular point. This diffeomorphism transforms the curve $f(x,y)$ to the curve $f(x, y + \beta x^n/\alpha) = U_{2,0}(x^n, y + \beta x^n/\alpha) + xU_{2,1}(x^n, y + \beta x^n/\alpha) + x^2U_{2,2}(x^n, y + \beta x^n/\alpha) + \cdots + x^{n-1}U_{2,n-1}(x^n, y + \beta x^n/\alpha) + U_{3,0}(x^n, y + \beta x^n/\alpha) + \cdots$, where $U_{2,0}(x^n, \beta x^n/\alpha) = x^{2n}U_{2,0}(\alpha,\beta)/\alpha^2 = 0$, and if $xU_{2,1}(x^n, \beta x^n/\alpha) = x^{2n+1}U_{2,1}(\alpha,\beta)/\alpha^2 \neq 0$, then the monomial $x^{2n+1}U_{2,1}(\alpha,\beta)/\alpha^2$ deter-

$X_9(N_{16})$ (b.41) $\rightarrow$ $\{4A_0{}^1\}(\{4A_1\})$

FIGURE 7.9.

mines a singular point of type A_{2n} of the curve $f(x, y + \beta x^n/\alpha)$ and thus of the curve $f(x, y)$ at the origin.

3) If $U_{2,1}(\alpha, \beta) = 0$, then $f(x, y + \beta x^n/\alpha) = V_{2,0}(x^{n+1}, y) + xV_{2,1}(x^{n+1}, y) + x^2V_{2,2}(x^{n+1}, y) + \cdots + x^nV_{2,n}(x^{n+1}, y) + V_{3,0}(x^{n+1}, y) + \cdots$, and the statement of this item becomes obvious. □

Theorem 5. (a) *For any affine quartic curve $q = q(x, y)$ with a singular point X belonging to the set*

$$\{A_2,\, A_3,\, A_3^*,\, A_4,\, A_5,\, A_5^*,\, A_6,\, A_7,\, A_7^*,\, D_4^*,\, D_5,\, D_6,\, E_6,\, E_7,\, X_9^*,\, X_9^{**},\, 2A_1^i,\, 2A_2^i,\, 2A_3^i\},$$

there exists a continuous family $q_t(x, y)$, $t \in (0, \tau]$, of quartic curves with $q_0 = q$ such that for any $t \in (0, \tau]$ the curve $q_t(x, y)$ has a singular-simple point Y isotopic to X and having the same coordinates as X.

(b) *For any projective quartic curve $Q(x, y, z)$ with a singular point X lying on the line at infinity L_∞ and belonging to the set*

$$\{A_1,\, A_1^*,\, A_2,\, A_3,\, A_3^*,\, A_4,\, A_5,\, A_5^*,\, A_6,\, A_7,\, A_7^*,\, D_4,\, D_4^*,\, D_5,\, D_6,\, E_6,\, E_7,\, X_9,\, X_9^*,\, X_9^{**},\, 2A_1^i,\, 2A_2^i,\, 2A_3^i\},$$

there exists a continuous family $Q_t(x, y, z)$, $t \in (0, \tau]$, $\tau > 0$, of quartic curves with $Q_0 = Q$, and there exists a neighborhood U of the singular point X such that, for any $t \in (0, \tau]$, 1) any curve $Q_t(x, y, z)$ has, in U, a quasi-simple intersection with the line L_∞, and 2) the curve $\mathbf{R}Q_t \cap (U \setminus L_\infty)$ is isotopic to $\mathbf{R}Q_0 \cap (U \setminus L_\infty)$ in $U \setminus L_\infty$.

Proof. (a) Let the affine quartic curve $q = q(x, y)$ have a real singular point X and let the affine coordinate system be chosen such that the singular point X is placed at the origin. In each of the following cases (a.1)–(a.16) we exhibit a family $q_t(x, y)$, $t \in (0, \tau]$ of quartic curves with $q_0 = q$ that realizes the required isotopy of singular points. Neighborhoods of the real singular points are depicted in Figures 6.1–6.4.

(a.1) Deformation that carries singular point A_2 to A_0 ($A_2 \to A_0$). The quartic curve can be written as $q = y^2 - ax^3 + r(x, y)$ where $0 < a$ and $r(x, y)$ is the sum of monomials that correspond to the points lying above the line $3y+2x-6=0$ in Newton's polygon of q. Then, according to Bertini's theorem [**3**] (see also [**4**]), there exists $\tau > 0$ such that for any $t \in (0, \tau]$, the family $q_t(x,y) = t(y-x) + q$, where $q_0 = q$ realizes the required deformation.

The following cases (a.2)–(a.19) are analogous to the case (a.1) and can be read in the same manner.

(a.2) $A_4 \to A_0$; $q = (y-ax^2)^2+bxy^2+cx^3y+dy^3+ex^2y^2+fxy^3+gy^4$, where $0 < a$, $ab+c \neq 0$, see Lemma 4; $q_t = t(y-x)+q$.

(a.3) $A_6 \to A_0$; $q = (y-ax^2)^2+bxy^2+cx^3y+dy^3+ex^2y^2+fxy^3+gy^4$, where $0 < a$, $ab+c = 0$, $b^2-4(ad+e) = 0$, and $ab^3-2b^2(3a^2d+e)+a^2f \neq 0$ (see Lemma 4); $q_t = t(y-x)+q$.

(a.4) $D_4^* \to A_0$; $q = y(y^2+ax^2)+r$, where $0 < a$, r is the sum of monomials that correspond to the points lying above the line $y+x-3=0$ in Newton's polygon of q, and the greatest common factor of the polynomials $y(y^2+ax^2)$ and r is a constant; $q_t = t(y+bx)+q$, where $b \neq 0$.

(a.5) $E_6 \to A_0$; $q = y^3 - ax^4 + r$, where $a < 0$, r is the sum of monomials that correspond to the points lying above the line $4y+3x-12=0$ in Newton's polygon of q; $q_t = t(y+bx)+q$, and $b \neq 0$.

(a.6) $A_3 \to A_1$; $q = (y-ax^2)(y-bx^2)+r$, where $ab \neq 0$ and r is the sum of monomials that correspond to the points lying above the line $2y+x-4=0$ in Newton's polygon of q; $q_t = -tx^2+q$.

(a.7) $A_5 \to A_1$; $q = (y-ax^2)^2+bxy^2+cx^3y+dy^3+ex^2y^2+fxy^3+gy^4$, where $a < 0$, $ab+c = 0$ and $b^2-4(ad+e) > 0$ (see Lemma 4); $q_t = -tx^2+q$.

(a.8) $A_7 \to A_1$; $q = [y-ax^2+r]^2 - dy^4$, where $a < 0$, $r = bxy+cy^2$ and $0 < d$; $q_t = -tx^2+q$.

(a.9) $D_5 \to A_1$; $q = ay^3 - xy^2 + bx^3 + r$, where $a < 0$, $0 < b$, r is the sum of monomials that correspond to the points lying above the line $4y+3x-12=0$ in Newton's polygon of q; $q_t = -t(c^2y^2-x^2)+q$ and $a < c$.

(a.10) $E_7 \to A_1$; $q = y^3 - ax^3y + r$, where $a < 0$ and r is the sum of monomials that correspond to the points lying above the line $3y+2x-9=0$ in Newton's polygon of q; $q_t = -txy+q$.

(a.11) $X_9^* \to A_1$; $q = (y^2-x^2)(y^2+axy+bx^2)$, where $a^2-4b < 0$;

$q_t = t(y^2 - 2x^2) + q$.

(a.12) $D_6 \to D_4$; $q = xy^2 - ay^4 + bxy^3 + cx^2y^2 - dx^3y$, where $a < 0$, $d > 0$; $q_t = tx(y^2 - x^2) + q$.

(a.13) $A_3^* \to A_1^*$; $q = y^2 + ax^2y + bx^4 + r$, where $a^2 - 4b < 0$, r is the sum of monomials that correspond to the points lying above the line $2y + x - 4 = 0$ in Newton's polygon of q; $q_t = tx^2 + q$.

(a.14) $A_5^* \to A_1^*$; $q = (y - ax^2)^2 + bxy^2 + cx^3y + dy^3 + ex^2y^2 + fxy^3 + gy^4$, where $a < 0$, $ab + c = 0$ and $b^2 - 4(ad + e) < 0$ (see Lemma 4); $q_t = tx^2 + q$.

(a.15) $A_7^* \to A_1^*$; $q = [y - ax^2 + r]^2 + dy^4$, where $a < 0$, $r = bxy + cy^2$ and $d > 0$; $q_t = tx^2 + q$.

(a.16) $X_9^{**} \to A_1^*$; $q = (y^2 + x^2)(y^2 + axy + bx^2)$, where $a^2 - 4b < 0$; $q_t = t(y^2 + cx^2) + q$ where $0 < c$.

In the cases (a.17)–(a.19), we choose the affine coordinate system such that complex point pairs $\{2A_1^i\}$, $\{2A_2^i\}$ and $\{2A_3^i\}$ are placed at the points $(\sqrt{-1}, 0)$ and $(-\sqrt{-1}, 0)$. We smooth these pairs and $\{4A_0^i\}$ means four points of simple intersection of q_t and the axis $y = 0$.

(a.17) $\{2A_1^i\} \to \{4A_0^i\}$; $q = (x^2 + 1)^2 + (x^2 + 1)(ax + b)y + r(x)y^2 + s$, where a and b are integers, the greatest common factor of the polynomials $r(x)$ of degree 2 and $x^2 + 1$ is a constant, and s is the sum of the monomials that correspond to the points lying above the line $y - 2 = 0$ in Newton's polygon of the quartic curve q; $q_t = -t^2 + q$.

(a.18) $\{2A_2^i\} \to \{4A_0^i\}$; $q = (x^2 + 1)^2 + a(x^2 + 1)y^2 + r$, where a is an integer and r is the sum of the monomials that correspond to the points lying in Newton's polygon of the quartic curve above the line $y - 2 = 0$; $q_t = -t^2 + q$.

(a.19) $\{2A_3^i\} \to \{4A_0^i\}$; $q = (x^2 + 1)^2 + a(x^2 + 1)y^2 + by^4$, where a and b are integers; $q_t = -t^2 + q$.

(b) Let the projective quartic curve $Q(x, y, z)$ have a real singular point X and let the affine coordinate system (x, y) be chosen such that 1) the line at infinity L_∞ is $\{y = 0\}$, 2) $q = q(x, y)$ is the affine curve that corresponds to the projective curve $Q(x, y, z)$, and 3) the singular point X of the quartic curve is placed at the origin $(0, 0)$. In each of the following cases (b.1)–(b.41) we preserve the line at infinity as a component of a quintic curve and exhibit deformations of the quartic components. We use a double notation for singular points. For example, the notation $A_1^*(D_4^*)$ means that the quartic curve q has a singular point A_1^* and the quintic curve yq has a singular point D_4^*, both at the origin. The proofs of the following cases (b.1)–(b.42) are analogous to each other and can be read in the same manner as item (a.1). In all items we consider families q_t of quartic curves with $t \in (0, \tau]$ and $q_0 = q$. The neighborhood U can be chosen as a regular neighborhood of the singular point X of the curve q. Neighborhoods of the real singular points are depicted in Figures 7.1–7.9.

(b.1) $A_1^*(D_4^*) \to A_{-1}(A_0)$; $q = y^2 + ax^2 + r = 0$, where $a < 0$ and r is the sum of monomials that correspond to the points lying above the line $y + x - 2 = 0$ in Newton's polygon of the quartic curve; $q_t = t + q$.

(b.2) $A_3^*(D_6^*) \to A_{-1}(A_0)$; $q = y^4 + axy^2 + bx^2 + r$, where $a^2 - 4b < 0$ and r is the sum of monomials that correspond to the points lying above the line $y + 2x - 4 = 0$ in Newton's polygon of the quartic curve q; $q_t = t + q$.

(b.3) $A_3^*(J_{10}^*) \to A_{-1}(A_0)$; $q = y^2 + ax^4 + r$, where $a < 0$ and r is the sum of monomials that correspond to the points lying above the line $2y + x - 4 = 0$ in Newton's polygon of the quartic curve q; $q_t = t + q$.

(b.4) $A_5^*(D_8^*) \to A_{-1}(A_0)$; $q = (x - ay^2)^2 + byx^2 + cy^3x + dx^3 + ex^2y^2 + fyx^3 + gx^4$, where $a < 0$, $ab + c = 0$ and $b^2 - 4(ad + e) < 0$ (see Lemma 4); $q_t = t + q$.

(b.5) $A_5^*(J_{12}^*) \to A_{-1}(A_0)$; $q = (y - ax^2)^2 + bxy^2 + cx^3y + dy^3 + ex^2y^2 + fxy^3 + gy^4$, where $a < 0$, $ab + c = 0$ and $b^2 - 4(ad + e) < 0$ (see Lemma 4); $q_t = t + q$.

(b.6) $A_7^*(D_{10}^*) \to A_{-1}(A_0)$; $q = [x - ay^2 + r]^2 + dx^4$, where $a < 0$, $r = bxy + cx^2$, $d > 0$; $q_t = t + q$.

(b.7) $A_7^*(J_{14}^*) \to A_{-1}(A_0)$; $q = [y - ax^2 + r]^2 + dy^4$, where $a < 0$, $r = bxy + cy^2$, $d > 0$; $q_t = t + q$.

(b.8) $X_9^{**}(N_{16}^{**}) \to A_{-1}(A_0)$; $q = (y^2 + x^2)(y^2 + a^2x^2)$; $q_t = t + q$.

(b.9) $A_0^3(A_5) \to A_0^1(A_1)$; $q = y + ax^3 + r$, where r is the sum of monomials that correspond to the points lying above the line $3y + x - 3 = 0$ in Newton's polygon of the quartic curve q; $q_t = tx + q$.

(b.10) $A_2(E_7) \to A_0^1(A_1)$; $q = y^2 - ax^3 + r$, where $a < 0$, r is the sum of monomials that correspond to the points lying above the line

$3y+2x-6=0$ in Newton's polygon of the quartic curve q; $q_t = -tx+q$.

(b.11) $D_4^*(X_9^*) \to A_0^1(A_1)$; $q = x(y^2+axy+bx^2)+r$, where $a^2-4b<0$, r is the sum of monomials that correspond to the points lying above the line $14y+10x-35=0$ in Newton's polygon of the quartic curve q; $q_t = t(x+ay^2)+q$.

(b.12) $E_6(X_{11}) \to A_0^1(A_1)$; $q = y^4 - x^3 + r$, where r is the sum of monomials that correspond to the points lying above the line $3y+4x-12=0$ in Newton's polygon of the quartic curve q; $q_t = t(ax+by)+q$ where $a \neq 0$.

(b.13) $A_0^4(A_7) \to A_0^2(A_3)$; $q = y - x^4 + r$, where r is the sum of monomials that correspond to the points lying above the line $4y+x-4=0$ in Newton's polygon of the quartic curve q; $q_t = -tx^2+q$.

(b.14) $A_2(D_5) \to A_0^2(A_3)$; $q = y^3 - x^2 + r$, where r is the sum of monomials that correspond to the points lying above the line $2y+3x-6=0$ in Newton's polygon of the quartic curve q; $q_t = ty+q$.

(b.15) $A_4(D_7) \to A_0^2(A_3)$; $q = (x-ay^2)^2+byx^2+cy^3x+dx^3+ex^2y^2+fyx^3+gx^4$, where $a<0$, $ab+c \neq 0$ (see Lemma 4); $q_t = -ty+q$.

(b.16) $A_4(J_{11}) \to A_0^2(A_3)$; $q = (y-ax^2)^2+bxy^2+cx^3y+dy^3+ex^2y^2+fxy^3+gy^4$, where $a<0$, $ab+c \neq 0$ (see Lemma 4); $q_t = t(y-hx^2)+q$ where $h<a$.

(b.17) $A_6(D_9) \to A_0^2(A_3)$; $q = (x-ay^2)^2+byx^2+cy^3x+dx^3+ex^2y^2+fyx^3+gx^4$, where $a<0$, $ab+c=0$, $b^2-4(ad+e)=0$, and $ab^3-2b^2(3a^2d+e)+a^2f \neq 0$ (see Lemma 4); $q_t = -ty+q$.

(b.18) $A_6(J_{13}) \to A_0^2(A_3)$; $q = (y-ax^2)^2+bxy^2+cx^3y+dy^3+ex^2y^2+fxy^3+gy^4$, where $a<0$, $ab+c=0$, $b^2-4(ad+e)=0$, and $ab^3-2b^2(3a^2d+e)+a^2f \neq 0$ (see Lemma 4); $q_t = t(y-hx^2)+q$ where $h<a$.

(b.19) $D_4^*(X_{11}'^*) \to A_0^2(A_3)$; $q = y(y^2+axy+bx^2)+cx^4+r$, where $a^2-4b<0$, $bc<0$, r is the sum of monomials that correspond to the points lying above the line $4y+3x-12=0$ in Newton's polygon of the quartic curve q; $q_t = t(y-x^2)+q$.

(b.20) $E_6(X_{13}) \to A_0^2(A_3)$; $q = y^3 - x^4 + r$, where r is the sum of monomials that correspond to the points lying above the line $4y+3x-12=0$ in Newton's polygon of the quartic curve q; $q_t = t(y-x^2)+q$.

(b.21) $A_1(D_4) \to 2A_0^1(2A_1)$; $q = y^2 - x^2 + r$, where r is the sum of monomials that correspond to the points lying above the line $y+x-2=0$ in Newton's polygon of the quartic curve q; $q_t = t+q$.

(b.22) $A_1(D_8) \to 2A_0^1(2A_1)$; $q = y^3 - axy + x^4 + r$, where $a<0$, r is the sum of monomials that correspond to the points lying above the two segments that connect the pairs of points (0,3), (1,1) and (1,1), (4,0) in Newton's polygon of the quartic curve q; $q_t = -t+q$.

(b.23) $A_3(D_6) \to 2A_0^1(2A_1)$; $q = (x+ay^2)(x+by^2)+r$, where r is the sum of monomials that correspond to the points lying above the line $y+2x-4=0$ in Newton's polygon of the quartic curve q; $q_t = -t+q$.

(b.24) $A_3(J_{10}) \to 2A_0^1(2A_1)$; $q = (y+ax^2)(y+bx^2)+r$, where $ab<0$, r is the sum of monomials that correspond to the points lying above the line $2y+x-4=0$ in Newton's polygon of the quartic curve q; $q_t = t+q$.

(b.25) $A_5(D_8) \to 2A_0^1(2A_1)$; $q = (x-ay^2)^2 + byx^2 + cy^3x + dx^3 + ex^2y^2 + fyx^3 + gx^4$, where $a<0$, $ab+c=0$ and $b^2 - 4(ad+e) > 0$ (see Lemma 4); $q_t = -t+q$.

(b.26) $A_7(D_{10}) \to 2A_0^1(2A_1)$; $q = [x-ay^2+r]^2 - dx^4$, where $a<0$, $r = bxy + cy^2$, $0<d$; $q_t = -t+q$.

(b.27) $D_5(X_{12}) \to 2A_0^1(2A_1)$; $q = y^4 - yx^2 + ax^4 + r$, where $a<0$ and r is the sum of monomials that correspond to the points lying above the two segments that connect the pairs of points $(0,4),(1,2)$ and $(1,2),(4,0)$ in Newton's polygon of the quartic curve q; $q_t = t+q$.

(b.28) $D_5(X'_{12}) \to 2A_0^1(2A_1)$; $q = ay^4 - xy^2 + x^4 + r$, where r is the sum of monomials that correspond to the points lying above the two segments that connect the pairs of points (0,4), (2,1) and (2,1), (4,0) in Newton's polygon of the quartic curve q; $q_t = -t+q$.

(b.29) $X_9^*(N_{16}^*) \to 2A_0^1(2A_1)$; $q = (y^2-x^2)(y^2+ayx+bx^2)$, where $a^2-4b<0$; $q_t = t+q$.

(b.30) $A_1(D_6) \to \{A_0^1, A_0^2\}(\{A_1, A_3\})$; $q = ay^3 - xy + ax^3 + r$, where $0<a$ and r is the sum of monomials that correspond to the points lying above the two segments that connect the pairs of points (0,3), (1,1) and (1,1), (3,0) in Newton's polygon of the quartic curve q; $q_t = -t(y-bx^2)+q$ where $b>a$.

(b.31) $D_5(X_{10}) \to \{2A_0^1, A_0^2\}(\{A_1, A_3\})$; $q = y^4 - yx^2 + ax^3 + r$, where $a<0$ and r is the sum of monomials that correspond to the points lying above the line $3y+4x-12=0$ in Newton's polygon of the quartic curve q; $q_t = -t^2(bty-2ax^2)+ty(y-bx)+q$ where $b>a$.

(b.32) $E_7(X'_{12}) \to \{2A_0^1, A_0^2\}(\{A_1, A_3\})$; $q = x(y^3-x^2)+r$, where r is the sum of monomials that correspond to the points lying above the line $2y+3x-9=0$ in Newton's polygon of the quartic curve q; $q_t = t^2y+t(y^4-x^2)+q$.

(b.33) $A_3(J_{10}) \to 2A_0^2(2A_3)$; $q=(y-ax^2)(y-bx^2)+r$, where $0<a$, $0<b$, $a\neq b$ and r is the sum of monomials that correspond to the points lying above the line $2y+x-4=0$ in Newton's polygon of the quartic curve q; $q_t = t^2-2t\sqrt{ab}\,x^2+q$.

(b.34) $A_5(J_{12}) \to 2A_0^2(2A_3)$; $q=(y-ax^2)^2+bxy^2+cx^3y+dy^3+ex^2y^2+fxy^3+gy^4$, where $0<a$, $ab+c=0$ and $b^2-4(ad+e)>0$ (see Lemma 4); $q_t=t^2-2atx^2+q$.

(b.35) $A_7(J_{14}) \to 2A_0^2(2A_3)$; $q=[y-ax^2+r]^2-dy^4$, where $0<a$, $r=bxy+cy^2$, $0<d$; $q_t=t^2-2atx^2+q$.

(b.36) $D_5(X_{12}) \to 2A_0^2(2A_3)$; $q=y^4-x^2y+ax^4+r$, where $0<a$ and r is the sum of monomials that correspond to the points lying above the two segments that connect the pairs of points (0,4), (2,1) and (2,1), (4,0) in Newton's polygon of the quartic curve q; $q_t=t^2-2t\sqrt{a}\,x^2+q$.

(b.37) $D_4(X_9) \to 3A_0^1(3A_1)$; $q=x(y^2-a^2x^2)+r$, where r is the sum of monomials that correspond to the points lying above the line $y+x-3=0$ in Newton's polygon of q; $q_t=tx+q$.

(b.38) $D_6(X'_{11}) \to 3A_0^1(3A_1)$; $q=xy^3-yx^2+ax^3+r$, where $a<0$, r is the sum of monomials that correspond to the points lying above the two segments that connect the pairs of points (1,3), (2,1) and (2,1), (3,0) in Newton's polygon of the quartic curve q; $q_t=t(y-bx)+q$ where $b>a$.

(b.39) $D_6(X'_{13}) \to \{2A_0^1, A_0^2\}(\{2A_1, A_3\})$; $q=xy^3-yx^2+x^4+r$, where r is the sum of monomials that correspond to the points lying above the two segments that connect the pairs of points (1,3), (2,1) and (2,1), (4,0) in Newton's polygon of the quartic curve q; $q_t = -t^3(y-cx^2)-t(y-ax^2)(x-by^2)+q$ where $c>a>1$, $b>1$.

(b.40) $D_4(X'_{11}) \to \{2A_0^1, A_0^2\}(\{2A_1, A_3\})$; $q=y^3-yx^2+cx^2+r=0$, where $0<c$ and r is the sum of monomials that correspond to the points lying above the line $4y+3x-12=0$ in Newton's polygon of the quartic curve q; $q_t=-t^3(y-dx^2)+t(y^2+axy+bx^2)+f$, where

$0 < c < -b/a < d$, $a < -1$.

(b.41) $X_9(N_{16}) \to \{4A_0^1\}(\{4A_1\})$; $q = (y^2 - x^2)(y^2 - ax^2) = 0$, where $0 < a < 1$; $q_t = t[(2a+1)y - 3ax^2 + 2at] + q$.

In the remaining cases (b.42)–(b.45), we choose the affine coordinate system such that the line at infinity is $y = 0$ and the complex point pairs $\{2A_0^{i,2}\}(\{2A_3^i\})$, $\{2A_1^i\}(\{2D_4^i\})$, $\{2A_2^i\}(\{2D_5^i\})$ and $\{2A_3^i\}(\{2D_6^i\})$ are placed at the points $(\sqrt{-1}, 0)$ and $(-\sqrt{-1}, 0)$.

(b.42) $\{2A_0^{i,2}\}(\{2A_3^i\}) \to \{4A_0^{i,1}\}(\{4A_1^i\})$; $q = (x^2+1)^2 + r(x)y + s$, where the greatest common factor of the polynomials $r(x)$ of degree 3 and x^2+1 is a constant, and s is the sum of monomials that correspond to the points lying above the line $y - 1 = 0$ in Newton's polygon of the quartic curve q; $q_t = -t^2 + q$. Note that instead of this deformation we can just perturb the line at infinity and remove any tangency at its complex points. From a geometrical point of view this is simpler.

(b.43) $\{2A_1^i\}(\{2D_4^i\}) \to \{4A_0^{i,1}\}(\{4A_1^i\})$; $q = (x^2+1)^2 + (x^2+1)(ax+b)y + r(x)y^2 + s$, where $a \neq 0$, the greatest common factor of the polynomial $r(x)$ of degree 2 and x^2+1 is a constant, and s is the sum of monomials that correspond to the points lying above the line $y - 2 = 0$ in Newton's polygon of the quartic curve q; $q_t = -t^2 + q$.

(b.44) $\{2A_2^i\}(\{2D_5^i\}) \to \{4A_0^{i,1}\}(\{4A_1^i\})$; $q = (x^2+1)^2 + a(x^2+1)y^2 + r$, where a is an integer, and r is the sum of monomials that correspond to the points lying above the line $y - 2 = 0$ in Newton's polygon of the quartic curve q; $q_t = -t^2 + q$.

(b.45) $\{2A_3^i\}(\{2D_6^i\}) \to \{4A_0^{i,1}\}(\{4A_1^i\})$; $q = (x^2+1)^2 + a(x^2+1)y^2 + by^4$, where a and b are integers; $q_t = -t^2 + q$. $\square$

Theorem 6. *Each isotopy class of real affine quartic curves without multiple components contains a singular-simple curve.*

Proof. Let f_0 be an affine quartic curve without multiple components. We will prove that there exists a singular-simple curve f_τ such that $\mathbf{R}f_\tau$ is isotopic to $\mathbf{R}f_0$. The curve f_0 can have singular points both in $\mathbf{R}^2 = \mathbf{R}P^2 \setminus L_\infty$ and on the line at infinity L_∞. This suggests that we consider the projective reducible quintic curve zF_0 where $z = 0$ is the equation of the line at infinity L_∞ and $F_0(x, y, z)$ is a homogeneous polynomial of degree 4 such that $F_0(x, y, 1) = f_0(x, y)$. It is clear that deformations of individual singular points of the quintic curve zF_0 lying in $\mathbf{R}^2 = \mathbf{R}P^2 \setminus L_\infty$ are provided by Theorem 5(a) and deformations of individual singular points of zF_0 lying on the line at infinity L_∞ are provided by Theorem 5(b). According to Theorem 3 there exists a family of curves $\{zF_t\}_{t\in(0,\tau]}$ that realizes the required isotopy between $\mathbf{R}(zF_0)$ and $\mathbf{R}(zF_\tau)$ if the curve zF_0 satisfies the inequality of Theorem 3.

Theorem 3 treats real and complex singular point on an equal footing and implies that only the sum of the Milnor numbers of the singular points counts. For example, we may ignore the distinction between A_1, A_1^*, A_1^i; it suffices to treat the case A_1. This enables us to avoid enumerating many cases of real singular points with real and complex branches and complex conjugate singular points of quintic curves zF_0. The sets of singular points of such reducible quintic curves can be divided into two parts: singular points of the quartic curve that do not lie on the line L_∞ and singular points that appear as the result of the intersection of the quartic curve and the line L_∞. For the purposes of enumerating the sets of singular points, the cases of irreducible and reducible quartic curves will be presented separately. Let $\mathbf{X}$ be the set of singular points of the curve F_0 and $\mathbf{Y}$ be the set of singular points of the curve zF_0.

0. In this item we enumerate all sets of singular points that a quartic curve F_0 can have.

0.1. In [**31**], [**12**], [**13**], it was proved that irreducible projective quartic curves can have only the following sets of singular points.

0.1.0. 0-point set: $\mathbf{X} = \varnothing$ corresponds to nonsingular quartic curves.

0.1.1. 1-point sets: $\mathbf{X} = \{A_1\}, \{A_2\}, \{A_3\}, \{A_4\}, \{A_5\}, \{A_6\}, \{D_4\}, \{D_5\}, \{E_6\}$.

0.1.2. 2-point sets: $\mathbf{X} = \{2A_1\}, \{A_1, A_2\}, \{A_1, A_3\}, \{A_1, A_4\}, \{2A_2\}, \{A_2, A_3\}, \{A_2, A_4\}$.

0.1.3. 3-point sets: $\mathbf{X} = \{3A_1\}, \{2A_1, A_2\}, \{A_1, 2A_2\}, \{3A_2\}$.

0.2. Based on the singular-isotopy classification of plane reducible projective quartic curves [**29**], it is easy to enumerate the sets of singular points of reducible projective quartic curves.

0.2.1. 1-point sets: $\mathbf{X} = \{A_5\}, \{A_7\}, \{E_7\}, \{D_6\}, \{X_9\}$.

0.2.2. 2-point sets: $\mathbf{X} = \{A_1, A_5\}, \{A_1, D_4\}, \{A_1, D_5\}, \{A_1, D_6\}, \{A_2, A_5\}, \{2A_3\}$.

0.2.3. 3-point sets: $\mathbf{X} = \{3A_1\}, \{2A_1, A_3\}, \{2A_1, D_4\}$, $\{A_1, A_2, A_3\}$, $\{A_1, 2A_3\}$.

0.2.4. 4-point sets: $\mathbf{X} = \{4A_1\}, \{3A_1, A_2\}$, $\{3A_1, A_3\}, \{3A_1, D_4\}$.

0.2.5. 5-point sets: $\mathbf{X} = \{5A_1\}$.

0.2.6. 6-point sets: $\mathbf{X} = \{6A_1\}$.

There are two possibilities for the intersection of a quartic curve F_0 and the line at infinity L_∞: either the quartic curve does have singular point(s) lying on the line or does not. In the second case the quintic curve zF_0 has singular points that appear as a result of the intersection of the line at infinity and the quartic curve, and the singular points of such an intersection can be either $\{4A_1\}$ or $\{2A_1, A_3\}$ or $\{A_1, A_5\}$, or $\{2A_3\}$ or $\{A_7\}$. The existence of the sets of singular points of the quintic curves enumerated below follows from Gudkov's classification of forms of irreducible quartic curves [**10**]–[**14**], [**26**] and from the classification of forms of reducible quartic curves [**29**].

1. The set $\mathbf{X}$ of singular points of the quartic curve F_0 consists of one singular point.

1.1. First we consider quintic curves zF_0 such that the quartic component is irreducible and the singular point of F_0 does not lie on the line L_∞. We take $\mathbf{X}$ from item 0.1.1. The quintic curve zF_0 has the following sets of singular points $\mathbf{Y}$.

1.1.1. If $\mathbf{X} = \{A_1\}$, then $\mathbf{Y} = \{5A_1\}$, $\{3A_1, A_3\}$, $\{2A_1, A_5\}$, $\{A_1, 2A_3\}$, $\{A_1, A_7\}$.

1.1.2. If $\mathbf{X} = \{A_2\}$, then $\mathbf{Y} = \{4A_1, A_2\}$, $\{2A_1, A_3, A_2\}$, $\{A_1, A_2, A_5\}$, $\{2A_3, A_2\}$, $\{A_2, A_7\}$.

1.1.3. If $\mathbf{X} = \{A_3\}$, then $\mathbf{Y} = \{4A_1, A_3\}$, $\{2A_1, 2A_3\}$, $\{A_1, A_3, A_5\}$, $\{3A_3\}$, $\{A_3, A_7\}$.

1.1.4. If $\mathbf{X} = \{A_4\}$, then $\mathbf{Y} = \{4A_1, A_4\}$, $\{2A_1, A_3, A_4\}$, $\{A_1, A_4, A_5\}$, $\{2A_3, A_4\}$, $\{A_4, A_7\}$.

1.1.5. If $\mathbf{X} = \{A_5\}$, then $\mathbf{Y} = \{4A_1, A_5\}$, $\{2A_1, A_3, A_5\}$, $\{A_1, 2A_5\}$, $\{2A_3, A_5\}$, $\{A_5, A_7\}$.

1.1.6. If $\mathbf{X} = \{A_6\}$, then $\mathbf{Y} = \{4A_1, A_6\}$, $\{2A_1, A_3, A_6\}$, $\{A_1, A_5, A_6\}$.

1.1.7. If $\mathbf{X} = \{D_4\}$, then $\mathbf{Y} = \{4A_1, D_4\}$, $\{2A_1, A_3, D_4\}$, $\{A_1, A_5, D_4\}$, $\{2A_3, D_4\}$, $\{A_7, D_4\}$.

1.1.8. If $\mathbf{X} = \{D_5\}$, then $\mathbf{Y} = \{4A_1, D_5\}$, $\{2A_1, A_3, D_5\}$, $\{A_1, A_5, D_5\}$, $\{2A_3, D_5\}$, $\{A_7, D_5\}$.

1.1.9. If $\mathbf{X} = \{E_6\}$, then $\mathbf{Y} = \{4A_1, E_6\}$, $\{2A_1, A_3, E_6\}$, $\{A_1, A_5, E_6\}$, $\{2A_3, E_6\}$, $\{A_7, E_6\}$.

1.2. We consider quintic curves such that the quartic component is irreducible and a singular point of F_0 lies on the line L_∞. The quintic curve zF_0 has the following sets of singular points $\mathbf{Y}$.

1.2.1. If $\mathbf{X} = \{A_1\}$, then $\mathbf{Y} = \{2A_1, D_4\}$, $\{A_1, D_6\}$, $\{D_8\}$, $\{A_3, D_4\}$.

1.2.2. If $\mathbf{X} = \{A_2\}$, then $\mathbf{Y} = \{2A_1, D_5\}$, $\{A_1, E_7\}$, $\{A_3, D_5\}$.

1.2.3. If $\mathbf{X} = \{A_3\}$, then $\mathbf{Y} = \{2A_1, D_6\}$, $\{J_{10}\}$, $\{A_3, D_6\}$.

1.2.4. If $\mathbf{X} = \{A_4\}$, then $\mathbf{Y} = \{2A_1, D_7\}$, $\{J_{11}\}$, $\{A_3, D_7\}$.

1.2.5. If $\mathbf{X} = \{A_5\}$, then $\mathbf{Y} = \{2A_1, D_8\}$, $\{J_{12}\}$, $\{A_3, D_8\}$.

1.2.6. If $\mathbf{X} = \{A_6\}$, then $\mathbf{Y} = \{2A_1, D_9\}$, $\{J_{13}\}$, $\{A_3, D_9\}$.

1.2.7. If $\mathbf{X} = \{D_4\}$, then $\mathbf{Y} = \{A_1, X_9\}$, $\{X'_{11}\}$.

1.2.8. If $\mathbf{X} = \{D_5\}$, then $\mathbf{Y} = \{A_1, X_{10}\}$, $\{X_{12}\}$, $\{X'_{12}\}$.

1.2.9. If $\mathbf{X} = \{E_6\}$, then $\mathbf{Y} = \{A_1, X_{11}\}$, $\{X_{13}\}$.

1.3. We consider quintic curves such that the quartic component is reducible and the singular point of F_0 does not lie on the line L_∞. We take $\mathbf{X}$ from item 0.2.1. The quint curve zF_0 has the following sets of singular points $\mathbf{Y}$.

1.3.1. If $\mathbf{X} = \{A_5\}$, then $\mathbf{Y} = \{4A_1, A_5\}$, $\{2A_1, A_3, A_5\}$, $\{A_1, 2A_5\}$.

1.3.2. If $\mathbf{X} = \{A_7\}$, then $\mathbf{Y} = \{4A_1, A_7\}$, $\{2A_1, A_3, A_7\}$.

1.3.3. If $\mathbf{X} = \{E_7\}$, then $\mathbf{Y} = \{4A_1, E_7\}$, $\{2A_1, A_3, E_7\}$, $\{A_1, A_5, E_7\}$.

1.3.4. If $\mathbf{X} = \{D_6\}$, then $\mathbf{Y} = \{4A_1, D_6\}$, $\{2A_1, A_3, D_6\}$, $\{A_1, A_5, D_6\}$.

1.3.5. If $\mathbf{X} = \{X_9\}$, then $\mathbf{Y} = \{4A_1, X_9\}$.

1.4. We consider quintic curves such that the quartic component is reducible and the singular point of F_0 lies on the line L_∞. The quintic curve zF_0 has the following sets of singular points $\mathbf{Y}$.

1.4.1. If $\mathbf{X} = \{A_5\}$, then $\mathbf{Y} = \{2A_1, D_8\}$, $\{A_3, D_8\}$.

1.4.2. If $\mathbf{X} = \{A_7\}$, then $\mathbf{Y} = \{2A_1, D_10\}$, $\{J_{14}\}$.

1.4.3. If $\mathbf{X} = \{E_7\}$, then $\mathbf{Y} = \{A_1, X'_{12}\}$.

1.4.4. If $\mathbf{X} = \{D_6\}$, then $\mathbf{Y} = \{A_1, X'_{11}\}$, $\{X_{13}\}$.

1.4.5. If $\mathbf{X} = \{X_9\}$, then $\mathbf{Y} = \{N_{16}\}$.

2. The set $\mathbf{X}$ of singular points of the quartic curve consists of two singular points.

2.1. We consider quintic curves such that the quartic component is irreducible and the singular points of F_0 do not lie on the line L_∞. We take $\mathbf{X}$ from item 0.1.2. The quintic curve zF_0 has the following sets of singular points $\mathbf{Y}$.

2.1.1. If $\mathbf{X} = \{2A_1\}$, then $\mathbf{Y} = \{6A_1\}$, $\{4A_1, A_3\}$, $\{3A_1, A_5\}$, $\{2A_1, 2A_3\}$, $\{2A_1, A_7\}$.

2.1.2. If $\mathbf{X} = \{A_1, A_2\}$, then $\mathbf{Y} = \{5A_1, A_2\}$, $\{3A_1, A_2, A_3\}$, $\{2A_1, A_2, A_5\}$, $\{A_1, A_2, 2A_3\}$, $\{A_1, A_2, A_7\}$.

2.1.3. If $\mathbf{X} = \{A_1, A_3\}$, then $\mathbf{Y} = \{5A_1, A_3\}$, $\{3A_1, 2A_3\}$, $\{2A_1, A_3, A_5\}$, $\{A_1, 3A_3\}$, $\{A_1, A_3, A_7\}$.

2.1.4. If $\mathbf{X} = \{A_1, A_4\}$, then $\mathbf{Y} = \{5A_1, A_4\}$, $\{3A_1, A_3, A_4\}$, $\{2A_1, A_4, A_5\}$, $\{A_1, 2A_3, A_4\}$, $\{A_1, A_4, A_7\}$.

2.1.5. If $\mathbf{X} = \{2A_2\}$, then $\mathbf{Y} = \{4A_1, 2A_2\}$, $\{2A_1, A_3, 2A_2\}$, $\{A_1, 2A_2, A_5\}$, $\{2A_2, 2A_3\}$, $\{2A_2, A_7\}$.

2.1.6. If $\mathbf{X} = \{A_2, A_3\}$, then $\mathbf{Y} = \{4A_1, A_2, A_3\}$, $\{2A_1, A_2, 2A_3\}$, $\{A_1, A_2, A_3, A_5\}$.

2.1.7. If $\mathbf{X} = \{A_2, A_4\}$, then $\mathbf{Y} = \{4A_1, A_2, A_4\}$, $\{2A_1, A_2, A_3, A_4\}$, $\{A_1, A_2, A_4, A_5\}$.

2.2. We consider quintic curves such that the quartic component is irreducible and one of the singular points of F_0 lies on the line L_∞. The quintic curve zF_0 has the following sets of singular points $\mathbf{Y}$.

2.2.1. If $\mathbf{X} = \{2A_1\}$, then $\mathbf{Y} = \{3A_1, D_4\}$, $\{2A_1, D_6\}$, $\{A_1, D_8\}$, $\{A_1, A_3, D_4\}$.

2.2.2. If $\mathbf{X} = \{A_1, A_2\}$, then $\mathbf{Y} = \{2A_1, A_2, D_4\}$, $\{A_1, A_2, D_6\}$, $\{A_2, D_8\}$, $\{A_2, A_3, D_4\}$, $\{3A_1, D_5\}$, $\{2A_1, E_7\}$, $\{A_1, A_3, D_5\}$.

2.2.3. If $\mathbf{X} = \{A_1, A_3\}$, then $\mathbf{Y} = \{2A_1, A_3, D_4\}$, $\{A_1, A_3, D_6\}$, $\{A_3, D_8\}$, $\{2A_3, D_4\}$, $\{3A_1, D_6\}$, $\{A_1, J_{10}\}$, $\{A_1, A_3, D_6\}$.

2.2.4. If $\mathbf{X} = \{A_1, A_4\}$, then $\mathbf{Y} = \{2A_1, A_4, D_4\}$, $\{A_1, A_4, D_6\}$, $\{A_4, D_8\}$, $\{A_3, A_4, D_4\}$, $\{3A_1, D_7\}$, $\{A_1, J_{11}\}$, $\{A_1, A_3, D_7\}$.

2.2.5. If $\mathbf{X} = \{2A_2\}$, then $\mathbf{Y} = \{2A_1, A_2, D_5\}$, $\{A_1, A_2, E_7\}$, $\{A_3, A_2, D_5\}$.

2.2.6. If $\mathbf{X} = \{A_2, A_3\}$, then $\mathbf{Y} = \{2A_1, A_3, D_5\}$, $\{A_1, A_3, E_7\}$, $\{2A_3, D_5\}$, $\{2A_1, A_2, D_6\}$, $\{A_2, J_{10}\}$, $\{A_2, A_3, D_6\}$.

2.2.7. If $\mathbf{X} = \{A_2, A_4\}$, then $\mathbf{Y} = \{2A_1, A_4, D_5\}$, $\{A_1, A_4, E_7\}$, $\{A_3, A_4, D_5\}$, $\{2A_1, A_2, D_7\}$, $\{A_2, J_{11}\}$.

2.3. We consider quintic curves such that the quartic component is irreducible and both singular points of F_0 lie on the line L_∞. The quintic curve zF_0 has the following sets of singular points $\mathbf{Y}$.

2.3.1. If $\mathbf{X} = \{2A_1\}$, then $\mathbf{Y} = \{2D_4\}$.

2.3.2. If $\mathbf{X} = \{A_1, A_2\}$, then $\mathbf{Y} = \{D_4, D_5\}$.

2.3.3. If $\mathbf{X} = \{A_1, A_3\}$, then $\mathbf{Y} = \{D_4, D_6\}$.

2.3.4. If $\mathbf{X} = \{A_1, A_4\}$, then $\mathbf{Y} = \{D_4, D_7\}$.

2.3.5. If $\mathbf{X} = \{2A_2\}$, then $\mathbf{Y} = \{2D_5\}$.

2.3.6. If $\mathbf{X} = \{A_2, A_3\}$, then $\mathbf{Y} = \{D_5, D_6\}$.

2.3.7. If $\mathbf{X} = \{A_2, A_4\}$, then $\mathbf{Y} = \{D_5, D_7\}$.

2.4. We consider quintic curves such that the quartic component is reducible and the singular points of F_0 do not lie on the line L_∞. We take $\mathbf{X}$ from item 0.2.2. The quintic curve zF_0 has the following sets of singular points $\mathbf{Y}$.

2.4.1. If $\mathbf{X} = \{A_1, A_5\}$, then $\mathbf{Y} = \{5A_1, A_5\}$, $\{3A_1, A_3, A_5\}$, $\{A_1, 2A_3, A_5\}$.

2.4.2. If $\mathbf{X} = \{A_1, D_4\}$, then $\mathbf{Y} = \{5A_1, D_4\}$, $\{3A_1, A_3, D_4\}$, $\{2A_1, A_5, D_4\}$.

2.4.3. If $\mathbf{X} = \{A_1, D_5\}$, then $\mathbf{Y} = \{5A_1, D_5\}$, $\{3A_1, A_3, D_5\}$, $\{2A_1, A_5, D_5\}$.

2.4.4. If $\mathbf{X} = \{A_1, D_6\}$, then $\mathbf{Y} = \{5A_1, D_6\}$, $\{3A_1, A_3, D_6\}$.

2.4.5. If $\mathbf{X} = \{A_2, A_5\}$, then $\mathbf{Y} = \{4A_1, A_2, A_5\}$, $\{2A_1, A_2, A_3, A_5\}$.

2.4.6. If $\mathbf{X} = \{2A_3\}$, then $\mathbf{Y} = \{4A_1, 2A_3\}$, $\{2A_1, 3A_3\}$.

2.5. We consider quintic curves such that the quartic component is reducible and one of the singular points of F_0 lies on the line L_∞. The quintic curve zF_0 has the following sets of singular points $\mathbf{Y}$.

2.5.1. If $\mathbf{X} = \{A_1, A_5\}$, then $\mathbf{Y} = \{2A_1, A_5, D_4\}$, $\{A_1, A_5, D_6\}$, $\{3A_1, D_8\}$, $\{A_1, A_3, D_8\}$, $\{A_1, J_{12}\}$.

2.5.2. If $\mathbf{X} = \{A_1, D_4\}$, then $\mathbf{Y} = \{2A_1, 2D_4\}$, $\{A_3, 2D_4\}$, $\{A_1, D_4, D_6\}$, $\{D_4, D_8\}$, $\{2A_1, X_9\}$, $\{A_1, X'_{11}\}$.

2.5.3. If $\mathbf{X} = \{A_1, D_5\}$, then $\mathbf{Y} = \{2A_1, D_4, D_5\}$, $\{A_1, D_5, D_6\}$, $\{D_5, D_8\}$, $\{2A_1, X_{10}\}$, $\{A_1, X'_{12}\}$.

2.5.4. If $\mathbf{X} = \{A_1, D_6\}$, then $\mathbf{Y} = \{2A_1, D_4, D_6\}$, $\{A_1, 2D_6\}$, $\{2A_1, X'_{11}\}$.

2.5.5. If $\mathbf{X} = \{A_2, A_5\}$, then $\mathbf{Y} = \{2A_1, A_5, D_5\}$, $\{A_1, A_5, E_7\}$, $\{2A_1, A_2, D_6\}$.

2.5.6. If $\mathbf{X} = \{2A_3\}$, then $\mathbf{Y} = \{2A_1, A_3, D_6\}$, $\{A_3, J_{10}\}$.

2.6. We consider quintic curves such that the quartic component is reducible and both singular points of F_0 lie on the line L_∞. The quintic curve zF_0 has the following set of singular points $\mathbf{Y}$.

2.6.1. If $\mathbf{X} = \{A_1, A_5\}$, then $\mathbf{Y} = \{D_4, D_6\}$.

2.6.2. If $\mathbf{X} = \{A_1, D_4\}$, then the quintic curve has the line L_∞ as a double component.

2.6.3. If $\mathbf{X} = \{A_1, D_5\}$, then the quintic curve has the line L_∞ as a double component.

2.6.4. If $\mathbf{X} = \{A_1, D_6\}$, then the quintic curve has the line L_∞ as a double component.

2.6.5. If $\mathbf{X} = \{A_2, A_5\}$, then $\mathbf{Y} = \{D_5, D_8\}$.

2.6.6. If $\mathbf{X} = \{2A_3\}$, then $\mathbf{Y} = \{2D_6\}$.

3. The set $\mathbf{X}$ of singular points of the quartic curve consists of three singular points.

3.1. We consider quintic curves such that the quartic component is irreducible and the singular points of F_0 do not lie on the line L_∞. We take $\mathbf{X}$ from item 0.1.3. The quintic curve zF_0 has the following sets of singular points $\mathbf{Y}$.

3.1.1. If $\mathbf{X} = \{3A_1\}$, then $\mathbf{Y} = \{7A_1\}$, $\{5A_1, A_3\}$, $\{4A_1, A_5\}$, $\{3A_1, 2A_3\}$, $\{3A_1, A_7\}$.

3.1.2. If $\mathbf{X} = \{2A_1, A_2\}$, then $\mathbf{Y} = \{6A_1, A_2\}$, $\{4A_1, A_2, A_3\}$, $\{3A_1, A_2, A_5\}$, $\{2A_1, A_2, 2A_3\}$, $\{2A_1, A_2, A_7\}$.

3.1.3. If $\mathbf{X} = \{A_1, 2A_2\}$, then $\mathbf{Y} = \{5A_1, 2A_2\}$, $\{3A_1, 2A_2, A_3\}$, $\{2A_1, 2A_2, A_5\}$, $\{A_1, 2A_2, 2A_3\}$, $\{A_1, 2A_2, A_7\}$.

3.1.4. If $\mathbf{X} = \{3A_2\}$, then $\mathbf{Y} = \{4A_1, 3A_2\}$, $\{2A_1, A_3, 3A_2\}$.

3.2. We consider quintic curves such that the quartic component is irreducible and one of the singular points of F_0 lies on the line L_∞. The quintic curve zF_0 has the following sets of singular points $\mathbf{Y}$.

3.2.1. If $\mathbf{X} = \{3A_1\}$, then $\mathbf{Y} = \{4A_1, D_4\}$, $\{3A_1, D_6\}$, $\{2A_1, D_8\}$, $\{2A_1, A_3, D_4\}$.

3.2.2. If $\mathbf{X} = \{2A_1, A_2\}$, then $\mathbf{Y} = \{3A_1, A_2, D_4\}$, $\{2A_1, A_2, D_6\}$, $\{A_1, A_2, D_8\}$, $\{A_1, A_2, A_3, D_4\}$, $\{4A_1, D_5\}$, $\{3A_1, E_7\}$, $\{2A_1, A_3, D_5\}$.

3.2.3. If $\mathbf{X} = \{A_1, 2A_2\}$, then $\mathbf{Y} = \{2A_1, 2A_2, D_4\}$, $\{A_1, 2A_2, D_6\}$, $\{3A_1, A_2, D_5\}$, $\{2A_1, A_2, E_7\}$, $\{A_1, A_2, A_3, D_5\}$.

3.2.4. If $\mathbf{X} = \{3A_2\}$, then $\mathbf{Y} = \{2A_1, 2A_2, D_5\}$, $\{A_1, 2A_2, E_7\}$.

3.3. We consider quintic curves such that the quartic component is irreducible and two singular points of F_0 lie on the line L_∞. The quintic curve zF_0 has the following sets of singular points $\mathbf{Y}$.

3.3.1. If $\mathbf{X} = \{3A_1\}$, then $\mathbf{Y} = \{A_1, 2D_4\}$.

3.3.2. If $\mathbf{X} = \{2A_1, A_2\}$, then $\mathbf{Y} = \{A_2, 2D_4\}$, $\{A_1, D_4, D_5\}$.

3.3.3. If $\mathbf{X} = \{A_1, 2A_2\}$, then $\mathbf{Y} = \{A_2, D_4, D_5\}$, $\{A_1, 2D_5\}$.

3.3.4. If $\mathbf{X} = \{3A_2\}$, then $\mathbf{Y} = \{A_2, 2D_5\}$.

3.4. We consider quintic curves such that the quartic component is reducible and the singular points of F_0 do not lie on the line L_∞. We take $\mathbf{X}$ from item 0.2.3. The quintic curve zF_0 has the following sets of singular points $\mathbf{Y}$.

3.4.1. If $\mathbf{X} = \{3A_1\}$, then $\mathbf{Y} = \{7A_1\}$, $\{5A_1, A_3\}$, $\{4A_1, A_5\}$.

3.4.2. If $\mathbf{X} = \{2A_1, A_3\}$, then $\mathbf{Y} = \{6A_1, A_3\}$, $\{4A_1, 2A_3\}$, $\{2A_1, 3A_3\}$.

3.4.3. If $\mathbf{X} = \{2A_1, D_4\}$, then $\mathbf{Y} = \{6A_1, D_4\}$, $\{4A_1, A_3, D_4\}$.

3.4.4. If $\mathbf{X} = \{A_1, A_2, A_3\}$, then $\mathbf{Y} = \{5A_1, A_2, A_3\}$, $\{3A_1, A_2, 2A_3\}$, $\{2A_1, A_2, A_3, A_5\}$.

3.4.5. If $\mathbf{X} = \{A_1, 2A_3\}$, then $\mathbf{Y} = \{5A_1, 2A_3\}$, $\{3A_1, 3A_3\}$.

3.5. We consider quintic curves such that the quartic component is reducible and one of the singular points of F_0 lies on the line L_∞. The quintic curve zF_0 has the following sets of singular points $\mathbf{Y}$.

3.5.1. If $\mathbf{X} = \{3A_1\}$, then $\mathbf{Y} = \{4A_1, D_4\}$, $\{2A_1, A_3, D_4\}$, $\{3A_1, D_6\}$, $\{2A_1, D_8\}$.

3.5.2. If $\mathbf{X} = \{2A_1, A_3\}$, then $\mathbf{Y} = \{3A_1, A_3, D_4\}$, $\{A_1, 2A_3, D_4\}$, $\{2A_1, A_3, D_6\}$, $\{A_1, A_3, D_8\}$, $\{4A_1, D_6\}$.

3.5.3. If $\mathbf{X} = \{2A_1, D_4\}$, then $\mathbf{Y} = \{3A_1, 2D_4\}$, $\{2A_1, D_4, D_6\}$, $\{3A_1, X_9\}$, $\{2A_1, X'_{11}\}$.

3.5.4. If $\mathbf{X} = \{A_1, A_2, A_3\}$, then $\mathbf{Y} = \{2A_1, A_2, A_3, D_4\}$, $\{3A_1, A_3, D_5\}$, $\{2A_1, A_3, E_7\}$, $\{3A_1, A_2, D_6\}$, $\{A_1, A_2, A_3, D_6\}$.

3.5.5. If $\mathbf{X} = \{A_1, 2A_3\}$, then $\mathbf{Y} = \{2A_1, 2A_3, D_4\}$, $\{3A_1, A_3, D_6\}$.

3.6. We consider quintic curves such that the quartic component is reducible and two singular points of F_0 lie on the line L_∞. The quintic curve zF_0 has the following sets of singular points $\mathbf{Y}$.

3.6.1. If $\mathbf{X} = \{3A_1\}$, then the quintic curve has the line L_∞ as a double component;

3.6.2. If $\mathbf{X} = \{2A_1, A_3\}$, then $\mathbf{Y} = \{A_3, 2D_4\}$, $\{A_1, D_4, D_6\}$.

3.6.3. If $\mathbf{X} = \{2A_1, D_4\}$, then $\mathbf{Y} = \{3D_4\}$.

3.6.4. If $\mathbf{X} = \{A_1, A_2, A_3\}$, then $\mathbf{Y} = \{A_3, D_4, D_5\}$, $\{A_1, D_5, D_6\}$.

3.6.5. If $\mathbf{X} = \{A_1, 2A_3\}$, then $\mathbf{Y} = \{A_1, 2D_6\}$.

4. The set $\mathbf{X}$ of singular points of the quartic curve consists of four singular points. In this case the quartic curve F_0 decomposes either into two conics intersecting transversally, or into a line and cuspidal cubic curve intersecting transversally, or into two lines and an irreducible conic with only one of the lines tangent to the conic, or four lines, three of which are concurrent. We take $\mathbf{X}$ from item 0.2.4.

4.1. We consider quintic curves such that the quartic component is reducible and the singular points of F_0 do not lie on the line L_∞. The quintic curve zF_0 has the following sets of singular points $\mathbf{Y}$.

4.1.1. If $\mathbf{X} = \{4A_1\}$, then $\mathbf{Y} = \{8A_1\}$, $\{6A_1, A_3\}$, $\{4A_1, 2A_3\}$, $\{5A_1, A_3, A_5\}$.

4.1.2. If $\mathbf{X} = \{3A_1, A_2\}$, then $\mathbf{Y} = \{7A_1, A_2\}$, $\{5A_1, A_2, A_3\}$, $\{4A_1, A_2, A_5\}$.

4.1.3. If $\mathbf{X} = \{3A_1, A_3\}$, then $\mathbf{Y} = \{7A_1, A_3\}$, $\{5A_1, 2A_3\}$.

4.1.4. If $\mathbf{X} = \{3A_1, D_4\}$, then $\mathbf{Y} = \{7A_1, D_4\}$.

4.2. We consider quintic curves such that the quartic component is reducible and one of the singular points of F_0 lies on the line L_∞. The quintic curve zF_0 has the following sets of singular points $\mathbf{Y}$.

4.2.1. If $\mathbf{X} = \{4A_1\}$, then $\mathbf{Y} = \{5A_1, D_4\}$, $\{4A_1, D_6\}$, $\{3A_1, D_8\}$.

4.2.2. If $\mathbf{X} = \{3A_1, A_2\}$, then $\mathbf{Y} = \{4A_1, A_2, D_4\}$, $\{2A_1, A_2, A_3, D_4\}$, $\{3A_1, A_2, D_6\}$, $\{2A_1, A_2, D_8\}$, $\{5A_1, D_5\}$, $\{4A_1, E_7\}$.

4.2.3. If $\mathbf{X} = \{3A_1, A_3\}$, then $\mathbf{Y} = \{5A_1, A_3, D_4\}$, $\{3A_1, A_3, D_6\}$, $\{5A_1, D_6\}$.

4.2.4. If $\mathbf{X} = \{3A_1, D_4\}$, then $\mathbf{Y} = \{4A_1, 2D_4\}$, $\{4A_1, X_9\}$.

4.3. We consider quintic curves such that the quartic component is reducible and two singular points of F_0 lie on the line L_∞. The quintic curve zF_0 has the following sets of singular points $\mathbf{Y}$.

4.3.1. If $\mathbf{X} = \{4A_1\}$, then $\mathbf{Y} = \{2A_1, 2D_4\}$.

4.3.2. If $\mathbf{X} = \{3A_1, A_2\}$, then $\mathbf{Y} = \{A_1, A_2, 2D_4\}$, $\{2A_1, D_4, D_5\}$.

4.3.3. If $\mathbf{X} = \{3A_1, A_3\}$, then $\mathbf{Y} = \{2A_1, D_4, D_6\}$.

4.3.4. If $\mathbf{X} = \{3A_1, D_4\}$, then the quintic curve has the line L_∞ as a double component.

5. The quartic curve has five singular points. We take $\mathbf{X} = \{5A_1\}$ from item 0.2.5. In this case the quartic curve decomposes into two lines and an irreducible conic intersecting transversally.

5.1. If F_0 is reducible and the singular points of F_0 do not lie on the line L_∞, then $\mathbf{Y} = \{9A_1\}$.

5.2. If F_0 is reducible and one of the singular points of F_0 lies on the line L_∞, then $\mathbf{Y} = \{6A_1, D_4\}$, $\{5A_1, D_6\}$.

5.3. If F_0 is reducible and two singular points of F_0 lie on the line L_∞, then $\mathbf{Y} = \{3A_1, 2D_4\}$.

6. The quartic curve has six singular points. We take $\mathbf{X} = \{6A_1\}$ from item 0.2.6. In this case the curve decomposes into four lines.

6.1. If F_0 is reducible and the singular points of F_0 do not lie on the line L_∞, then $\mathbf{Y} = \{10A_1\}$.

6.2. If F_0 is reducible and one of the singular points of F_0 lies on the line L_∞, then $\mathbf{Y} = \{7A_1, D_4\}$.

6.3. If F_0 is reducible and two singular points of F_0 lie on the line L_∞, then the quintic curve has the line L_∞ as a double component.

The quintic curves from items 2.6.2–2.6.4, 3.6.1, 4.3.4 and 6.3 have multiple components and so are considered separately. The quintic curves from the following items with the indicated sets of singular points already have quartic components that represent singular-simple curves:

item 1.1.1 with $\{5A_1\}$,

item 1.1.7 with $\{4A_1, D_4\}$,

item 1.3.5 with $\{4A_1, X_9\}$,

item 2.1.1 with $\{6A_1\}$,

item 2.4.2 with $\{5A_1, D_4\}$,

item 3.1.1 with $\{7A_1\}$,

item 3.4.1 with $\{7A_1\}$,

item 4.1.1 with $\{8A_1\}$,

item 4.1.4 with $\{7A_1, D_4\}$.

For the remainder of the quintic curves here one can see that only one projective quintic curve, from item 1.4.5, does not satisfy the inequality from Theorem 3. This exceptional quintic curve zF_0 consists of five concurrent lines (one of these lines is L_∞), and so has one singular point N_{16}. For this quintic curve we apply the deformation of item (b.39) of Theorem 5. The isotopy class of the corresponding affine quartic curve $f_0(x,y) = F_0(x,y,1)$ contains all affine quartic curves of the family f_t with $t \in (0,\tau]$ provided by Theorem 5. For each other projective quintic curve zF_0 (different from the quintic curve of item 1.4.5) we apply Theorem 3 and the prescribed deformations from Theorem 5. The isotopy class of the corresponding affine quartic curve $f_0(x,y)$ contains all affine quartic curves of the family f_t with $t \in (0,\tau]$ provided by Theorem 3. □

Corollary 6.1. *Each isotopy class of real projective quartic curves without multiple components contains a singular-simple curve.*

Corollary 6.2. *The isotopy classification of all real projective quartic curves is equivalent to the isotopy classification of real projective quartic curves whose singular points, if any, are of types* A_1, A_1^*, D_4 *or* X_9.

Corollary 6.3. *The topological classification of pairs of all real affine quartic curves is equivalent to the topological classification of pairs of real affine quartic curves whose singular points, if any, are of types* A_1, A_1^*, D_4 *or* X_9*, and with no singular points on the line at infinity.*

Corollary 6.4. *The isotopy classification of all real affine quartic curves is equivalent to the isotopy classification of real affine quartic curves whose singular points, if any, are of types* A_1, A_1^*, D_4 *or* X_9 *and with no singular points on the line at infinity.*

3. Isotopy classification of affine quartic curves. To get an isotopy classification of some given set of algebraic curves, the following procedure is traditional. In the first step one enumerates the admissible isotopy classes of topological curves that satisfy the theorems of Bezout and Harnack and perhaps another restriction. A priori there are two kinds of admissible classes of topological curves: classes that contain algebraic curves from the given set and classes that don't. Thus, the second step in solving the problem is to prove the existence of the curves (usually just to construct these algebraic curves) for the first case and to prove that such curves do not exist for the second one.

In our first step we enumerate the affine admissible isotopy classes by considering all possible intersections of a topological line, representing the line at infinity, with representatives of all projective isotopy classes of quartic curves from Figure 3. By applying Bezout's theorem to the intersection of a projective algebraic curve of degree d and the line at infinity, it follows that the number N of noncompact connected components of the affine algebraic curve can be less than or equal to d. By applying Harnack's theorem to affine curves, the total number of real connected components of an affine curve of degree d can be less than or equal to $g+1$ if $N=[1-(-1)^d]/2$, and can be less than or equal to $g+N$ if $[1-(-1)^d]/2 < N < d$, where g is the genus of the corresponding projective curve. To calculate the genus we use the formula $g=(d-1)(d-2)/2-g(z_i)$ from [**30**], where $g(z_i)$ is the genus of the singular point z_i of the projective curve, and the sum runs over all singular points of the curve. To calculate the genus of a singular point we use the formula $g(z)=\{\kappa(z)-\sum[g(P_j-1)]\}/2$ from [**30**], where $\kappa(z)$ is the class of the singular point z, $g(P_j)$ is the order of the branch P_j [**46**], and the sum runs over all branches with center at singular point z. In the case of reducible quartic curves one can apply Harnack's inequality and the formula for the calculation of the genus for each irreducible algebraic component. Thus, we obtain that a compact affine quartic curve has no more than 4 connected components, and a noncompact affine quartic curve has no more than 7 connected components.

During this enumeration we divide the set of singular-simple quartic curves into seven divisions with respect to their behavior at infinity (the 1st–6th divisions) and the presence of multiple algebraic components (the 7th division). This is justified by Theorem 6.

The 1*st division.* The affine quartic curves of this division intersect the line at infinity at 4 real points. These quartic curves have 8 real branches going to infinity along 4 asymptotic lines in opposite directions.

The 2*nd division.* The affine quartic curves of this division have 2 real points of intersection and 1 point of tangency with the line at infinity. These quartic curves have 6 real branches going to infinity, 4 of which go along 2 asymptotic lines in opposite directions and two of which are parabolic branches.

The 3*rd division.* The affine quartic curves of this division have 2 real points of tangency with the line at infinity. These quartic curves have 4 real branches going to infinity; all of them are parabolic branches.

The 4*th division.* The affine quartic curves of this division have 2 real and 2 complex points of intersection with the line at infinity. These quartic curves have 4 real branches going to infinity along 2 asymptotic lines in opposite directions.

The 5*th division.* The affine quartic curves of this division have 1 point of tangency and 2 complex points of intersection with the line at infinity. These quartic curves have 2 real branches going to infinity; both of them are parabolic branches.

The 6*th division.* The affine quartic curves of this division have 4 complex points of intersection with the line at infinity. These quartic curves do not have real branches going to infinity.

The 7*th division.* The affine quartic curves of this division have multiple algebraic components.

One property of affine quartic curves that we place in the foundation of the enumeration is the decomposition into irreducible algebraic components. There are 11 kinds of such decompositions of projective quartic curves, where l, c, cb and q denote an affine line, irreducible conic, cubic and quartic curve, respectively:

(1) four lines $l_1l_2l_3l_4$,

(2) two lines and a conic l_1l_2c,

(3) a line and a cubic curve lcb,

(4) two conics c_1c_2,

(5) a quartic curve q,

(6) a 4-fold line l^4,

(7) 1- and 3-fold lines $l_1l_2^3$,

(8) two 1-fold lines and a 2-fold line $l_1l_2l_3^2$,

(9) two 2-fold lines $l_1^2l_2^2$,

(10) a 2-fold line and an irreducible conic l^2c, and

(11) a 2-fold conic curve c^2.

The complete isotopy classification of affine quartic curves contains all curves: both reducible and irreducible, and singular and nonsingular curves. In this general situation it is more appropriate to put the topological classification of triples $(\mathbf{R}^2, \mathbf{R}f, \operatorname{Sing} f)$ in the foundation of the isotopy classification of affine quartic curves in the same manner as we have done for projective quartic curves. Recall that $\mathbf{R}f$ is the set of real points of an affine curve f, and we denote the set of its real singular points as $\mathrm{Sing}\, f$. We take the isotopy classification of the triples into consideration because the addition of the third member $\mathrm{Sing}\,(f)$ simplifies the enumeration of isotopy classes. The set of (admissible) isotopy classes with the same kind of irreducible algebraic components and with the same set of real singular points is called a *grade* of quartic curves. Each grade can be described by a pair $(\mathbf{R}f, \mathrm{Sing}\,(f))$. One can see that there are 32 grades of affine quartic curves without multiple algebraic components and 12 grades of curves with multiple components. The real affine quartic curves have the same grades as projective quartic curves. These grades were enumerated in the Introduction and we apply them to affine quartic curves. Note that the 1st–6th divisions contain the 1st–32nd grades and the 7th division contains the 33rd–44th grades.

There are two isotopy classes in the set of all homeomorphisms $\mathbf{R}^2 \to \mathbf{R}^2$. One of the classes contains the identity map $\mathrm{id}\,(x, y) = (x, y)$, and the other one contains the reflection $\mathrm{ref}\,(x, y) = (-x, y)$. Obviously, homeomorphisms from the first isotopy class $[id]$ preserve the orientation of the plane and from the second class [ref] reverse the orientation. If $h \in [\mathrm{ref}]$ and C is a curve in $\mathbf{R}^2$, then either the curves C and $h(C)$ are isotopic or are not. In the first case the isotopy class $[C]$ is called *reflectable* and otherwise *nonreflectable*. We denote isotopy classes of affine quartic curves by means of numbers and letters. We denote the admissible classes that contain quartic curves by numbers (for example $1, \dots, 5, 6^{\pm}, \dots$) and denote the admissible classes that do not contain quartic curves by capital letters (for example, $A, B^{\pm}, \dots$). A number n means a reflectable class. A number $n^{\pm}$ means two nonreflectable isotopy classes n^+ and n^-. We draw a picture only for the class n^+. One can obtain the picture for class n^- by means of the reflection of n^+.

There are two classifications of interest. One is the isotopy classification and the other is the topological classification of pairs. The latter refers to the equivalence relation where C_1 and C_2 are equivalent provided the pairs $(\mathbf{R}^2, C_1)$ and $(\mathbf{R}^2, C_2)$ are topologically equivalent. The topological classification of pairs is distinct from the isotopy classification, but the latter is easily obtained from the former. It turns out that 131 out of the 516 homeomorphism types of pairs (described in Theorem 7) are such that their mirror images (i.e., the result of reflection with respect to any line in the plane) are not isotopic to the initial pair. This phenomenon can be regarded as new in the sense that for affine curves of degree lower than four there is no distinction between the topological classification of pairs and the isotopy classification. If a denotes the number of reflectable classes and $2b$ the number of non-reflectable classes, then the topological classification of pairs contains $a+b$ classes, and the isotopy classification contains $a+2b$ classes. Both of these classifications are presented in the following theorem.

Theorem 7. (a) *There are* 667 *distinct admissible* (397 *reflectable and* 270 *nonreflectable*) *isotopy classes for affine quartic curves.*

(b) *There are* 647 *distinct isotopy classes* (385 *reflectable and* 262 *nonreflectable*) *of affine quartic curves.*

(c) *There are* 516 *distinct equivalence classes of topological pairs* $(\mathbf{R}^2, C)$, *where* C *is an affine quartic curve.*

Proof. According to Theorem 6 of Section 2, each isotopy class of affine quartic curves contains a singular-simple curve. This allows us to classify just the singular-simple quartic curves and simplifies the enumeration of admissible isotopy classes substantially. It means that it suffices to consider affine quartic curves that have only singular-simple points of the types A_1, A_1^*, D_4 and X_9 in $\mathbf{R}^2$ and that have transversal intersection with the line at infinity in real or complex points or simple tangency with the line at infinity in real points.

We remark here that the existence of the irreducible affine quartic curves can be verified by referring to Gudkov's projective classification of forms [**16**], [**17**], [**18**]–[**24**], [**10**]–[**14**], [**26**], which takes into account the arrangement of the flex points. We also remark that the existence of the reducible affine quartic curves consisting of a line and an irreducible cubic curve can be verified by referring to one of Newton's classifications of cubic curves [**38**]. However, a much shorter proof is obtained by independently constructing the quartic representatives from the affine admissible classes (denoted by numbers), and this is what we do. In the notation $m^{\pm}(n^{\pm})$ under the pictures, the first integer $m^{\pm}$ refers to the number of the admissible isotopy class and the second integer $n^{\pm}$, in parentheses, means that the quartic curve from the admissible class $m^{\pm}$ can be constructed from a quartic curve from admissible isotopy class $n^{\pm}$ either by means of Theorem 3 and a small deformation of a singular point preserving, if necessary, other singular points, or by contracting an oval of a quartic curve to a point, or by moving the line at infinity. Below we describe such constructions, which allow us to generate, in tree-like fashion, the quartic curves that realize all of the affine isotopy classes.

All restrictions are treated in Lemma 8.

The 1st division. The quartic curve has 4 real points of intersection with the line at infinity and thus it has 4 asymptotes.

To obtain an admissible class for affine quartic curves of the n-th affine grade, first we take a projective topological curve C from an isotopy class of the n-th projective grade depicted in Figure 3, second, we consider all isotopy types of the different unions $C \cup L_\infty$ that satisfy Bezout's theorem and, third, we declare L_∞ as the line at infinity and obtain an affine topological curve $C \setminus L_\infty$ as a representative for the admissible class of affine quartic curves. The admissible classes of the 1st division are shown in Figure 8. Recall that the open disk is a model of the affine plane and the line at infinity, just chosen, is the boundary circle, and that $1p, 2p, \dots, 66p$ refer to the isotopy classes of the projective quartic curves depicted in Figure 3.

The 1st–21st grades represent admissible isotopy classes for reducible quartic curves.

The 1st–6th grades represent admissible isotopy classes for reducible quartic curves that consist of 4 line components.

The 1st grade $(\mathbf{R}(l_1l_2l_3l_4), \{X_9\})$ contains 1 reflectable admissible class: 1, obtained from the class $1p$ by an obvious selection of the line at infinity. The existence of such a quartic curve is obvious.

The 2nd grade $(\mathbf{R}(l_1l_2l_3l_4), \{D_4, 3A_1\})$ contains 1 reflectable admissible class: 2, obtained from the class $2p$ by an obvious selection of the line at infinity. If we move one of the lines of a curve of the 1st grade off the singular point X_9, we obtain the required quartic curve.

The 3rd grade $(\mathbf{R}(l_1l_2l_3l_4), \{6A_1\})$ contains 1 reflectable admissible class: 3, obtained from the class $3p$ by an obvious selection of the line at infinity. The quartic curve is easily constructed by moving one of the lines of a quartic curve from the 2nd grade off the singular point D_4.

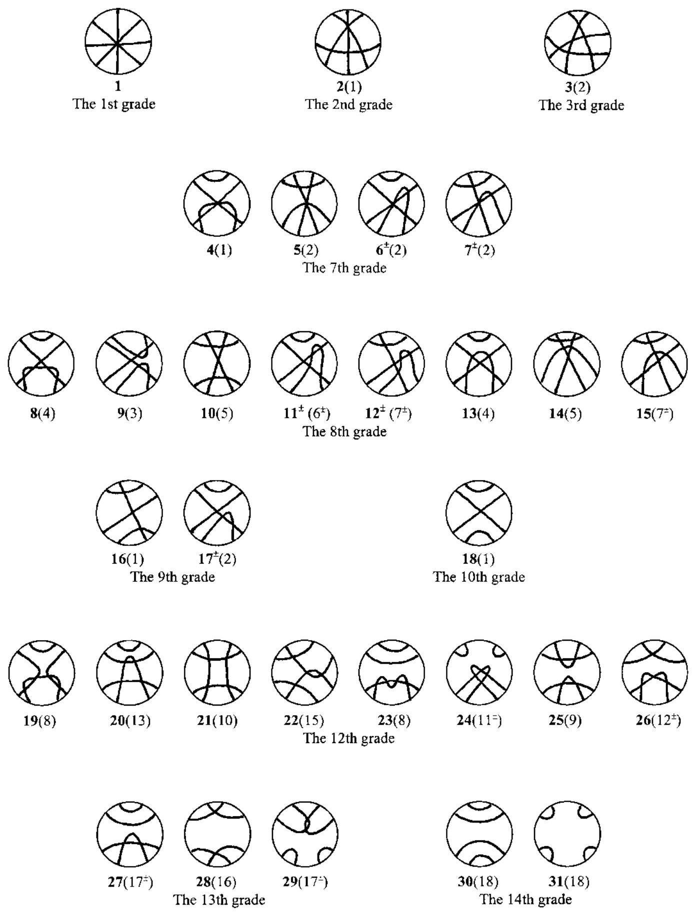


FIGURE 8 (beginning). The 1st division.

The 4th grade $(\mathbf{R}(l_1l_2l_3l_4), \{A_1, A_1^*\})$, the 5th grade $(\mathbf{R}(l_1l_2l_3l_4), \{2A_1^*\})$, and the 6th grade $(\mathbf{R}(l_1l_2l_3l_4), \{4A_1^*\})$ don't contain admissible classes.

The 7th–10th grades represent admissible isotopy classes for reducible quartic curves that consist of 2 line and 1 conic components.

The 7th grade $(\mathbf{R}(l_1l_2c), \{D_4, 2A_1\})$ contains 6 new admissible classes (2 reflectable and 4 nonreflectables classes): 4, 5, $6^{\pm}$, and $7^{\pm}$. *Enu-*

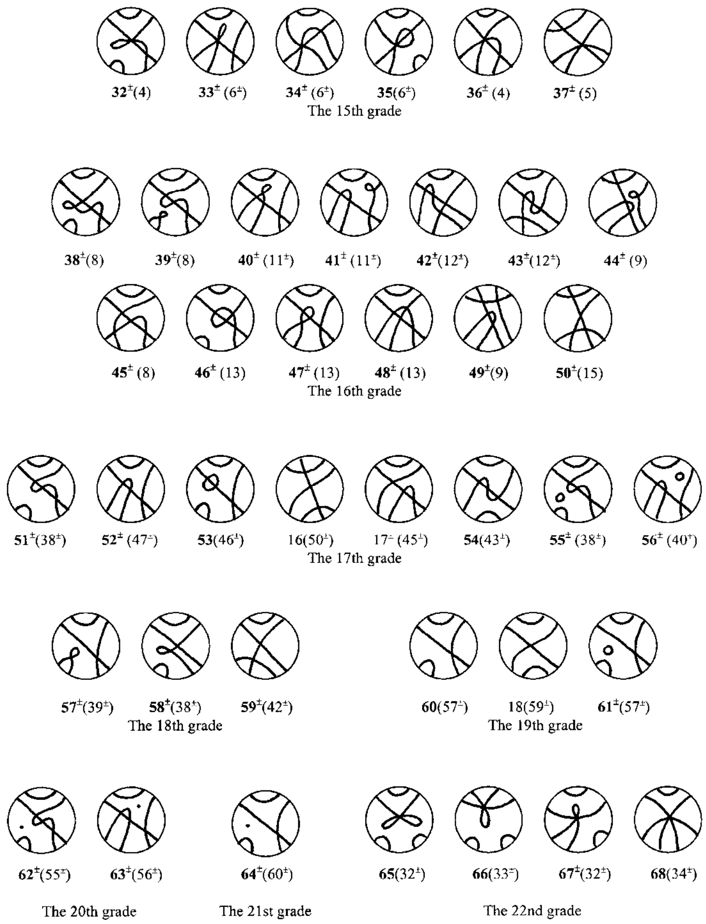


FIGURE 8 (continued). The 1st division.

meration: These classes can be obtained from the class $7p$ by means of appropriate selections of the line at infinity. *Construction*: The arrangements of a hyperbola and two lines shown in the figure for the 7th grade are obvious. However, the quartic curves from these affine admissible classes can be constructed by means of a small deformation of one of the singular points of quartic curves from the 1st and 2nd grades.

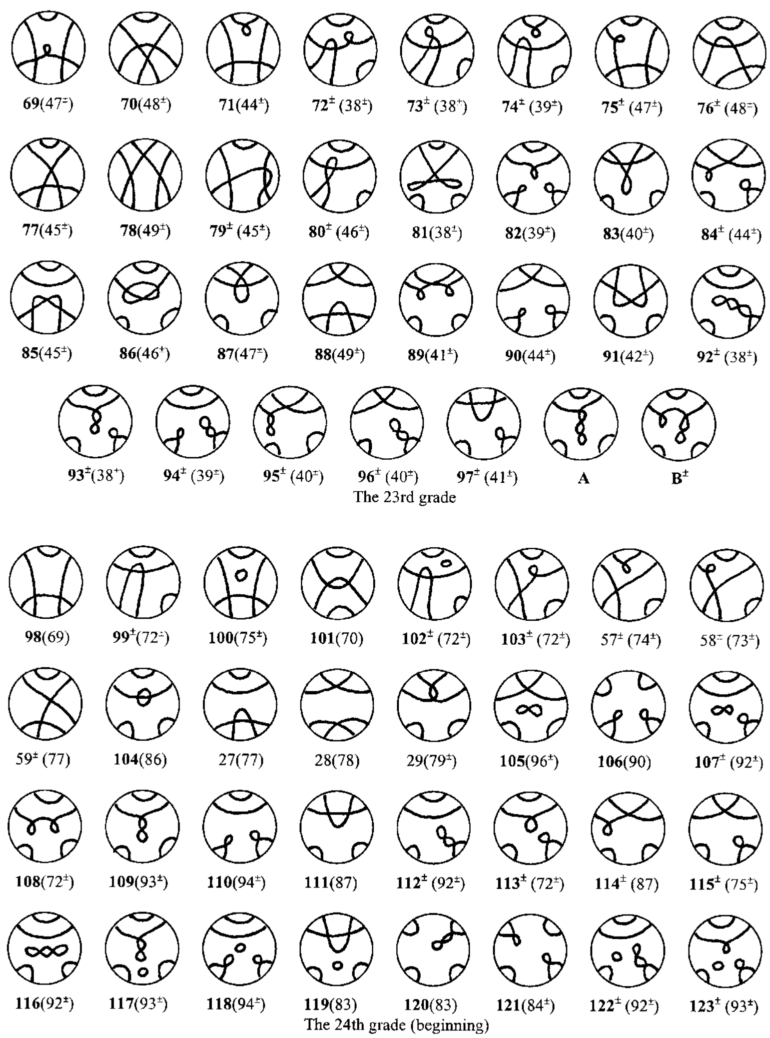


FIGURE 8 (continued). The 1st division.

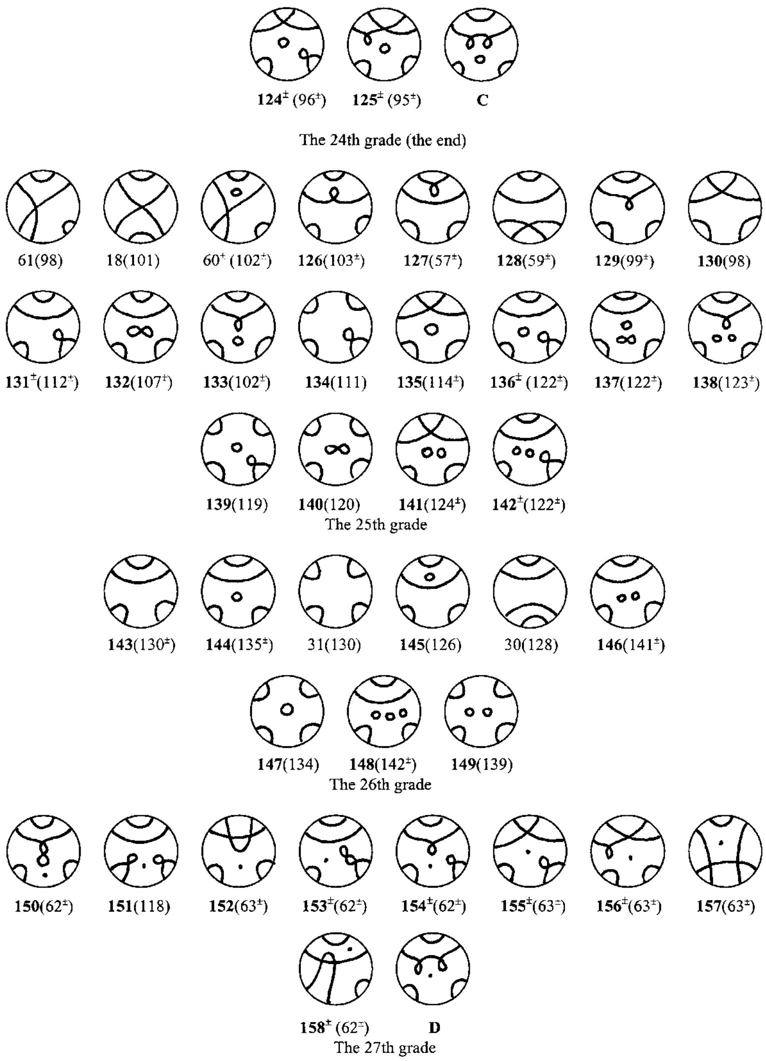


FIGURE 8 (continued). The 1st division.

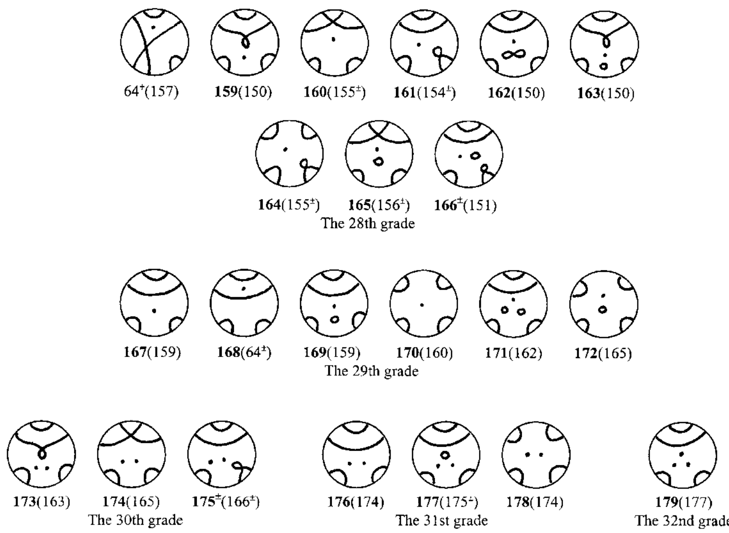


FIGURE 8 (conclusion). The 1st division.

The 8th grade $(\mathbf{R}(l_1l_2c), \{5A_1\})$ contains 10 admissible classes (6 reflectable and 4 nonreflectable): 8, 9, 10, $11^{\pm}$, $12^{\pm}$, 13, 14 and 15. *Enumeration*: The classes 8–$12^{\pm}$ can be obtained from the class $8p$, and the classes 13–15 from the class $9p$ by means of appropriate selections of the line at infinity. *Construction*: The arrangements of a hyperbola and two lines show in the figure for this grade are obvious. However, the quartic curves from these affine admissible classes can be constructed by means of small deformations of one of the double points of a quartic curve from the 3rd grade or the 3-fold point of quartic curves from the 7th grade.

The 9th grade $(\mathbf{R}(l_1l_2c), \{3A_1\})$ contains 3 admissible classes (1 reflectable and 2 nonreflectable): 16 and $17^{\pm}$. *Enumeration*: These classes can be obtained from the class $10p$ by means of appropriate selections of the line at infinity. *Construction*: The arrangements of a hyperbola and two lines shown in the figure for this grade are obvious. However, the quartic curves from these affine admissible classes are constructed by means of small deformations of the 4-fold point of a quartic curve from the 1st grade or the 3-fold point of a quartic curve from the 2nd grade.

The 10th grade $(\mathbf{R}(l_1l_2c), \{A_1\})$ contains one reflectable admissible class: 18. *Enumeration*: This class can be obtained from the class $12p$ by an obvious selection of the line at infinity. *Construction*: The arrangement of a hyperbola and two lines shown in the figure for this grade is obvious. However, a quartic curve from this affine admissible class can be constructed by means of a small deformation of the 4-fold point of a quartic curve from the 1st grade.

The 11th grade $(\mathbf{R}(l_1 l_2 c), \{A_1^*\})$ does not contain admissible classes.

The 12th–14th grades represent admissible isotopy classes for reducible quartic curves that consist of 2 conic components.

The 12th grade $(\mathbf{R}(c_1 c_2), \{4A_1\})$ contains 8 reflectable admissible classes: 19–26. *Enumeration*: The classes 19–22 can be obtained from the class $16p$, and the classes 23–26 from the class $17p$ by means of appropriate selections of the line at infinity. *Construction*: The arrangements of two hyperbolas shown in the figure for this grade are obvious. However, the quartic curves from these affine admissible classes can be constructed by means of small deformations of the point of intersection of the two line components of quartic curves from the 8th grade.

The 13th grade $(\mathbf{R}(c_1 c_2), \{2A_1\})$ contains 3 reflectable admissible classes: 27, 28 and 29. *Enumeration*: These classes can be obtained from the class $18p$ by means of appropriate selections of the line at infinity. *Construction*: The arrangements of two hyperbolas shown in the figure for this grade are obvious. However, the quartic curves from these affine admissible classes can be constructed by means of small deformations of the point of intersection of the two line components of quartic curves from the 9th grade.

The 14th grade $(\mathbf{R}(c_1 c_2), \varnothing)$ contains 2 reflectable admissible classes: 30 and 31. *Enumeration*: These classes can be obtained from the classes $21p$ and $22p$, respectively, by obvious selections of the line at infinity. *Construction*: The arrangements of two hyperbolas shown in the figure for this grade are obvious. However, the quartic curves from these affine admissible classes can be constructed by means of small deformations of the point of intersection of the two line components of quartic curves from the 10th grade.

The 15th–21st grades represent admissible isotopy classes for reducible quartic curves that consist of 1 line and 1 cubic component.

The 15th grade $(\mathbf{R}(lcb), \{D_4, A_1\})$ contains 11 admissible classes (1 reflectable and 10 nonreflectable): $32^{\pm}$, $33^{\pm}$, $34^{\pm}$, 35, $36^{\pm}$, $37^{\pm}$. *Enumeration*: The classes $32^{\pm}-34^{\pm}$ can be obtained from the class $23p$, and the classes 35, $36^{\pm}$, $37^{\pm}$ from the class $24p$ by means of appropriate selections of the line at infinity. *Construction*: The quartic curves from these affine admissible classes can be constructed by means of small deformations of one of the double points of quartic curves from the 7th grade.

The 16th grade $(\mathbf{R}(lcb), \{4A_1\})$ contains 26 nonreflectable admissible classes: $38^{\pm} - 50^{\pm}$. *Enumeration*: The classes $38^{\pm} - 44^{\pm}$ can be obtained from the class $24p$, and the classes $45^{\pm} - 50^{\pm}$ from $25p$ by means of appropriate selections of the line at infinity. *Construction*: The quartic curves from these affine admissible classes can be constructed by means of small deformations of one of the double points of quartic curves from the 8th grade. To preserve a line component, one must choose a double point that differs from the point of intersection of the two line components.

The 17th grade $(\mathbf{R}(lcb), \{3A_1\})$ contains 3 old (1 reflectable and 2 nonreflectable) admissible classes: 16, $17^{\pm}$, and 10 new (2 reflectable and 8 nonreflectable) admissible classes: $51^{\pm}$, $52^{\pm}$, 53, 54, $55^{\pm}$, $56^{\pm}$. *Enumeration*: The classes $51^{\pm}$ and $52^{\pm}$ can be obtained from the class $27p$, the classes 53, 16 and $17^{\pm}$ from class $10p$, and the classes 54, $55^{\pm}$ and $56^{\pm}$ from the class $28p$ by means of appropriate selections of the line at infinity. *Construction*: The quartic curves from these affine admissible classes can be constructed by means of small deformations of one of the double points of quartic curves from the 16th grade. To preserve a line component, one must choose a double point that differs from the points of intersection of the line and cubic curve.

The 18th grade $(\mathbf{R}(lcb), \{2A_1\})$ contains 6 new nonreflectable admissible classes: $57^{\pm}$, $58^{\pm}$, $59^{\pm}$. *Enumeration*: These classes can be obtained from the projective class $29p$ by means of appropriate selections of the line at infinity. *Construction*: The quartic curves from these affine admissible classes can be constructed by means of small deformations of one of the double points of quartic curves from the 9th grade. To preserve a line component, one must choose a double point that differs from the point of intersection of the two line components.

The 19th grade $(\mathbf{R}(lcb), \{A_1\})$ contains 1 old reflectable admissible class: 18, and 3 new (1 reflectable and 2 nonreflectable) admissible classes: $60^{\pm}$, 61. *Enumeration*: The classes $60^{\pm}$ can be obtained from the class $11p$ and the classes 18 and 61 from the class $12p$ by means of appropriate selections of the line at infinity. *Construction*: The quartic curves from these affine admissible classes can be constructed by means of small deformations of one of the double points of quartic curves from the 18th grade. To preserve a line component, one must choose a double point that differs from the point of intersection of the line and cubic curve.

The 20th grade $(\mathbf{R}(lcb), \{3A_1, A_1^*\})$ contains 4 new nonreflectable admissible classes: $62^{\pm}$ and $63^{\pm}$. *Enumeration*: These classes can be obtained from the projective class $30p$ by means of appropriate selections of the line at infinity. *Construction*: The quartic curves from these affine admissible classes can be constructed by means of contracting the oval of the cubic component of these quartic curves to a point. If $lcb = 0$ is an equation of a quartic curve from classes $55^{\pm}$ or $56^{\pm}$ such that the signs of l and cb are different inside the oval, then the segment of curves with equation $l(cb + tl) = 0$, $t \in [0, t_0]$, provides the required contraction for some $t_0 > 0$ and connects quartic curves of classes $55^{\pm}$ and $56^{\pm}$ with $62^{\pm}$ and $63^{\pm}$, respectively.

The 21st grade $(\mathbf{R}(lcb), \{A_1, A_1^*\})$ contains 2 nonreflectable admissible classes: $64^{\pm}$. *Enumeration*: They can be obtained from the class $4p$ by an obvious selection of the line at infinity. *Construction*: The quartic curves from these affine admissible classes can be constructed by means of contracting the oval of the cubic component of these quartic curves to a point in the same manner as in the 20th grade. Another way to construct these curves is parallel motion of a line component of quartic curves from the 20th grade.

The 22–32nd grades represent admissible classes of irreducible affine quartic curves.

The 22nd grade $(\mathbf{R}q, \{D_4\})$ contains 5 new (3 reflectable and 2 nonreflectable) admissible classes: 65, 66, $67^{\pm}$, and 68. *Enumeration*: The classes 65 and 66 can be obtained from the class $31p$, and the classes $67^{\pm}$ and 68 from the class $32p$ by means of appropriate selections of the line at infinity. *Construction*: The quartic curves from these affine admissible classes can be constructed by means of suitable deformations of the double point of intersection of a line and cubic curve of the 15th grade.

The 23rd grade $(\mathbf{R}q, \{3A_1\})$ contains 48 new (16 reflectable and 32 nonreflectable) admissible classes: 69–71, $72^{\pm} - 76^{\pm}$, 77, 78, $79^{\pm}$, $80^{\pm}$, 81–83, $84^{\pm}$, 85–91, $92^{\pm}$–$97^{\pm}$, A and $B^{\pm}$. *Enumeration*: The classes 69–$76^{\pm}$ can be obtained from the class $33p$, the classes 77–$80^{\pm}$ from the class $34p$, the classes 81–$84^{\pm}$ from the class $35p$, the classes 85–88 from the class $36p$, and the classes 89–$97^{\pm}$, A and $B^{\pm}$ from the class $37p$ by means of appropriate selections of the line at infinity. *Construction*: The classes 69–$97^{\pm}$ contain quartic curves. The quartic curves from these affine admissible classes can be constructed by means of suitable deformations of one of the double points of intersection of a line and cubic curve of the 16th grade. *Restriction*: The classes A and $B^{\pm}$ don't contain quartic curves (see Lemma 8).

The 24th grade $(\mathbf{R}q, \{2A_1\})$ contains 9 old (3 reflectable and 6 nonreflectable) admissible classes: 27–29, $57^{\pm}$–$59^{\pm}$ and 40 new (18 reflectable and 22 nonreflectable) admissible classes 98, $98^{\pm}$, 100, 101, $102^{\pm}$, $103^{\pm}$, 104–106, $107^{\pm}$, 108–111, $112^{\pm}$–$115^{\pm}$, 116–121, $122^{\pm}$–$125^{\pm}$ and C. *Enumeration*: The classes 98 and $99^{\pm}$ can be obtained from the class $38p$, the classes 100–$102^{\pm}$ from the class $39p$, the classes $103^{\pm}$, $57^{\pm}$–$59^{\pm}$ from the class $29p$, the classes 104, 27–29 from the class $18p$, the classes 105–$107^{\pm}$ from the class $40p$, the classes 108–115 from the class $41p$, the classes 116–$125^{\pm}$ and C from the class $42p$ by means of appropriate selection of the line at infinity. *Construction*: The classes 27–29, $57^{\pm}$–$59^{\pm}$, 98–$125^{\pm}$ contain quartic curves. The quartic curves from these affine admissible classes can be constructed by means of suitable deformations of one of the double points of a quartic curve of the 23rd grade. The double point in each case can be found in figures of the 23rd grade. *Restriction*: The class C doesn't contain quartic curves (see Lemma 8).

The 25th grade $(\mathbf{R}q, \{A_1\})$ contains 4 old (2 reflectable and 2 nonreflectable) admissible classes: 61, 18, $60^{\pm}$ and 20 new (14 reflectable and 6 nonreflectable) admissible classes, 126–130, $131^{\pm}$, 132–135, $136^{\pm}$, 137–141, and $142^{\pm}$. *Enumeration*: The class 61 can be obtained from the class $11p$, the classes 18 and $60^{\pm}$ from the class $12p$, the classes 126–128 from the class $43p$, the classes 129–$131^{\pm}$ from the class $44p$, the classes 132–$136^{\pm}$ from the class $45p$, the classes 137–$142^{\pm}$ from the class $46p$ by means of appropriate selection of the line at infinity. *Construction*: The quartic curves from these affine admissible classes can be constructed by means of suitable deformations of one of the double points of a quartic curve of 24th grade. The double point in each case can be found in figures of the 24th grade.

The 26th grade $(\mathbf{R}q, \varnothing)$ contains 2 old reflectable admissible classes: 30, 31, and 7 new reflectable, 143–149. All of them contain quartic curves. *Enumeration*: The class 143 can be obtained from the class $20p$, the classes 144 and 31 from the class $21p$, the classes 145 and 30 from the class $22p$, the classes 146 and 147 from the class $47p$, the classes 148 and 149 from the class $48p$ by means of appropriate selection of the line at infinity. *Construction*: The quartic curves from these affine admissible classes can be constructed by means of suitable deformations of the double point of a quartic curve of the 25th grade.

The 27th grade $(\mathbf{R}q, \{2A_1, A_1^*\})$ contains 15 new (5 reflectable and 10 nonreflectable) admissible classes: 150–152, $153^{\pm}$–$156^{\pm}$, 157, $158^{\pm}$, and D. *Enumeration*: The classes 150–$156^{\pm}$ and D can be obtained from the class $49p$, and the classes 157 and $158^{\pm}$ from the class $50p$ by means of appropriate selection of the line at infinity. *Construction*: The classes 150–$156^{\pm}$, 157 and $158^{\pm}$ contain quartic curves. Quartic curves from these affine admissible classes, with the exception of 151, can be constructed by means of suitable deformations of one of the nonisolated double points of a quartic curve of the 20th grade. The double point in each case can be found in figures of the 20th grade. A curve of the class 151 can be constructed by means of contracting to a point the oval of a quartic curve of class 118 of the 24th grade. Let $q = 0$ be an equation of the quartic curve and $l = 0$ an equation of the line connecting the double points of the quartic curve. Let the signs of q and l be different inside the quartic oval; then the segment of curves with equation $q + tl^3 = 0$, $t \in [0, t_0]$, provides the required contraction for some $t_0 > 0$. *Restriction*: The class D doesn't contain quartic curves (see Lemma 8).

The 28th grade $(\mathbf{R}q\{A_1, A_1^*\})$ contains 2 old nonreflectable admissible classes: $64^{\pm}$, and 9 new (7 reflectable and 2 nonreflectable) 159–165, $166^{\pm}$. *Enumeration*: The class $64^{\pm}$ can be obtained from the class $4p$, the classes 159–161 can be obtained from the class $51p$, and the classes 162–$166^{\pm}$ can be obtained from the class $52p$ by means of appropriate selection of the line at infinity. *Construction*: The quartic curves from these affine admissible classes can be constructed by means of suitable deformations of one of the double points of quartic curves of the 27th grade. The double point in each case can be found in figures of the 27th grade.

The 29th grade $(\mathbf{R}q, \{A_1^*\})$ contains 6 new reflectable admissible classes: 167–172. *Enumeration*: The class 167 can be obtained from the class $14p$, the class 168 from the class $15p$, the classes 169 and 170 from the class $53p$, and the classes 171 and 172 from the class $54p$ by means of appropriate selection of the line at infinity. *Construction*: The quartic curves from these affine admissible classes can be constructed by means of suitable deformations of the double point of quartic curves of the 28th grade.

The 30th grade $(\mathbf{R}q, \{A_1, 2A_1^*\})$ contains 4 new (2 reflectable and 2 nonreflectable) admissible classes: $173, 174, 175^{\pm}$. *Enumeration*: These classes can be obtained from the class $55p$ by means of appropriate selection of the line at infinity. *Construction*: These classes contain quartic curves. The quartic curves from these affine admissible classes can be constructed by means of contracting to a point the oval of the quartic curves of classes 163, 165 and $166^{\pm}$. Let $q = 0$ be an equation of the quartic curve and $l = 0$ an equation of the line connecting the double points of the quartic curve. Let the signs of q and l be different inside the quartic oval; then the segment of curves with equation $q + tl^3 = 0$, $t \in [0, t_0]$, provides the required contraction for some $t_0 > 0$.

The 31st grade $(\mathbf{R}q, \{2A_1^*\})$ contains 3 new reflectable admissible classes: 176–178. *Enumeration*: The class 176 can be obtained from the class $56p$, the classes 177 and 178 from the class $57p$ by means of appropriate selection of the line at infinity. *Construction*: The quartic curves from these affine admissible classes can be constructed by means of suitable deformations of the double point of quartic curves of the 30th grade.

The 32nd grade $(\mathbf{R}q, \{3A_1^*\})$ contains 1 new reflectable admissible class: 179. *Enumeration*: The class 179 can be obtained from the class $59p$ by means of appropriate selection of the line at infinity. *Construction*: This class contains quartic curves. The quartic curves from these affine admissible classes can be constructed by means of contracting to a point the oval of a quartic curve of class 177. Let

$q = 0$ be an equation of the quartic curve and $l = 0$ be an equation of the line connecting the double points of the quartic curve. Let the signs of q and l be different inside the quartic oval; then the segment of curves with equation $q + tl^3 = 0$, $t \in [0, t_0]$, provides the required contraction for some $t_0 > 0$.

In conclusion, the 1st division contains 254 admissible (112 reflectable and 142 nonreflectable) classes. The 5 admissible (3 reflectable and 2 nonreflectable) classes $A, B^{\pm}, C$ and D do not contain quartic curves.

The 2nd division. The quartic curves have 2 real points of intersection and 1 point of tangency with the line at infinity, and thus have 2 two-sided asymptotes and 2 parabolic branches.

Let us consider a noncompact nonsingular (homeomorphic to an open segment) component of an affine curve that goes to infinity along two asymptotes. The complement of such a component in the affine plane consists of two components homeomorphic to a disk. If one of the components does not contain any other points of the curve, then the component of the curve is called a *simple arc*.

Let us consider a quartic curve of the 2nd division. Such a curve has 2 real points of intersection and 1 point of tangency with the line at infinity. If we make a small rotation of the line at infinity around a point of intersection with the corresponding projective curve in the direction which replaces the point of tangency with two real points of intersection, then we obtain a curve that has acquired a new simple arc and belongs to the 1st division. The point here is that each curve of the 2nd division can be constructed from a curve of the 1st division that has a simple arc. Thus, to construct a curve of the 2nd division it is enough to rotate the line at infinity around a point of intersection with the curve that is not an endpoint of a simple arc until the new line at infinity becomes tangent to that simple arc. In Figure 9 one may see the admissible classes and their quartic representatives. We show in parentheses the numbers of the admissible classes that we use for construction. We show all quartic curves that can be obtained from the 1st division, even if they represent classes obtained earlier.

The 1st–6th grades don't contain admissible classes.

The 7th–21st grades represent admissible isotopy classes for reducible quartic curves. All these admissible classes contain quartic curves.

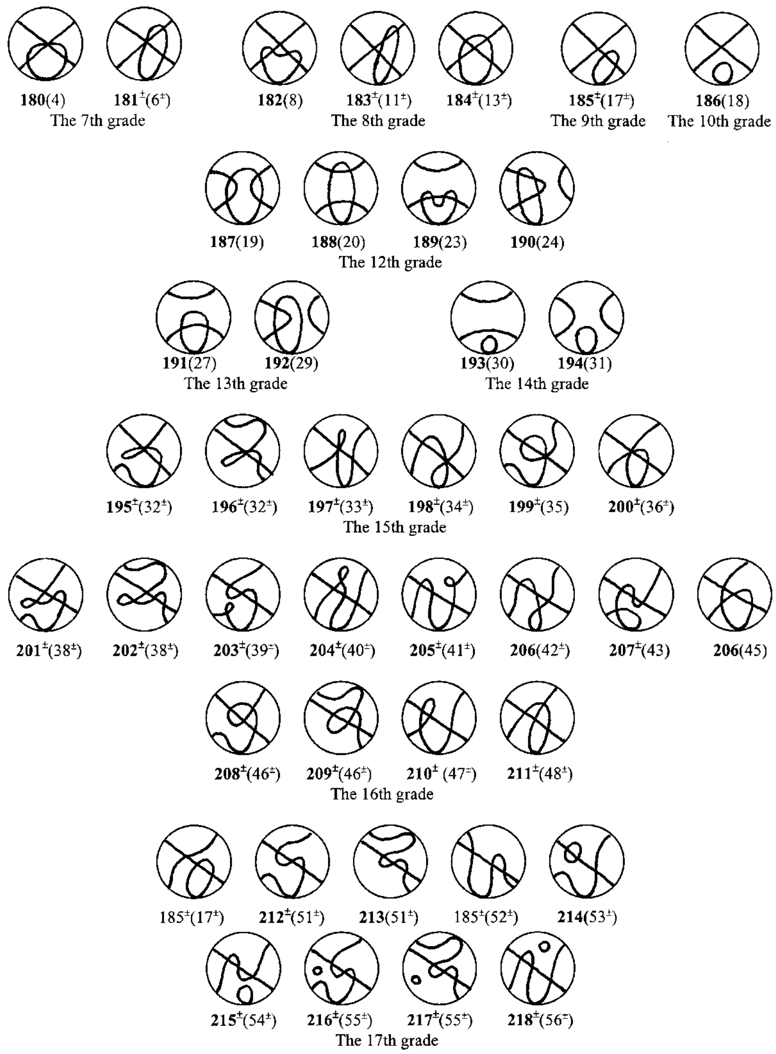


FIGURE 9 (beginning). The 2nd division.

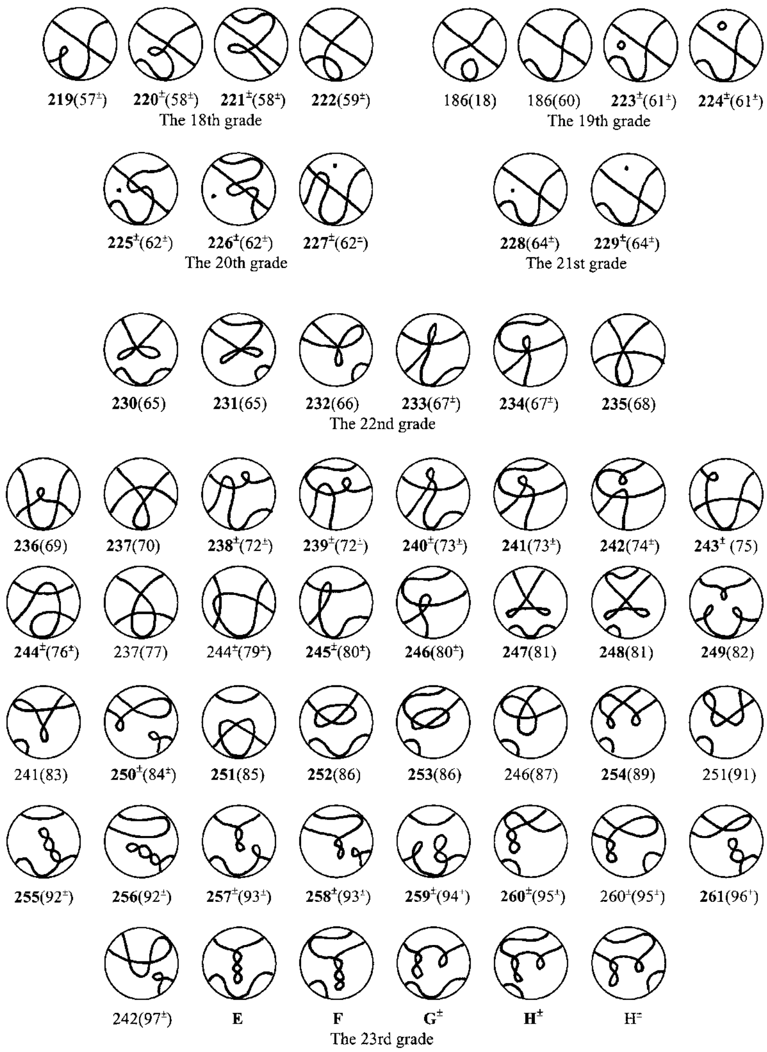
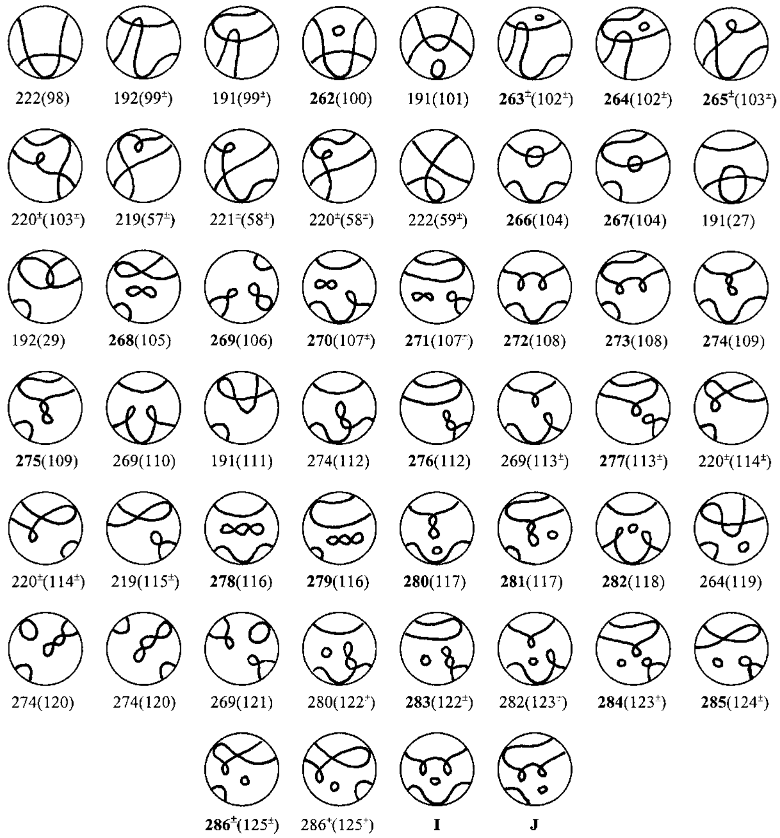
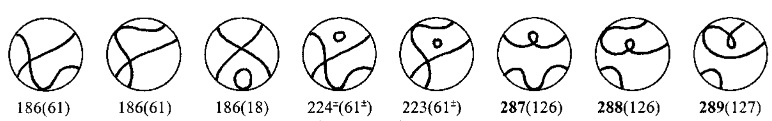


FIGURE 9 (continued). The 2nd division.



The 24th grade



The 25th grade (beginning)

FIGURE 9 (continued). The 2nd division.

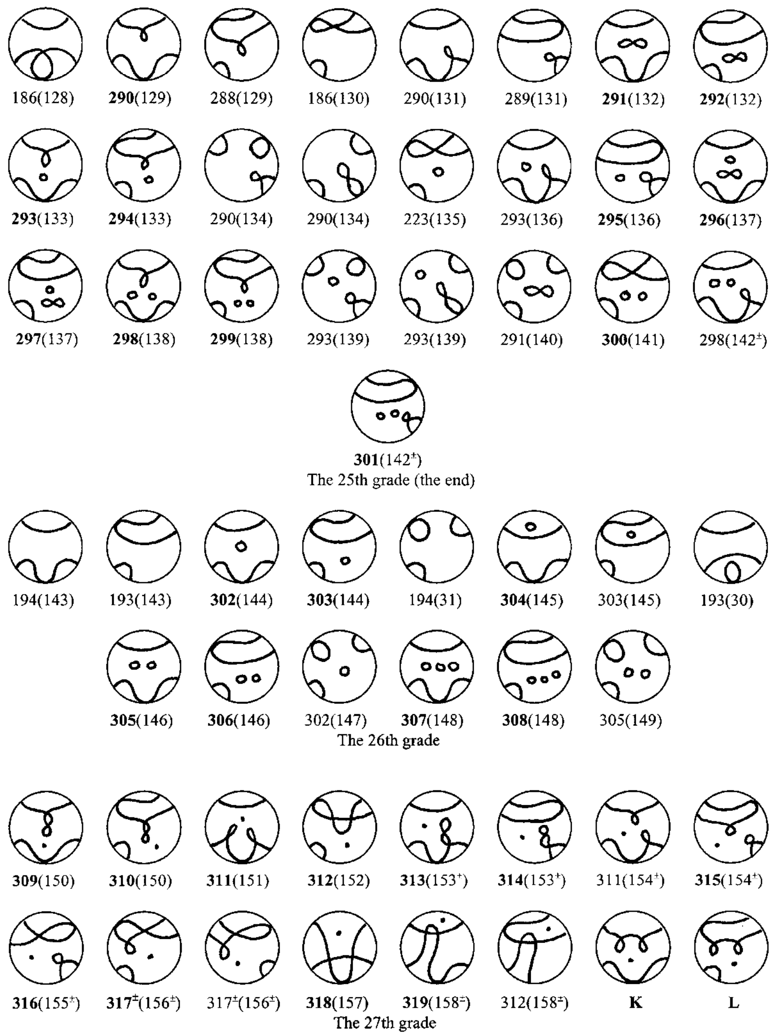


FIGURE 9 (continued). The 2nd division.

The 7th grade $(\mathbf{R}(l_1 l_2 c), \{D_4, 2A_1\})$ contains 3 new (1 reflectable and 2 nonreflectable) admissible classes: 180 and $181^{\pm}$.

The 8th grade $(\mathbf{R}(l_1 l_2 c), \{5A_1\})$ contains 5 new (1 reflectable and 4 nonreflectable) admissible classes: 182, $183^{\pm}$ and $184^{\pm}$.

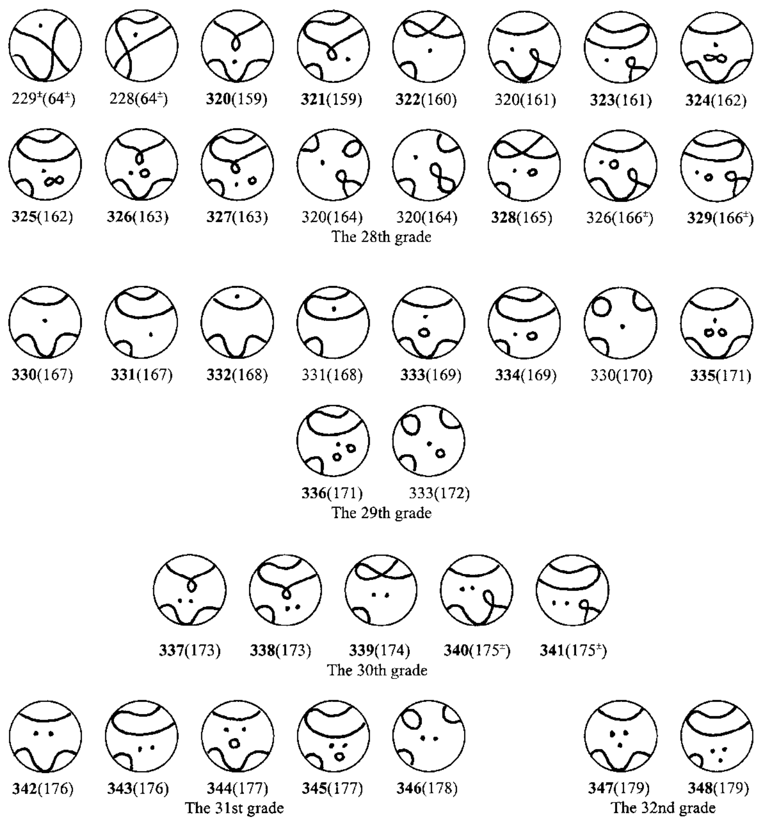


FIGURE 9 (conclusion). The 2nd division.

The 9th grade $(\mathbf{R}(l_1l_2c), \{3A_1\})$ contains 2 new nonreflectable admissible classes: $185^{\pm}$.

The 10th grade $(\mathbf{R}(l_1l_2c), \{A_1\})$ contains 1 new reflectable admissible class: 186.

The 11th grade $(\mathbf{R}(l_1l_2c), \{A_1^*\})$ does not contain admissible classes.

The 12th grade $(\mathbf{R}(c_1c_2), \{4A_1\})$ contains 4 new reflectable admissible classes: 187–190.

The 13th grade $(\mathbf{R}(c_1c_2), \{2A_1\})$ contains 2 new reflectable admissible classes: 191 and 192.

The 14th grade $(\mathbf{R}(c_1c_2), \varnothing)$ contains 2 new reflectable admissible classes: 193 and 194.

The 15th grade $(\mathbf{R}(lcb), \{D_4, A_1\})$ contains 12 new nonreflectable admissible classes: $195^{\pm}$–$200^{\pm}$.

The 16th grade $(\mathbf{R}(lcb), \{4A_1\})$ contains 21 new (1 reflectable and 20 nonreflectable) admissible classes: $201^{\pm}$–$205^{\pm}$, 206, $207^{\pm}$–$211^{\pm}$.

The 17th grade $(\mathbf{R}(lcb), \{3A_1\})$ contains 2 old nonreflectable admissible classes: $185^{\pm}$ and 12 new (2 reflectable and 10 nonreflectable) admissible classes: $212^{\pm}$, 213, 214, $215^{\pm}$–$218^{\pm}$.

The 18th grade $(\mathbf{R}(lcb), \{2A_1\})$ contains 6 new (2 reflectable and 4 nonreflectable) admissible classes: 219, $220^{\pm}$, $221^{\pm}$ and 222.

The 19th grade $(\mathbf{R}(lcb), \{A_1\})$ contains 1 old reflectable admissible class: 185 and 4 new nonreflectable admissible classes: $223^{\pm}$ and $224^{\pm}$.

The 20th grade $(\mathbf{R}(lcb), \{3A_1, A_1^*\})$ contains 6 new nonreflectable admissible classes: $225^{\pm}$–$227^{\pm}$.

The 21st grade $(\mathbf{R}(lcb), \{A_1, A_1^*\})$ contains 3 new (1 reflectable and 2 nonreflectable) admissible classes: 228 and $229^{\pm}$.

The 22nd grade $(\mathbf{R}q, \{D_4\})$ contains 6 new reflectable admissible classes: 230–235.

The 23rd grade $(\mathbf{R}q, \{3A_1\})$ contains 46 new (18 reflectable and 28 nonreflectable) admissible classes: 236, 237, $238^{\pm}$–$240^{\pm}$, 241, 242, $243^{\pm}$–$245^{\pm}$, 246–249, $250^{\pm}$, 251–256, $257^{\pm}$–$260^{\pm}$, 261, E, F, $G^{\pm}$, $H^{\pm}$. *Restriction*: The 2 reflectable classes E and F and the 4 nonreflectable classes $G^{\pm}$ and $H^{\pm}$ don't contain quartic curves (see Lemma 8).

The 24th grade $(\mathbf{R}q, \{2A_1\})$ contains 8 old (4 reflectable and 4 nonreflectable) admissible classes: 222, 192, 191, $220^{\pm}$, 219 and $221^{\pm}$ and 30 new (24 reflectable and 6 nonreflectable) admissible classes: 262, $263^{\pm}$, 264, $265^{\pm}$, 266–282, $286^{\pm}$, I and J. *Restriction*: The 2 reflectable classes I and J don't contain quartic curves (see Lemma 8).

The 25th grade $(\mathbf{R}q, \{A_1\})$ contains 4 old (2 reflectable and 2 nonreflectable) admissible classes: 186, 223 and $224^{\pm}$ and 15 new reflectable admissible classes: 287–301.

The 26th grade $(\mathbf{R}q, \varnothing)$ contains 2 old admissible classes: 194 and 193 and 7 new reflectable admissible classes: 302–308.

The 27th grade $(\mathbf{R}q, \{2A_1, A_1^*\})$ contains 14 new (12 reflectable and 2 nonreflectable) admissible classes: 309–316, $317^{\pm}$, 318, 319, K and L. *Restriction*: The 2 reflectable classes K and L don't contain quartic curves (see Lemma 8).

The 28th grade $(\mathbf{R}q, \{A_1, A_1^*\})$ contains 3 old (1 reflectable and 2 nonreflectable) admissible classes: 228 and $229^{\pm}$ and 10 new reflectible admissible classes: 320–329.

The 29th grade $(\mathbf{R}q, \{A_1^*\})$ contains 7 new reflectable admissible classes: 330–336.

The 30th grade $(\mathbf{R}q, \{A_1, 2A_1^*\})$ contains 5 new reflectable admissible classes: 337–341.

The 31st grade $(\mathbf{R}q, \{2A_1^*\})$ contains 5 new reflectable admissible classes: 342–346.

The 32nd grade $(\mathbf{R}q, \{3A_1^*\})$ contains 2 new reflectable admissible classes: 347 and 348.

In conclusion, the 2nd division contains 227 new (127 reflectable and 100 nonreflectable) admissible classes. The 10 admissible (6 reflectable and 4 nonreflectable) classes $E, F, G^{\pm}, H^{\pm}, I, J, K, L$ do not contain quartic curves.

The 3rd division. These quartic curves have 2 real points of tangency with the line at infinity and thus have 4 parabolic asymptotic branches.

Let us consider a quartic curve of the 3rd division. Such a curve F has 2 points of tangency with the line at infinity L_∞. Let U_1 and U_2 be regular neighborhoods of the points of tangency a_1 and a_2, respectively. Let us slide the line at infinity so that it is tangent to $\mathbf{R}f \cap U_1$ and intersects $\mathbf{R}f \cap U_2$ in two real points. We then obtain a quartic curve that has acquired a new simple arc and belongs to the 2nd division. The point here is that each curve of the 3rd division can be constructed from a curve of the 2nd division that has a simple arc. Thus, to construct a curve of the 3rd division, it is enough to slide the line at infinity along the curve of the 2nd division until the line at infinity becomes tangent to that simple arc. In Figure 10 one may see the admissible classes and their quartic representatives. We show in parentheses the numbers of the admissible classes that we use for construction. We show all

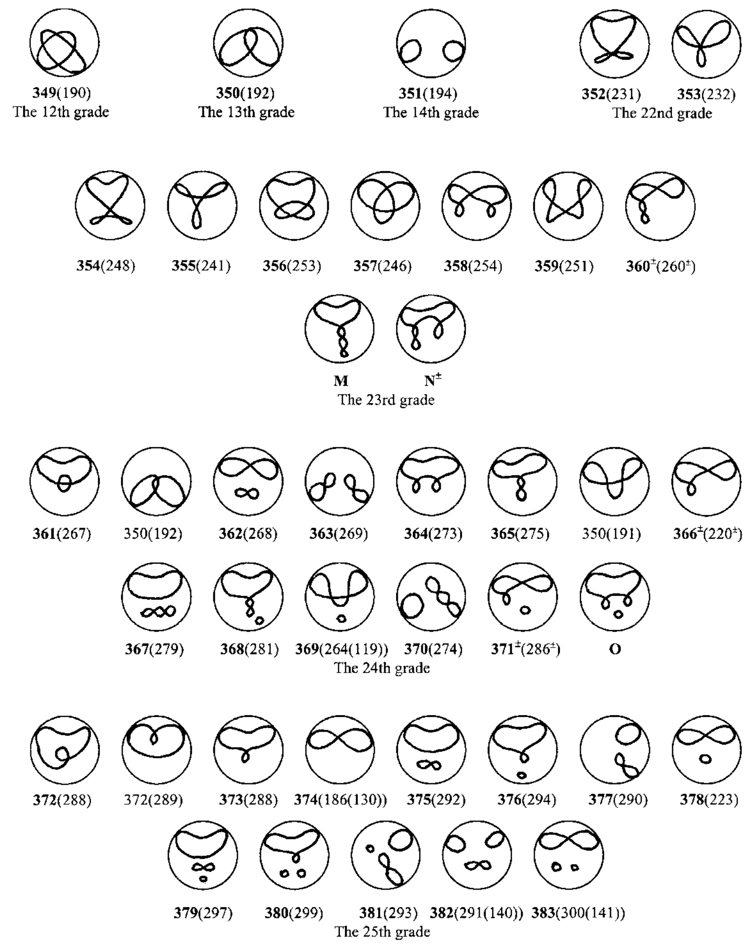


FIGURE 10 (beginning). The 3rd division.

quartic curves that can be obtained from the 2nd division, even if they represent classes obtained earlier.

The 1st–11th grades don't contain admissible classes.

The 12th–14th grades represent admissible isotopy classes for reducible quartic curves. All these admissible classes contain quartic curves.

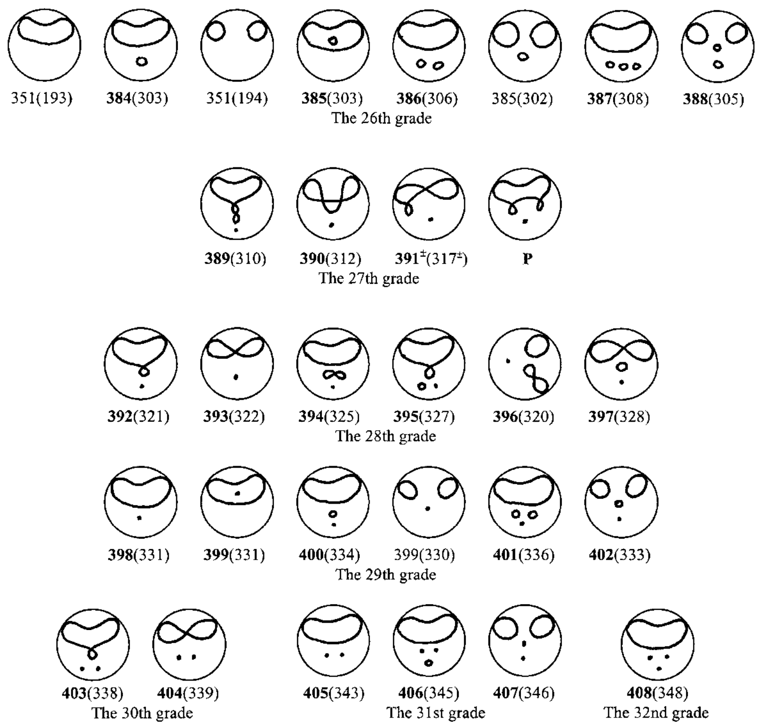


FIGURE 10 (conclusion). The 3rd division.

The 12th grade $(\mathbf{R}(c_1c_2), \{4A_1\})$ contains 1 new reflectable admissible class: 349.

The 13th grade $(\mathbf{R}(c_1c_2), \{2A_1\})$ contains 1 new reflectable admissible class: 350.

The 14th grade $(\mathbf{R}(c_1c_2), \varnothing)$ contains 1 new reflectable admissible class: 351.

The 15th–21st grades don't contain admissible classes.

The 22nd grade $(\mathbf{R}q, \{D_4\})$ contains 2 reflectable admissible classes: 352 and 353.

The 23rd grade $(\mathbf{R}q, \{3A_1\})$ contains 11 new (7 reflectable and 4 nonreflectable) admissible classes: 354–359, $360^{\pm}$, M and $N^{\pm}$. *Restriction*: The 1 reflectable class M and 2 nonreflectable classes $N^{\pm}$ don't contain quartic curves (see Lemma 8).

The 24th grade $(\mathbf{R}q, \{2A_1\})$ contains 1 old reflectable admissible class: 350 and 14 new (10 reflectable and 4 nonreflectable) admissible classes: 361–365, $366^{\pm}$, 367–370, $371^{\pm}$ and O. *Restriction*: The 1 reflectable class O doesn't contain quartic curves (see Lemma 8).

The 25th grade $(\mathbf{R}q, \{A_1\})$ contains 12 new reflectable admissible classes: 372–383.

The 26th grade $(\mathbf{R}q, \varnothing)$ contains 1 old reflectable admissible class: 351 and 5 new reflectable admissible classes: 384–388.

The 27th grade $(\mathbf{R}q, \{2A_1, A_1^*\})$ contains 5 new (3 reflectable and 2 nonreflectable) admissible classes: 389, 390, $391^{\pm}$ and P. *Restriction*: The 1 reflectable class P doesn't contain quartic curves (see Lemma 8).

The 28th grade $(\mathbf{R}q, \{A_1, A_1^*\})$ contains 6 new reflectable admissible classes: 392–397.

The 29th grade $(\mathbf{R}q, \{A_1^*\})$ contains 5 new reflectable admissible classes: 398–402.

The 30th grade $(\mathbf{R}q, \{A_1, 2A_1^*\})$ contains 2 new reflectable admissible classes: 403 and 404.

The 31st grade $(\mathbf{R}q, \{2A_1^*\})$ contains 3 new reflectable admissible classes: 405–407.

The 32nd grade $(\mathbf{R}q, \{3A_1^*\})$ contains 1 new reflectable admissible class: 408.

In conclusion, the 3rd division contains 69 new (59 reflectable and 10 nonreflectable) admissible classes. The 5 admissible (3 reflectable and 2 nonreflectable) classes M, $N^{\pm}$, O, P do not contain quartic curves.

The 4th division. These quartic curves have 2 real and 2 complex points of intersection with the line at infinity. We enumerate the admissible classes and construct quartic curves in the same manner as in the 1st division. In Figure 11 one may see the admissible classes and their quartic representatives. We show in parentheses the numbers of the admissible classes that we use for construction. We show all quartic curves, even if they represent classes obtained earlier.

The 1st–3rd grades don't contain admissible classes.

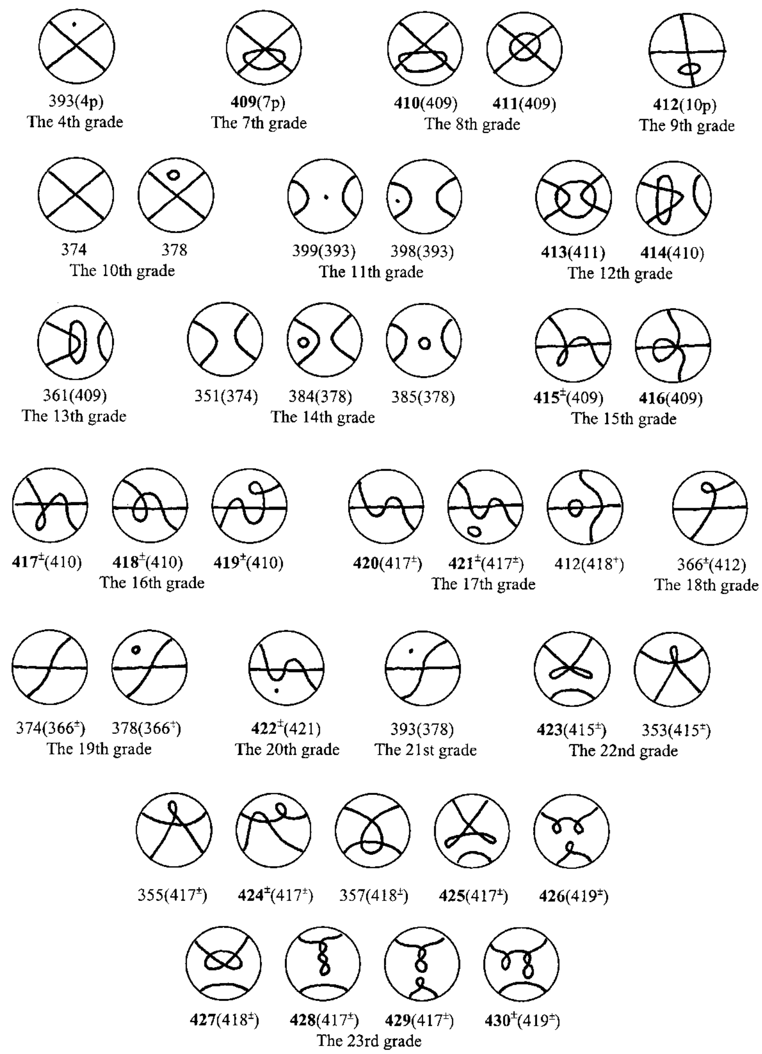


FIGURE 11 (beginning). The 4th division.

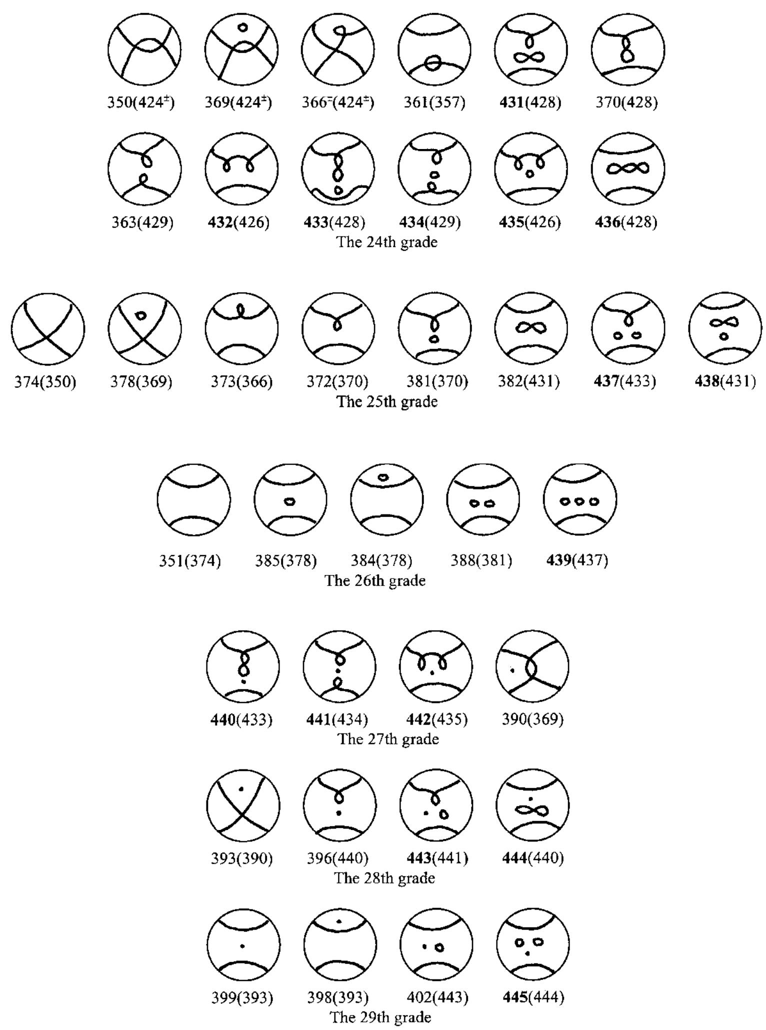
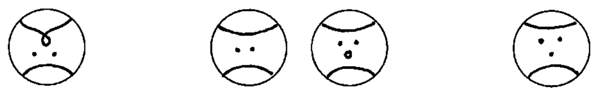


FIGURE 11 (continued). The 4th division.



446(443) The 30th grade — 407(446) **447**(446) The 31st grade — **448**(446) The 32nd grade

FIGURE 11 (conclusion). The 4th division.

The 4th grade $(\mathbf{R}(l_1 l_2 l_3 l_4), \{X_9\})$ contains 1 old reflectable admissible class: 393, obtained from the class $4p$ by an obvious selection of the line at infinity. The existence of such a quartic curve is obvious.

The 5th and 6th grades don't contain admissible classes.

The 7th grade $(\mathbf{R}(l_1 l_2 c), \{D_4, 2A_1\})$ contains 1 new reflectable admissible class: 409, obtained from the class $7p$ by means of an obvious selection of the line at infinity. The existence of such a quartic curve is obvious.

The 8th grade $(\mathbf{R}(l_1 l_2 c), \{5A_1\})$ contains 2 new reflectable admissible classes: 410 and 411, obtained from the classes $8p$ and $9p$, respectively, by means of obvious selections of the line at infinity. The existence of such quartic curves is obvious.

The 9th grade $(\mathbf{R}(l_1 l_2 c), \{3A_1\})$ contains 1 new reflectable admissible classes: 412, obtained from the class $10p$ by means of an obvious selection of the line at infinity. The existence of such a quartic curve is obvious.

The 10th grade $(\mathbf{R}(l_1 l_2 c), \{A_1\})$ contains 2 old reflectable admissible classes: 374 and 378, obtained from the classes $11p$ and $12p$, respectively, by an obvious selection of the line at infinity. The existence of such quartic curves is obvious.

The 11th grade $(\mathbf{R}(l_1 l_2 c), \{A_1^*\})$ contains 2 old admissible classes 398 and 399, obtained from the classes $14p$ and $15p$, respectively, by an obvious selection of the line at infinity. The existence of such quartic curves is obvious.

The 12th grade $(\mathbf{R}(c_1 c_2), \{4A_1\})$ contains 2 new reflectable admissible classes: 413 and 414, obtained from the classes $16p$ and $17p$, respectively, by means of an obvious selection of the line at infinity. The existence of such quartic curves is obvious.

The 13th grade $(\mathbf{R}(c_1c_2), \{2A_1\})$ contains 1 old reflectable admissible class: 361, obtained from the class $18p$ by means of an obvious selection of the line at infinity. The existence of such a quartic curve is obvious.

The 14th grade $(\mathbf{R}(c_1c_2), \varnothing)$ contains 3 old reflectable admissible classes: 351, 384 and 385, obtained from the classes $20p$, $21p$ and $22p$, respectively, by means of obvious selections of the line at infinity. The existence of such quartic curves is obvious.

The 15th–21st grades represent admissible isotopy classes for reducible quartic curves that consist of 1 line and 1 cubic component.

The 15th grade $(\mathbf{R}(lcb), \{D_4, A_1\})$ contains 3 new (1 reflectable and 2 nonreflectable) admissible classes: $415^{\pm}$ and 416. *Enumeration*: They can be obtained from the classes $23p$ and $24p$ by means of obvious selections of the line at infinity. *Construction*: The quartic curves from these affine admissible classes can be constructed by means of small deformations of one of the double points of quartic curves of the 7th grade.

The 16th grade $(\mathbf{R}(lcb), \{4A_1\})$ contains 6 new nonreflectable admissible classes: $417^{\pm}$–$419^{\pm}$. *Enumeration*: They can be obtained from the classes $25p$ and $26p$ by means of appropriate selections of the line at infinity. *Construction*: The quartic curves from these affine admissible classes can be constructed by means of small deformations of one of the double points of quartic curves from the 8th grade. To preserve the line component one must choose a double point that differs from the point of intersection of the two line components.

The 17th grade $(\mathbf{R}(lcb), \{3A_1\})$ contains 1 old admissible class: 412, and 3 new (1 reflectable and 2 nonreflectable) admissible classes: 420 and $421^{\pm}$. *Enumeration*: The classes can be obtained from the classes $10p$, $27p$ and $28p$, respectively, by means of appropriate selections of the line at infinity. *Construction*: The quartic curves from these affine admissible classes can be constructed by means of small deformations of one of the double points of quartic curves from the 16th grade. To preserve a line component one must choose a double point that differs from the points of intersection of the line and cubic curve.

The 18th grade $(\mathbf{R}(lcb), \{2A_1\})$ contains 1 old reflectable admissible class: $366^{\pm}$. *Enumeration*: This class can be obtained from the class $29p$ by means of appropriate selection of the line at infinity. *Construction*: A quartic curve from this affine admissible class can be constructed by means of small deformations of one of the double points of a quartic curve from the 9th grade. To preserve a line component one must choose a double point that differs from the point of intersection of the two line components.

The 19th grade $(\mathbf{R}(lcb), \{A_1\})$ contains 2 old reflectable admissible classes: 374 and 378. *Enumeration*: These classes can be obtained from the classes $11p$ and $12p$ by means of appropriate selections of the line at infinity. *Construction*: The quartic curves from these affine admissible classes can be constructed by means of small deformations of one of the double points of quartic curves from the 18th grade. To preserve a line component one must choose a double point that differs from the points of intersection of the line and cubic curve.

The 20th grade $(\mathbf{R}(lcb), \{3A_1, A_1^*\})$ contains 2 new nonreflectable admissible classes: $422^{\pm}$. *Enumeration*: These classes can be obtained from the projective class $30p$ by means of appropriate selections of the line at infinity. *Construction*: The quartic curves from these affine admissible classes can be constructed by means of contracting the oval of the cubic component to a point. If $lcb = 0$ is an equation of a quartic curve from class $421^{\pm}$ such that the signs of l and cb are different inside the oval, then the segment of curves with equation $l(cb + tl) = 0$, $t \in [0, t_0]$, provides the required contraction for some $t_0 > 0$ and constructs quartic curves of classes $422^{\pm}$ from those of classes $421^{\pm}$.

The 21st grade $(\mathbf{R}(lcb), \{A_1, A_1^*\})$ contains 1 old reflectable admissible class: 393. *Enumeration*: This class can be obtained from the class $4p$ by an obvious selection of the line at infinity. *Construction*: A quartic curve from this affine admissible class can be constructed by means of contracting the oval of the cubic component of a quartic curve of 19th grade to a point in the same way as in the 20th grade. Another way to construct this curve is parallel motion of a line component of a quartic curve from the 20th grade.

The 22nd grade $(\mathbf{R}q, \{D_4\})$ contains 1 old reflectable class: 353, and 1 new reflectable admissible class: 423. *Enumeration*: The classes 423 and 353 can be obtained from the classes $31p$ and $32p$ respectively by means of appropriate selections of the line at infinity. *Construction*: The quartic curves from these affine admissible classes can be constructed by means of suitable deformations of the double point of intersection of the line and cubic curve of the 15th grade.

The 23rd grade $(\mathbf{R}q, \{3A_1\})$ contains 2 old reflectable classes: 355 and 357, and 9 new (5 reflectable and 4 nonreflectable) admissible classes: $424^{\pm}$, 425–429 and $430^{\pm}$. *Enumeration*: The classes 355 and $424^{\pm}$ can be obtained from the class $33p$, the class 357 from $34p$, the classes 425 and 426 from class $35p$, class 427 from class $36p$, classes 428, 429 and $430^{\pm}$ from $37p$ by means of appropriate selections of the line at infinity. *Construction*: The quartic curves from these affine admissible classes can be constructed by means of suitable deformations of one of the double points of intersection of the line and cubic curve of the 16th grade.

The 24th grade $(\mathbf{R}q, \{2A_1\})$ contains 7 old (5 reflectable and 2 nonreflectable) admissible classes: 350, 369, $366^{\pm}$, 361, 370 and 363, and 6 new reflectable admissible classes: 431–436. *Enumeration*: The class 350 can be obtained from the class $38p$, the class 369 from $39p$, the classes $366^{\pm}$ from $29p$, the class 361 from $18p$, the class 431 from $40p$, the classes 370, 363 and 432 from $41p$, the classes 433–436 from $42p$ by means of appropriate selections of the line at infinity. *Construction*: The quartic curves from these affine admissible classes can be constructed by means of suitable deformations of one of the double points of quartic curves of the 23rd grade. The double point in each case can be found in figures of the 23rd grade.

The 25th grade $(\mathbf{R}q, \{A_1\})$ contains 6 old reflectable admissible classes: 374, 378, 373, 372, 381 and 382, and 2 new reflectable classes: 437 and 438. *Enumeration*: The class 374 can be obtained from the class $11p$, the class 378 from the class $12p$, the class 373 from $43p$, the class 372 from the class $44p$, the classes 381 and 382 from the class $45p$, the classes 437 and 438 from the class $46p$ by means of appropriate selections of the line at infinity. *Construction*: The quartic curves from these affine admissible classes can be constructed by means of suitable deformations of one of the double points of quartic curves of the 24th grade. The double point in each case can be found in figures of the 24th grade.

The 26th grade $(\mathbf{R}q, \varnothing)$ contains 4 old reflectable admissible classes: 351, 385, 384 and 388, and 1 new reflectable admissible class: 439. *Enumeration*: The class 351 can be obtained from the class $20p$, the class 385 from $22p$, the class 384 from $21p$, the class 388 from $47p$, the class 439 from $48p$ by means of appropriate selections of the line at infinity. *Construction*: The quartic curves from these affine admissible classes can be constructed by means of suitable deformations of the double point of quartic curves of the 25th grade.

The 27th grade $(\mathbf{R}q, \{2A_1, A_1^*\})$ contains 1 old reflectable admissible class: 390 and 3 new reflectable admissible classes: 440–442. *Enumeration*: The classes 440–442 can be obtained from the class $49p$, the class 390 from $50p$ by means of appropriate selections of the line at infinity. *Construction*: The quartic curves from these affine admissible classes can be constructed by means of contracting to a point the oval of quartic curves of the 24th grade in the same manner as in the 27th grade of the 1st division.

The 28th grade $(\mathbf{R}q\{A_1, A_1^*\})$ contains 2 old reflectable admissible classes: 393 and 396 and 2 new reflectable admissible classes: 443 and 444. *Enumeration*: The class 393 can be obtained from the class $4p$, the class 396 from $51p$ and the classes 443 and 444 from $52p$ by means of appropriate selections of the line at infinity. *Construction*: The quartic curves from these affine admissible classes can be constructed by means of suitable deformations of one of the double points of quartic curves of the 27th grade. The double point in each case can be found in figures of the 27th grade.

The 29th grade $(\mathbf{R}q, \{A_1^*\})$ contains 3 old reflectable admissible classes: 399, 398 and 402 and 1 new reflectable admissible class: 445. *Enumeration*: The class 399 can be obtained from the class $14p$, the class 398 from $15p$, the class 402 from $53p$ and the class 445 from $54p$ by means of appropriate selections of the line at infinity. *Construction*: The quartic curves from these affine admissible classes can be constructed by means of suitable deformations of the double point of quartic curves of the 28th grade.

The 30th grade $(\mathbf{R}q, \{A_1, 2A_1^*\})$ contains 1 new reflectable admissible class: 446. *Enumeration*: This class can be obtained from the class $55p$ by means of appropriate selection of the line at infinity. *Construction*: A quartic curve from this affine admissible class can be constructed by means of contracting to a point the oval of a quartic curve of class 443 in the same manner as in the 30th grade of the 1st division.

The 31st grade $(\mathbf{R}q, \{2A_1^*\})$ contains 1 old and 1 new reflectable admissible classes: 407 and 447. *Enumeration*: The class 407 can be obtained from the class $56p$, the class 447 from the class $57p$ by means of appropriate selections of the line at infinity. *Construction*: The quartic curves from these affine admissible classes can be constructed by means of suitable deformations of the double point of quartic curves of the 30th grade.

The 32nd grade $(\mathbf{R}q, \{3A_1^*\})$ contains 1 new reflectable admissible class: 448. *Enumeration*: The class 448 obtained from the class $59p$ by means of appropriate selection of the line at infinity. *Construction*: A quartic curve from this affine admissible class can be constructed by means of contracting to a point the oval of a quartic curve of class 447 in the same manner as in the 32nd grade of the 1st division.

In conclusion, the 4th division contains 48 new (32 reflectable and 16 nonreflectable) admissible classes. All admissible classes contain quartic curves.

The 5th division. These quartic curves have 1 real point of tangency with the line at infinity and thus 2 parabolic asymptotic branches.

Let us consider a quartic curve of the 5th division. Such a curve has 1 point of tangency with the line at infinity. If we move the line at infinity so that the point of tangency is replaced by two real points of intersection with the curve, then we obtain a curve from an admissible class of the 4th division. So to construct the curves from the 5th division it is enough to consider each curve of the 4th division that has a simple arc and move the line at infinity so that it becomes tangent to this simple arc. It is easy to see that for projective quartic curves there exists such a motion during which the line does not intersect other branches of the curve. In Figure 12 one may see the admissible classes and their quartic representatives. We show in parentheses the number of the admissible classes that we use for construction. We show all quartic curves that can be obtained from the 4th division, even if they represent classes obtained earlier.

The 1st–10th grades don't contain admissible classes.

The 11th–14th grades represent admissible isotopy classes for reducible quartic curves. All these admissible classes contain quartic curves.

The 11th grade $(\mathbf{R}(l_1 l_2 c), \{A_1^*\})$ contains 1 new reflectable admissible class: 449.

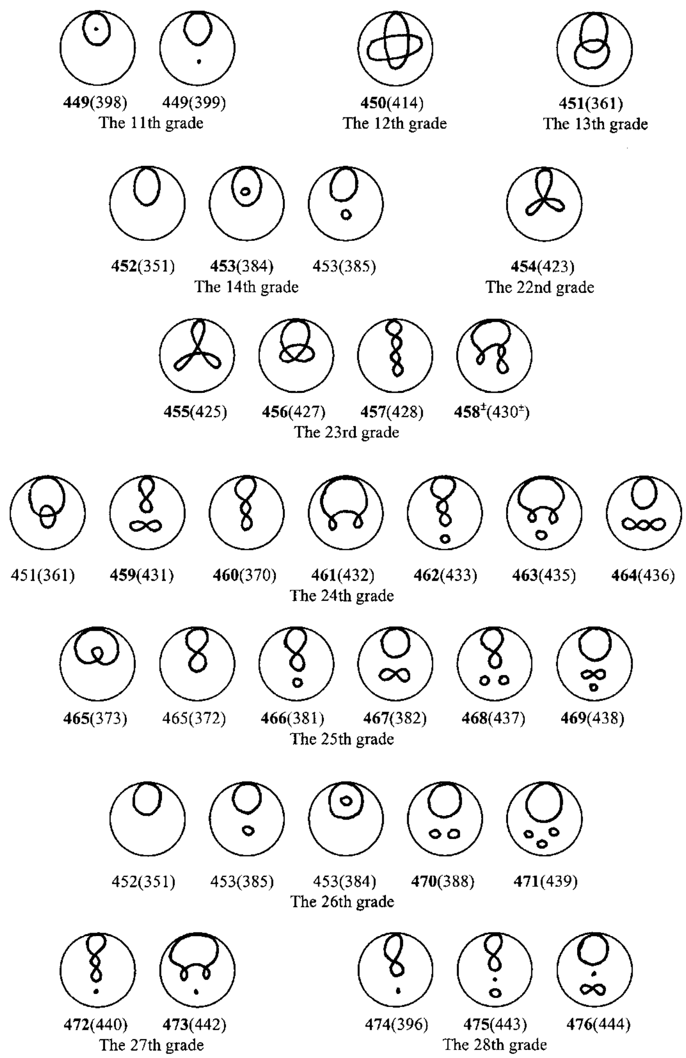
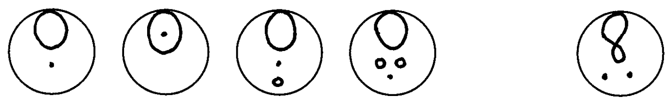
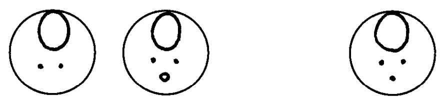


FIGURE 12 (beginning). The 5th division.



449(399) 449(398) **477**(402) **478**(445) **479**(446)

The 29th grade The 30th grade



480(407) **481**(447) **482**(448)

The 31st grade The 32nd grade

FIGURE 12 (conclusion). The 5th division.

The 12th grade $(\mathbf{R}(c_1c_2), \{4A_1\})$ contains 1 new reflectable admissible class: 450.

The 13th grade $(\mathbf{R}(c_1c_2), \{2A_1\})$ contains 1 new reflectable admissible class: 451.

The 14th grade $(\mathbf{R}(c_1c_2), \varnothing)$ contains 2 new reflectable admissible classes: 452 and 453.

The 15th–21st grades don't contain admissible classes.

The 22nd grade $(\mathbf{R}q, \{D_4\})$ contains 1 new reflectable admissible class: 454.

The 23rd grade $(\mathbf{R}q, \{3A_1\})$ contains 5 new (3 reflectable and 2 nonreflectable) admissible classes: 455–457 and $458^{\pm}$.

The 24th grade $(\mathbf{R}q, \{2A_1\})$ contains 1 old reflectable admissible class: 451 and 6 new reflectable admissible classes: 459–464.

The 25th grade $(\mathbf{R}q, \{A_1\})$ contains 5 new reflectable admissible classes: 465–469.

The 26th grade $(\mathbf{R}q, \varnothing)$ contains 2 old reflectable admissible classes: 452 and 453 and 2 new reflectable admissible classes: 470 and 471.

The 27th grade $(\mathbf{R}q, \{2A_1, A_1^*\})$ contains 2 new reflectable admissible classes: 472 and 473.

The 28th grade $(\mathbf{R}q, \{A_1, A_1^*\})$ contains 3 new reflectable admissible classes: 474–476.

The 29th grade $(\mathbf{R}q, \{A_1^*\})$ contains 1 old reflectable admissible class: 449 and 2 new reflectable admissible classes: 477 and 478.

The 30th grade $(\mathbf{R}q, \{A_1, 2A_1^*\})$ contains 1 new reflectable admissible class: 479.

The 31st grade $(\mathbf{R}q, \{2A_1^*\})$ contains 2 new reflectable admissible classes: 480 and 481.

The 32nd grade $(\mathbf{R}q, \{3A_1^*\})$ contains 1 new reflectable admissible class: 482.

In conclusion, the 5th division contains 35 new (33 reflectable and 2 nonreflectable) admissible classes. All admissible classes contain quartic curves.

The 6th division. These quartic curves do not have real points of intersection with the line at infinity.

To enumerate the admissible classes and construct their quartic representatives, it is enough to consider the isotopy classes of contractible projective curves of Figure 3. In Figure 13 one may see the affine admissible classes and their quartic representatives for the 6th division.

The 1st–4th grades don't contain admissible classes.

The 5th grade $(\mathbf{R}(l_1l_2l_3l_4), \{2A_1^*\})$ contains 1 new reflectable admissible class: 483.

The 6th grade $(\mathbf{R}(l_1l_2l_3l_4), \{4A_1^*\})$ contains 1 new reflectable admissible class: 484.

The 7th–10th grades don't contain admissible classes.

The 11th grade $(\mathbf{R}(l_1l_2c), \{A_1^*\})$ contains 3 new reflectable admissible classes: 485–487.

The 12th grade $(\mathbf{R}(c_1c_2), \{4A_1\})$ contains 1 new reflectable admissible class: 488.

The 13th grade $(\mathbf{R}(c_1c_2), \{2A_1\})$ contains 1 new reflectable admissible class: 489.

The 14th grade $(\mathbf{R}(c_1c_2), \varnothing)$ contains 4 new reflectable admissible classes: 490–493.

The 15th–21st grades don't contain admissible classes.

The 22nd grade $(\mathbf{R}q, \{D_4\})$ contains 1 new reflectable admissible class: 494.

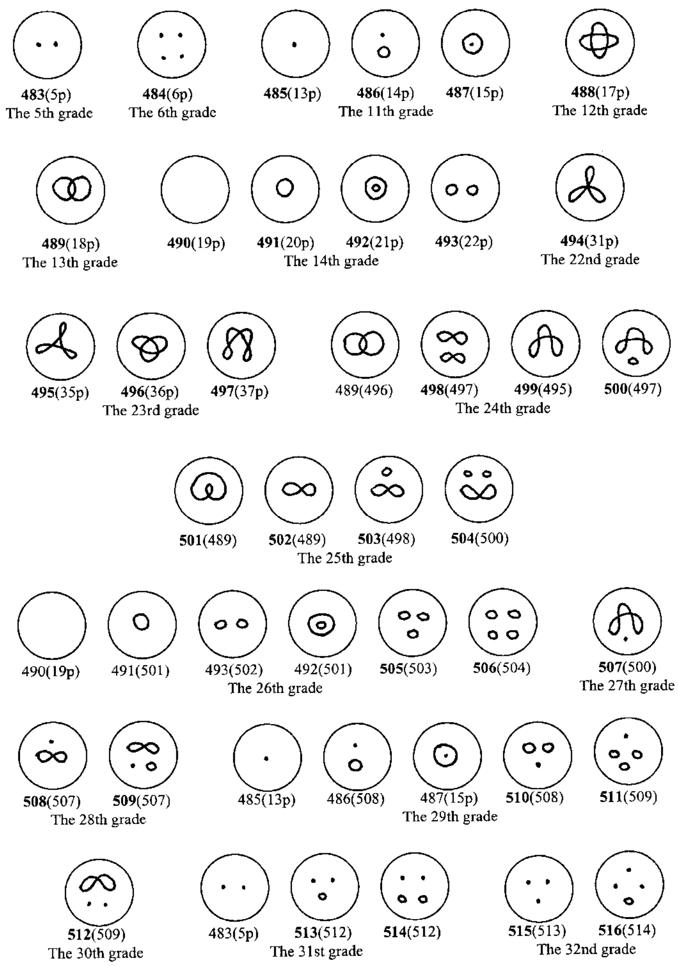


FIGURE 13. The 6th division.

The 23rd grade $(\mathbf{R}q, \{3A_1\})$ contains 3 new reflectable admissible classes: 495–497.

The 24th grade $(\mathbf{R}q, \{2A_1\})$ contains 1 old reflectable admissible class: 489 and 3 new reflectable admissible classes: 498–500.

The 25th grade $(\mathbf{R}q, \{A_1\})$ contains 4 new reflectable admissible classes: 501–504.

The 26th grade $(\mathbf{R}q, \varnothing)$ contains 4 old reflectable admissible classes: 490, 491, 493 and 492, and 2 new reflectable admissible classes: 505 and 506.

The 27th grade $(\mathbf{R}q, \{2A_1, A_1^*\})$ contains 1 new reflectable admissible class: 507.

The 28th grade $(\mathbf{R}q, \{A_1, A_1^*\})$ contains 2 new reflectable admissible classes: 508 and 509.

The 29th grade $(\mathbf{R}q, \{A_1^*\})$ contains 3 old reflectable admissible classes: 485–487, and 2 new reflectable admissible classes: 510 and 511.

The 30th grade $(\mathbf{R}q, \{A_1, 2A_1^*\})$ contains 1 new reflectable admissible class: 512.

The 31st grade $(\mathbf{R}q, \{2A_1^*\})$ contains 1 old reflectable admissible class: 483, and 2 new reflectable admissible classes: 513 and 514.

The 32nd grade $(\mathbf{R}q, \{3A_1^*\})$ contains 2 new reflectable admissible classes: 515 and 516.

In conclusion, the 6th division contains 34 new reflectable admissible classes. All admissible classes contain quartic curves.

The 7th division. This division contains admissible classes and their quartic representatives with multiple components. If a quartic curve has a multiple component, then the degree of such a component is no more than 2. We divide the affine quartic curves with multiple components into 12 grades as was done for the projective quartic curves. Thus, the enumeration and construction of such quartic curves is easy enough and we consider them without comments. In Figure 14, one may see the affine admissible classes and their quartic representatives for the 7th division.

The 33rd grade $(\mathbf{R}(l^4), \mathrm{Sing}\,(l) = \varnothing)$ contains 1 old reflectable admissidble class: 452.

The 34th grade $(\mathbf{R}(l_1 l_2^3), \mathrm{Sing}\,(l_1 l_2) = \{A_1\})$ contains 2 old reflectable admissible classes: 374 and 351.

The 35th grade $(\mathbf{R}(l_1^2 l_2^2), \mathrm{Sing}\,(l_1 l_2) = \{A_1\})$ contains 2 old reflectable admissible classes: 374 and 351.

The 36th grade $(\mathbf{R}(l_1^2 l_2^2), \mathrm{Sing}\,(l_1 l_2) = \{A_1^*\})$ contains 2 old reflectable

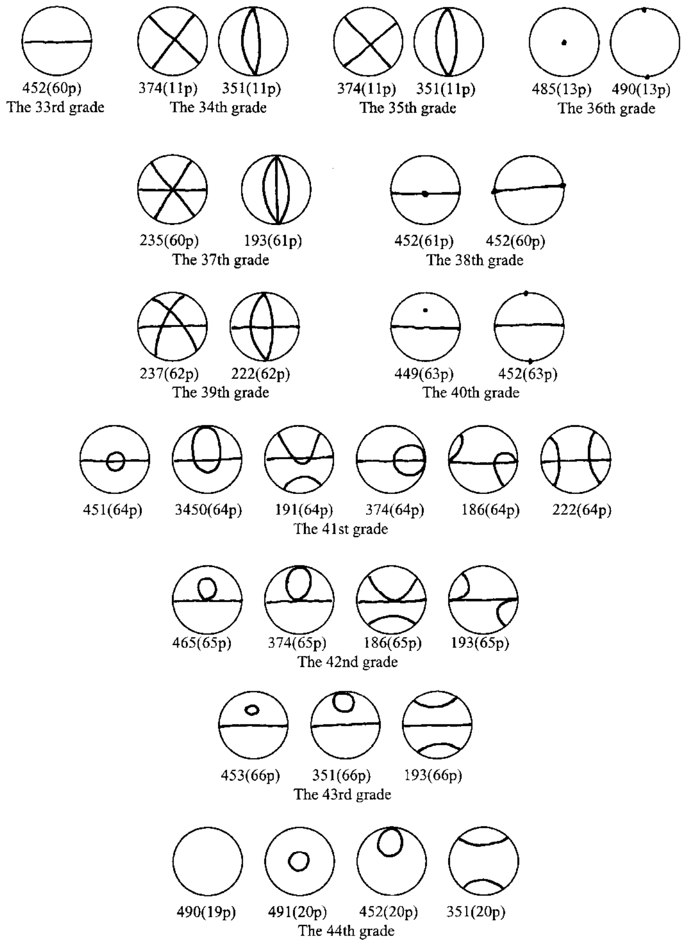


FIGURE 14. The 7th division.

admissible classes: 485 and 490.

The 37th grade $(\mathbf{R}(l_1 l_2 l_3^2), \mathrm{Sing}\,(l_1 l_2 l_3) = \{D_4\})$ contains 2 old reflectable admissible classes: 235 and 193.

The 38th grade $(\mathbf{R}(l_1 l_2 l_3^2), \mathrm{Sing}\,(l_1 l_2 l_3) = \{D_4^*\})$ contains 1 old reflectable admissible class: 452.

The 39th grade $(\mathbf{R}(l_1 l_2 l_3^2), \mathrm{Sing}\,(l_1 l_2 l_3) = \{3A_1\})$ contains 2 old reflectable admissible classes: 237 and 222.

The 40th grade $(\mathbf{R}(l_1 l_2 l_3^2), \mathrm{Sing}\,(l_1 l_2 l_3) = \{A_1^*\})$ contains 2 old reflectable admissible classes: 449 and 452.

The 41st grade $(\mathbf{R}(l^2 c), \mathrm{Sing}\,(lc) = \{2A_1\})$ contains 6 old reflectable admissible classes: 451, 350, 191, 374, 186 and 222.

The 42nd grade $(\mathbf{R}(l^2 c), \mathrm{Sing}\,(lc) = \{A_3\})$ contains 4 old reflectable admissible classes: 465, 374, 186 and 193.

The 43rd grade $(\mathbf{R}(l^2 c), \mathrm{Sing}\,(lc) = \varnothing)$ contains 3 old reflectable admissible classes: 453, 351 and 193.

The 44th grade $(\mathbf{R}(c^2), \mathrm{Sing}\,(c) = \varnothing)$ contains 4 old reflectable admissible classes: 465, 374, 186 and 193.

In conclusion, the 7th division does not contain new admissible classes.

In conclusion, the 1st–7th divisions contain 667 admissible (397 reflectable and 270 nonreflectable) isotopy classes. There are 20 admissible isotopy classes, $A, B^{\pm}, C, D, E, F,\ G^{\pm}, H^{\pm},\ I, J, K, L, M,\ N^{\pm}, O$ and P (12 reflectable and 8 nonreflectable), that do not contain quartic curves (see Lemma 8). So the isotopy classification of quartic curves contains 647 (385 reflectable and 262 nonreflectable) isotopy classes. The topological classification of pairs ($\mathbf{R}^2$, *quartic curve*) contains 516 classes. $\square$

4. Restrictions.

Lemma 8. *There do not exist polynomials $f(x, y)$ of degree 4 whose sets of real points $\mathbf{R}f$ realize representatives of the isotopy classes $A, B^{\pm}, C, D, E, F,\ G^{\pm}, H^{\pm}, I, K, L,\ M, N^{\pm}, O$ and P.*

Proof. Topological curves from the isotopy classes $A, B^{\pm}, C, D$ that satisfy the theorems of Bezout and Harnack for degree 4 are depicted in Figures 15.1–15.4, where L_∞ is the line at infinity and R is some auxiliary line. If f is a projective algebraic curve of degree d with singular points $z_1, \ldots, z_k$ of the type A_1 or A_2, then according to the first Plücker formula [**46**] the class m^* of the curve (the number of

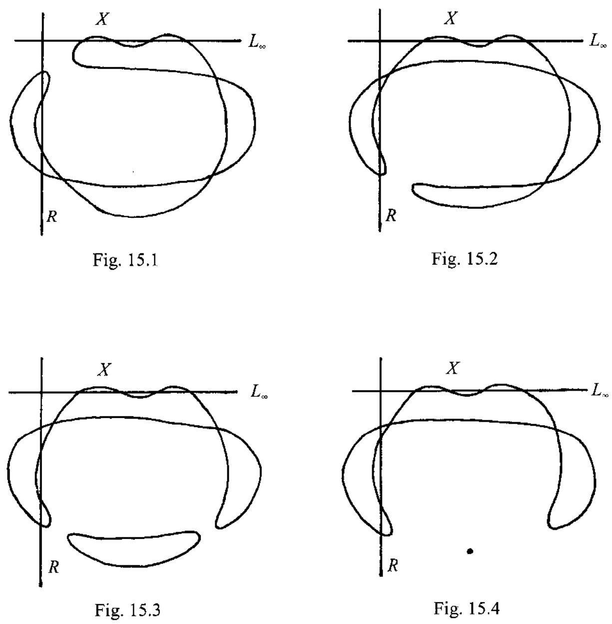


FIGURE 15.

tangent lines to the curve from a point counted properly) is

$$m^*(f) = d(d-1) - \sum_i \kappa(z_i)$$

where $\kappa(z_i)$ is the class of singular point z_i and the sum runs over all singular points of the curve. We consider quartic curves ($d = 4$) with singular points A_1 or A_1^*, for which $\kappa(A_1) = \kappa(A_1^*) = 2$.

We choose the auxiliary lines as follows. It is easy to see that each curve from the admissible classes $A, B^\pm, C, D$ has at least 4 points of inflection. Each curve has three simple arcs. At least two points of inflection lie on the segment X of the curve that is the union of three simple arcs, and at least two lie in the complement of the segment X. We draw the line R through a point of inflection that lies in the complement of X such that there are three points of intersection of the curve, and the line R is in a small enough neighborhood of the point of inflection.

If a quartic curve f belongs to one of the classes $A, B^{\pm}$ or D, then $m^*(f) = 6$. On the other hand, it is easy to see in Figures 15.1, 15.2 and 15.4 that each pencil of lines with center $L_\infty \cap R$ contains at least 8 real tangent lines to the curves. This contradiction proves that the isotopy classes $A, B^{\pm}$ and D do not contain quartic curves. In the same way one can prove that the isotopy classes $E, F, G^{\pm}, H^{\pm}$, $K, L, M, N^{\pm}$ and P do not contain quartic curves.

If a quartic curve f belongs to the class C, then $m^*(f) = 8$. On the other hand, it is easy to see in Figure 15.3 that the pencil of lines with center $L_\infty \cap R$ contains at least 10 real tangent lines to the curve. This contradiction proves that the isotopy class C does not contain quartic curves. In the same way one can prove that the isotopy classes I, J and O do not contain quartic curves. $\square$

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ENDNOTES

1. In the literature on real algebraic geometry, another definition of isotopy equivalence in $\mathbf{R}P^2$ is used. Two topological subspaces T_1 and T_2 in X are called *isotopy equivalent* if the pairs (X, T_1) and (X, T_2) are topologically equivalent. This definition is equivalent to the definition above because every homeomorphism from $\mathbf{R}P^2$ to $\mathbf{R}P^2$ is isotopic to the identity map.

2. According to tradition the words *an affine curve has singular points at infinity* mean that the corresponding projective curve has singular points on the line L_∞. The expressions *the line at infinity is tangent to affine curve*, or *an asymptote is tangent to the curve at an infinite point*, and the like, have an analogous meaning.

3. In this paper we also apply this definition to curves with multiple components. In this case we reduce such a curve to the corresponding curve without multiple components and then apply the definition.

4. Note that the definition of singular-isotopy equivalence differs from Gudkov's classification of the *algebraic-topological type* of an algebraic curve [**15**], [**8**], [**28**], but both definitions generate a one-to-one correspondence between the algebraic-topological types and classes of the singular-isotopy classification. Thus, both definitions lead to the same classification of irreducible conic, cubic and quartic curves.

5. According to Polotovskii (private communication), the numbers 92, 93 and 96 for the singular-isotopy classes of reducible projective quartic curves shown in [**27**], [**28**] and [**29**], respectively, are not correct.

6. This correct number can be found by counting coarse forms in [**10**]–[**14**]; it differs from the number of classes (396) shown in [**10**]. We explain the reason for the difference in [**37**].

7. The restriction in this definition to lines and conics is suggested by their connectedness. One can find a discussion of the subject and a problem in [**39**].

8. In the definition of admissible quartic curve, Gudkov requires satisfying the Klein-Viro formula in addition to Bezout's and Harnack's theorems.

9. We estimate this number in the following manner. We take into account only coarse forms: 349 irreducible plus 95 reducible ones. The average number of singular points of a quartic curve is about 2. The simplest projective quartic curve that has two complex-conjugate double points and does not have real points, generates one affine isotopy class, consisting of the empty quartic curve. We select as one of the most complicated projective quartic curves with two double points, the curve shown in Figure(2.12)28 in Gudkov's paper [**12**], which generates about 40 distinct affine isotopy classes. So the average number of affine isotopy classes that are generated by a projective quartic curve is about 20.5. Thus $(349 + 95) \times 20.5 = 9102$.

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