

SOME APPROXIMATION THEOREMS VIA STATISTICAL CONVERGENCE

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ABSTRACT. In this paper we prove some Korovkin and Weierstrass type approximation theorems via statistical convergence. We are also concerned with the order of statistical convergence of a sequence of positive linear operators.

1. Introduction and Background. Let E be a subset of \mathbf{N} , the set of all natural numbers. The density of E is defined by $\delta(E) := \lim_n (1/n) \sum_{j=1}^n \chi_E(j)$ whenever the limit exists, where χ_E is the characteristic function of E . The number sequence $\alpha = (\alpha_k)$ is statistically convergent to the number L if, for every $\varepsilon > 0$, $\delta\{k \in \mathbf{N} : |\alpha_k - L| \geq \varepsilon\} = 0$, [4], or equivalently there exists a subset $K \subseteq \mathbf{N}$ with $\delta(K) = 1$ and $n_0(\varepsilon)$ such that $k > n_0$ and $k \in K$ imply that $|\alpha_k - L| < \varepsilon$, [5], [10], [12]. In this case we write $st - \lim \alpha_k = L$. It is known that any convergent sequence is statistically convergent, but not conversely. For example, the sequence (α_n) defined by $\alpha_n = \sqrt{n}$ if n is square and $\alpha_n = 0$ otherwise has the property that $st - \lim \alpha_n = 0$.

Some basic properties of statistical convergence are exhibited in [2], [12], [13]. Over the years this concept has been examined in number theory [3], trigonometric series [14], probability theory [7], optimization [11], measure theory [10] and summability theory [2], [5], [6].

Statistical convergence has not been examined in approximation theory so far. It is the purpose of this paper to consider it in some problems of approximation theory. Especially we shall be concerned with the Korovkin approximation theory which deals with the problem of approximation of function f by the sequence $(A_n(f, x))$ where (A_n) is a sequence of positive linear operators [8], [9].

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This theory has important applications in the theory of polynomial approximation, in various areas of functional analysis, in numerical solutions of differential and integral equations, etc. (see [1]).

In Section 2, we prove Korovkin and Weierstrass type approximation theorems via statistical convergence while in Section 3 we are concerned with the order of statistical convergence of a sequence of positive linear operators. Section 4 is also devoted to a Korovkin type result for a sequence of positive linear operators acting on $L_p[a, b]$.

2. Korovkin and Weierstrass type approximation theorems via statistical convergence. In this section, using statistical convergence, we prove a Korovkin theorem and Weierstrass type approximation theorem.

In order to formulate the classical Korovkin's theorem, we pause to collect some notation.

By $C_M[a, b]$ we denote the space of all functions f which are continuous in every point of the interval $[a, b]$ and bounded on the entire line, that is,

$$|f(x)| \leq M_f, \quad -\infty < x < \infty,$$

where M_f is a constant depending on f . Let (A_n) be a sequence of positive linear operators (that is, $A_n(f, x) \geq 0$ if $f(x) \geq 0$), acting from $C_M[a, b]$ to the space $B[a, b]$ of all bounded functions on $[a, b]$. Recall that $B[a, b]$ is a Banach space with norm $\|f\|_B := \sup_{a \leq x \leq b} |f(x)|$, $f \in B[a, b]$. As usual we write $A_n(f, x)$ instead of $A_n(f(t), x)$.

With this terminology, the classical Korovkin [9] theorem claims that if the sequence of positive linear operators $A_n : C_M[a, b] \rightarrow B[a, b]$ satisfies the conditions

- (a) $\|A_n(1, x) - 1\|_B \rightarrow 0$, as $n \rightarrow \infty$,
- (b) $\|A_n(t, x) - x\|_B \rightarrow 0$, as $n \rightarrow \infty$,
- (c) $\|A_n(t^2, x) - x^2\|_B \rightarrow 0$, as $n \rightarrow \infty$,

then for any function $f \in C_M[a, b]$, we have

$$\|A_n(f, x) - f(x)\|_B \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

In analogy with this result, we have

Theorem 1. *If the sequence of positive linear operators $A_n : C_M[a, b] \rightarrow B[a, b]$ satisfies the conditions*

- (1) $st - \lim \|A_n(1, x) - 1\|_B = 0$
- (2) $st - \lim \|A_n(t, x) - x\|_B = 0$
- (3) $st - \lim \|A_n(t^2, x) - x^2\|_B = 0$

then for any function $f \in C_M[a, b]$, we have

- (4) $st - \lim \|A_n(f, x) - f(x)\|_B = 0.$

Proof. We follow the proof of the Korovkin theorem up to a certain stage. Since f is bounded on the whole real axis, we can write

$$|f(t) - f(x)| \leq 2M, \quad -\infty < t, x < \infty.$$

Also, since f is continuous on $[a, b]$, we have

$$|f(t) - f(x)| < \varepsilon$$

for all t, x satisfying $|t - x| \leq \delta$.

Therefore for all $t \in (-\infty, \infty)$ and all $x \in [a, b]$ we get

$$(5) \quad |f(t) - f(x)| < \varepsilon + \frac{2M}{\delta^2} (t - x)^2$$

where δ is a fixed real number.

Now we can write

$$A_n(f, x) - f(x) = A_n(f(t) - f(x), x) + f(x)(A_n(1, x) - 1)$$

and therefore

$$\begin{aligned} \|A_n(f, x) - f(x)\|_B &\leq \left(\varepsilon + M + \frac{2M}{\delta^2} \right) \|A_n(1, x) - 1\|_B \\ &\quad + \frac{4Mb}{\delta^2} \|A_n(t, x) - x\|_B + \frac{2M}{\delta^2} \|A_n(t^2, x) - x^2\|_B \\ &\leq K_1 (\|A_n(1, x) - 1\|_B + \|A_n(t, x) - x\|_B \\ &\quad + \|A_n(t^2, x) - x^2\|_B) \end{aligned}$$

where

$$K_1 = \max\left(\varepsilon + M + \frac{2M}{\delta^2}, \frac{4Mb}{\delta^2}\right).$$

The last inequality shows that, for any $\varepsilon' > 0$,

$$(6) \quad \begin{aligned} & |\{n \leq N : \|A_n(f, x) - f(x)\|_B \geq \varepsilon'\}| \\ & \leq \left| \left\{ n \leq N : \|A_n(1, x) - 1\|_B + \|A_n(t, x) - x\|_B \right. \right. \\ & \quad \left. \left. + \|A_n(t^2, x) - x^2\|_B \geq \frac{\varepsilon'}{K_1} \right\} \right|. \end{aligned}$$

Now write

$$D := \left\{ n : \|A_n(1, x) - 1\|_B + \|A_n(t, x) - x\|_B + \|A_n(t^2, x) - x^2\|_B \geq \frac{\varepsilon'}{K_1} \right\},$$

$$D_1 := \left\{ n : \|A_n(1, x) - 1\|_B \geq \frac{\varepsilon'}{3K_1} \right\},$$

$$D_2 := \left\{ n : \|A_n(t, x) - x\|_B \geq \frac{\varepsilon'}{3K_1} \right\},$$

$$D_3 := \left\{ n : \|A_n(t^2, x) - x^2\|_B \geq \frac{\varepsilon'}{3K_1} \right\}.$$

Then it is easy to see that $D \subset D_1 \cup D_2 \cup D_3$.

Thus (6) yields that

$$(7) \quad \begin{aligned} & |\{n \leq N : \|A_n(f, x) - f(x)\|_B \geq \varepsilon'\}| \\ & \leq \left| \left\{ n \leq N : \|A_n(1, x) - 1\|_B \geq \frac{\varepsilon'}{3K_1} \right\} \right| \\ & \quad + \left| \left\{ n \leq N : \|A_n(t, x) - x\|_B \geq \frac{\varepsilon'}{3K_1} \right\} \right| \\ & \quad + \left| \left\{ n \leq N : \|A_n(t^2, x) - x^2\|_B \geq \frac{\varepsilon'}{3K_1} \right\} \right| \end{aligned}$$

and, using (1), (2) and (3), we get (4) and the proof is complete.

We note that the sequence of classical Bernstein polynomials

$$B_n(f, x) := \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}; \quad 0 \leq x \leq 1,$$

satisfies the conditions of the Korovkin theorem, so it also satisfies the present Theorem 1. But we now exhibit an example of a sequence of positive linear operators satisfying the conditions of Theorem 1 but that does not satisfy the conditions of the classical Korovkin theorem.

Example 1. Consider the sequence (P_n) defined by $P_n : C_M[0, 1] \rightarrow B[0, 1]$, $P_n(f, x) = (1 + \alpha_n)B_n(f, x)$ where (B_n) is the sequence of Bernstein polynomials and (α_n) is the unbounded statistically convergent sequence given in Section 1. It is known that

$$B_n(1, x) = 1, \quad B_n(t, x) = x \quad \text{and} \quad B_n(t^2, x) = x^2 + \frac{x - x^2}{n}.$$

Hence, for the sequence (P_n) , conditions (1), (2) and (3) are evidently satisfied. So we have

$$st - \lim \|P_n(f, x) - f(x)\|_B = 0.$$

On the other hand, since $B_n(f, 0) = f(0)$, we have $P_n(f, 0) = (1 + \alpha_n)f(0)$ and therefore

$$(8) \quad \|P_n(f, x) - f(x)\| \geq |P_n(f, 0) - f(0)| = \alpha_n |f(0)|.$$

Combining (8) with the fact that $\limsup_{n \rightarrow \infty} \alpha_n = \infty$, we conclude that (P_n) does not satisfy the classical Korovkin theorem which proves the claim.

In Section 1 we have noted that a convergent sequence is statistically convergent but not conversely. So, the following result may be of interest.

Theorem 2. *Let the sequence of positive linear operators $A_n : C_M[a, b] \rightarrow B[a, b]$ satisfy conditions (2) and (3) of Theorem 1 and the condition*

$$(9) \quad \lim_{n \rightarrow \infty} \|A_n(1, x) - 1\|_B = 0.$$

Then for any function $f \in C_M[a, b]$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \|A_n(f, x) - f(x)\|_B = 0.$$

Proof. It follows from condition (9) that there exists a constant M_1 such that, for all n the inequality

$$\|A_n(1, x)\|_B \leq M_1$$

holds. Therefore, for any $f \in C_M[a, b]$ and any $n = 1, 2, 3, \dots$, we get

$$(10) \quad \|A_n(f, x) - f(x)\|_B \leq \|f\|_C \|A_n(1, x)\|_B + \|f\|_C \leq M(M_1 + 1).$$

Furthermore, since (9) implies (1) we immediately get by Theorem 1 that

$$(11) \quad st - \lim \|A_n(f, x) - f(x)\|_B = 0.$$

It is known [13] that any bounded statistically convergent sequence is Cesàro summable. Hence, (10) and (11) yield the result.

We conclude this section with the following observation. The Weierstrass approximation theorem asserts that if f is a continuous function on $[a, b]$, then there is a sequence of polynomials (P_n) such that

$$(12) \quad \lim_n \|P_n - f\|_{C[a,b]} = 0.$$

Hence we necessarily have

$$(13) \quad st - \lim \|P_n - f\|_{C[a,b]} = 0.$$

Note that (13) may be interpreted as saying that the sequence of polynomials (P_n) is “statistically uniformly convergent” to the continuous function f on $[a, b]$. One may now ask if there is a sequence of polynomials (P_n) such that (13) holds but (12) does not. Actually, the sequence (P_n) presented in Example 1 answers this question in the affirmative. We state it formally as follows:

Proposition 3. *If f is a continuous function on $[a, b]$, then there is a sequence of polynomials which is statistically uniformly convergent to f on this interval but not uniformly convergent.*

3. The order of statistical convergence. In this section we deal with the order of statistical convergence of a sequence of positive linear operators.

Definition 4. The number sequence $\alpha = (\alpha_k)$ is statistically convergent to the number L with degree $0 < \beta < 1$ if, for each $\varepsilon > 0$,

$$\lim_n \frac{|\{k \leq n : |\alpha_k - L| \geq \varepsilon\}|}{n^{1-\beta}} = 0.$$

In this case we write

$$\alpha_k - L = st - o(k^{-\beta}), \quad \text{as } k \rightarrow \infty.$$

We shall now find the order of statistical convergence of the sequence of positive linear operators in Theorem 1.

Theorem 5. *Let the sequence of positive linear operators $A_n : C[a, b] \rightarrow B[a, b]$ satisfy the conditions*

$$(14) \quad \|A_n(1, x) - 1\|_B = st - o(n^{-\beta_1})$$

$$(15) \quad \|A_n(t, x) - x\|_B = st - o(n^{-\beta_2})$$

$$(16) \quad \|A_n(t^2, x) - x^2\|_B = st - o(n^{-\beta_3})$$

as $n \rightarrow \infty$.

Then for any function $f \in C_M[a, b]$, we have

$$\|A_n(f, x) - f(x)\|_B = st - o(n^{-\beta}), \quad \text{as } n \rightarrow \infty,$$

where $\beta = \min(\beta_1, \beta_2, \beta_3)$.

Proof. As in the proof of Theorem 1 we can write inequality (7) as follows:

$$\begin{aligned} & \frac{|\{n \leq m : \|A_n(f, x) - f(x)\|_B \geq \varepsilon'\}|}{m^{1-\beta}} \\ & \leq \frac{|\{n \leq m : \|A_n(1, x) - 1\|_B \geq (\varepsilon'/3K_1)\}|}{m^{1-\beta_1}} \frac{m^{1-\beta_1}}{m^{1-\beta}} \\ & \quad + \frac{|\{n \leq m : \|A_n(t, x) - x\|_B \geq (\varepsilon'/3K_1)\}|}{m^{1-\beta_2}} \frac{m^{1-\beta_2}}{m^{1-\beta}} \\ & \quad + \frac{|\{n \leq m : \|A_n(t^2, x) - x^2\|_B \geq (\varepsilon'/3K_1)\}|}{m^{1-\beta_3}} \frac{m^{1-\beta_3}}{m^{1-\beta}} \end{aligned}$$

from which the desired result follows.

As an application we reconsider the classical Bernstein polynomials. Recalling the fact that

$$B_n(1, x) = 1, \quad B_n(t, x) = x \quad \text{and} \quad B_n(t^2, x) = x^2 + \frac{x - x^2}{n},$$

we see that (14) and (15) hold at once. Furthermore, the set

$$\{n : \|B_n(t^2, x) - x^2\|_B \geq \varepsilon'\} = \left\{n : \frac{1}{4n} \geq \varepsilon'\right\},$$

is a finite subset of the natural numbers, and hence (16) also holds.

So we have the following

Corollary 6. *If f is a continuous function on $[0, 1]$, then for Bernstein polynomials and $\beta \in (0, 1)$,*

$$\|B_n(f, x) - f(x)\|_B = st - o(n^{-\beta}), \quad \text{as } n \rightarrow \infty,$$

holds.

4. Statistical approximation by positive linear operators in $L_p[a, b]$. In this section we shall consider the sequence of positive linear operators $A_n : L_p[a, b] \rightarrow L_p[a, b]$ and study a Korovkin type theorem via statistical convergence. Here, of course, $L_p[a, b]$ has the usual meaning.

Theorem 7. *Let (A_n) be the sequence of positive linear operators $A_n : L_p[a, b] \rightarrow L_p[a, b]$ and let the sequence $\{\|A_n\|\}$ be uniformly bounded. If*

$$(17) \quad st - \lim \|A_n(t^\nu, x) - x^\nu\|_{L_p} = 0, \quad \nu = 0, 1, 2,$$

then for any function $f \in L_p[a, b]$, we have

$$st - \lim \|A_n(f, x) - f(x)\|_{L_p} = 0.$$

Proof. By (17), given $\varepsilon > 0$, there exist $n_\nu(\varepsilon)$, $\nu = 0, 1, 2$, and subsets K_ν , $\nu = 0, 1, 2$, of density 1 such that

$$(18) \quad \|A_n(t^\nu, x) - x^\nu\|_{L_p} < \varepsilon$$

holds for all $n \in K_\nu$ and $n > n_\nu$, $\nu = 0, 1, 2$.

Since $\delta(K_0 \cap K_1 \cap K_2) = 1$, the inequalities in (18) also hold for $n \in K := K_0 \cap K_1 \cap K_2$ and $n > \max\{n_0, n_1, n_2\}$.

By hypothesis, there is a constant $M > 0$ such that $\|A_n\|_{L_p \rightarrow L_p} \leq M$, $n = 1, 2, \dots$. Since $C[a, b]$ is dense in $L_p[a, b]$, given $f \in L_p[a, b]$, there exists $g \in C[a, b]$ such that $\|f - g\|_{L_p} < \varepsilon$. Hence we have

$$(19) \quad \begin{aligned} \|A_n(f, x) - f(x)\|_{L_p} &\leq \|A_n(f - g, x)\|_{L_p} \\ &\quad + \|A_n(g, x) - g(x)\|_{L_p} + \|f - g\|_{L_p} \\ &\leq \varepsilon(1 + M) + \|A_n(g, x) - g(x)\|_{L_p}. \end{aligned}$$

By the continuity of g on $[a, b]$, we get $|g(x)| \leq C$ for all $x \in [a, b]$ and for some C . Thus,

$$\|A_n(g, x) - g(x)\|_{L_p} \leq \|A_n(|g(t) - g(x)|, x)\|_{L_p} + C\|A_n(1, x) - 1\|_{L_p}.$$

Since $g \in C[a, b]$, we can write, as in (5), for all $t, x \in [a, b]$ that

$$|g(t) - g(x)| < \varepsilon + 2H\delta^{-2}(t - x)^2$$

where H and δ are positive constants. Hence

$$(20) \quad \begin{aligned} &\|A_n(|g(t) - g(x)|, x)\|_{L_p} \\ &\leq \varepsilon\|A_n(1, x)\|_{L_p} + \|A_n((t - x)^2, x)\|_{L_p} \\ &\leq \varepsilon(\|A_n(1, x) - 1\|_{L_p} + 1) + \|A_n(t^2, x) - x^2\|_{L_p} \\ &\quad + 2b\|A_n(t, x) - x\|_{L_p} + b^2\|A_n(1, x) - 1\|_{L_p}. \end{aligned}$$

It follows from (18), for $n \in K$ and $n > \max\{n_0, n_1, n_2\}$ that the last member of inequality (20) can be made as small as we please, so by (19), we conclude that

$$st - \lim \|A_n(f, x) - f(x)\|_{L_p} = 0,$$

whence the result.

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