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THE EXTENSION OF p-ADIC COMPACT OPERATORS

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ABSTRACT. This paper is devoted to study the non-Archimedean locally convex spaces X having the following property: For all non-Archimedean locally convex spaces $Z \supset Y$, every compact operator $T : Y \to X$ has an extension to a compact operator $\overline{T} : Z \to X$. The results obtained depend strongly on the spherical completeness of the ground field. On the other hand, the situation here is completely different from its Archimedean counterpart. Our results also lead to some new characterizations of spherically complete fields and of discretely valued fields.

Introduction. In [**16**], the authors characterize the (real or complex) Banach spaces X having the following property:

($*$) For all Banach spaces $Z \supset Y$ every compact operator $T : Y \to X$ has an extension to a compact operator $\overline{T} : Z \to X$.

They prove that X has property ($*$) if and only if it is an $\mathcal{L}_\infty$-space.

Our first goal was to study this property in the non-Archimedean case. It turned out that not only was the situation completely different here, but also that the answer was surprisingly simpler. We found, indeed, that when the ground field $\mathbf{K}$ is spherically complete, then every non-Archimedean Banach space X over $\mathbf{K}$ has property ($*$), and if $\mathbf{K}$ is not spherically complete, no $X \neq \{0\}$ has this property.

We therefore decided to look at the problem in the much more general frame of non-Archimedean locally convex spaces and called the property CEP (Definition 2.1). Here the situation is more complicated. If $\mathbf{K}$ is not spherically complete, still there are no nontrivial locally convex spaces over $\mathbf{K}$ with the CEP (Section 3). On the other hand, if $\mathbf{K}$ is spherically complete, lots of spaces have the CEP but not all of them (Section 4), and the situation is still different in the special case when the valuation on $\mathbf{K}$ is discrete (Section 5).

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Our results also lead to some new characterizations of spherically complete fields (Theorem 4.11) and of discretely valued fields (Theorem 5.3). Finally it turns out that the CEP is related to a locally convex version of the notion of weakly injective normed space introduced in [**23**, Theorem 4.9].

1. Preliminaries. Throughout, $\mathbf{K}$ is a non-Archimedean nontrivially valued field that is complete under the metric induced by its valuation $|.| : \mathbf{K} \to [0, \infty)$. Let X be a $\mathbf{K}$-vector space. By $\dim X$ we denote the algebraic dimension of X and by X^* its algebraic dual. A subset A of X is called *absolutely convex* if A is a module over the valuation ring. A set that is either empty or the translation of an absolutely convex set is called a *convex* set. The linear hull of a set $B \subset X$ is written $[B]$, its absolutely convex hull $\operatorname{co} B$. If A is an absolutely convex subset of X, we will denote by X_A the vector space $[A]$ equipped with the *Minkowski functional* p_A of A (i.e., for $x \in [A]$, $p_A(x) = \inf\{|\lambda| : x \in \lambda A\}$). If p is a (non-Archimedean) seminorm on X, we denote by X_p the normed space $X/\operatorname{Ker} p$ endowed with the norm given by $\|\pi_p(x)\| = p(x)$, $x \in X$, where $\pi_p : X \to X_p$ is the canonical quotient map. We say that p is a polar seminorm if $p = \sup\{|f| : f \in X^*, |f| \leq p\}$.

In the sequel X, Y, Z will be Hausdorff locally convex spaces over $\mathbf{K}$. $\mathcal{P}_X$ will denote the family of all (non-Archimedean) continuous seminorms on X. A complete metrizable locally convex space is called a *Fréchet* space. X is called *quasicomplete* if every closed bounded subset of X is complete. We say that X *contains a copy* of Y if X contains a linear subspace X_1 that is linearly homeomorphic to Y. If there is a continuous linear projection from X onto X_1, then we say that X_1 is *complemented* in X. If X' is the topological dual of X, we denote by $\sigma(X, X')$ the weak topology on X and by $\sigma(X', X)$ the weak*-topology on X' associated with the natural dual pair $\langle X, X' \rangle$. The canonical map $X \to (X')^*$ is denoted by J_X. X is called *polar* if its topology is defined by a family of polar seminorms. If $\mathbf{K}$ is spherically complete, then every locally convex space X over $\mathbf{K}$ is polar and every weakly convergent sequence in X is convergent for the original topology (see [**21**]); in particular, X is weakly sequentially complete if and only if X is sequentially complete.

A *compactoid* in X is a set $B \subset X$ such that for each zero neighborhood U in X, there is a finite set $F \subset X$ such that $B \subset U + \operatorname{co} F$.

An absolutely convex subset A of X is called *c-compact* if for every collection $\mathcal{C}$ of closed convex subsets of A with the finite intersection property we have $\cap\mathcal{C} \neq \varnothing$. If $\mathbf{K}$ is spherically complete and A is an absolutely convex subset of X, then A is c-compact and bounded if and only if A is compactoid and complete ([**20**, Theorem 9]). A linear operator $T : X \to Y$ is called *compact* if there is a zero neighborhood V in X such that $T(V)$ is compactoid in Y. We denote by $C(X,Y)$ the set of the compact operators from X to Y. Clearly $C(X,Y)$ is a linear subspace of $L(X,Y)$, the vector space of all continuous linear operators from X to Y endowed with the canonical norm given by

$$\|T\| = \sup\left\{\frac{\|Tx\|}{\|x\|} : x \in X, x \neq 0\right\}, \quad T \in L(X,Y).$$

A particularly interesting class of locally convex spaces is formed by the perfect sequence spaces endowed with the associated normal topology. Recall that a sequence space Λ is called *perfect* if $\Lambda^{**} = \Lambda$, where $\Lambda^* := \{(b_n)_n \in K^{\mathbf{N}} : \lim_n a_n b_n = 0 \text{ for all } (a_n)_n \in \Lambda\}$ is the Köthe-dual of Λ. Also, the *normal topology* on Λ is the topology $n(\Lambda,\Lambda^*)$ defined by the family of seminorms $\{p_b : b \in \Lambda^*\}$, where for each $b = (b_n)_n \in \Lambda^*$, p_b is defined by $p_b(a) = \sup_n |a_n b_n|$, $a = (a_n)_n \in \Lambda$. Every $a = (a_n)_n \in \Lambda$ can be written uniquely as $a = \sum_n a_n e_n$, where for each n, e_n is the sequence with 1 in the nth place and zeros elsewhere. For a perfect sequence Λ, a (separating) bilinear form on the dual pair (Λ,Λ^*) is defined by

$$\langle a,b\rangle = \sum_n a_n b_n, \quad a = (a_n)_n \in \Lambda,\ b = (b_n)_n \in \Lambda^*.$$

We shall denote by $\sigma(\Lambda,\Lambda^*)$ the weak topology on Λ associated with this dual pair. Since Λ is $n(\Lambda,\Lambda^*)$-complete, it follows that Λ is also $\sigma(\Lambda,\Lambda^*)$-sequentially complete.

This kind of space plays an important role in the set-up of p-adic quantum mechanics (see, e.g., [**9**], [**10**] and [**14**]). For instance, if B is an infinite matrix consisting of strictly positive real numbers b_n^k, $n,k \in \mathbf{N}$, satisfying $b_n^k \le b_n^{k+1}$ for all n,k, then the non-Archimedean Köthe space $K(B)$ associated with the matrix B and defined by

$$K(B) = \{(\alpha_n)_n \in K^{\mathbf{N}} : \lim_n |\alpha_n| b_n^k = 0,\ \text{for all } k = 1,2,\dots\}$$

is a perfect sequence space which is a Fréchet space (see [**5**]). For $b_n^k = k^n$, $K(B)$ is the space of entire functions on $\mathbf{K}$, which is needed for the definition of a non-Archimedean Laplace transform in [**9**] and [**14**], and for the definition of a non-Archimedean Fourier transform in [**10**].

Now let $(X, \|.\|)$ be a normed space over $\mathbf{K}$. By B_X we denote the closed unit ball of X, i.e., $B_X = \{x \in X : \|x\| \leq 1\}$. A sequence $(x_n)_n$ in X is called *t-orthogonal* ($t \in (0, 1]$) if

$$\|\lambda_1 x_1 + \cdots + \lambda_n x_n\| \geq t \max(\|\lambda_1 x_1\|, \ldots, \|\lambda_n x_n\|)$$

for all $\lambda_1, \ldots, \lambda_n \in \mathbf{K}$, $n \in \mathbf{N}$.

For terms that are still unexplained, see [**21**] and [**23**].

2. Definition and basic facts.

Definition 2.1. We say that X has the *compact extension property* (CEP for short) if, for every pair of locally convex spaces Y, Z with $Y \subset Z$ and every $T \in C(Y, X)$, there exists $\overline{T} \in C(Z, X)$ that extends T.

We will see that in the above definition it is enough to assume that Y and Z are normed spaces (Proposition 2.3). To this end, we need the following lemma.

Lemma 2.2. *If $T \in C(Y, X)$, then there exist $p \in \mathcal{P}_Y$ and $T_p \in C(Y_p, X)$ such that $T = T_p \circ \pi_p$.*

Proof. There exists an absolutely convex zero neighborhood U in Y such that $T(U)$ is compactoid in X. Let $p \in \mathcal{P}_Y$ be the Minkowski functional of U. Then $T_p : Y_p \to X$ defined by $T_p(\pi_p(y)) = T(y)$, $y \in Y$, satisfies the required conditions.

Proposition 2.3. *If, for every pair of normed spaces E, F with $E \subset F$ and for every $T \in C(E, X)$ there exists $\overline{T} \in C(F, X)$ that extends T, then for every pair of locally convex spaces Y, Z with $Y \subset Z$ and every $S \in C(Y, X)$, there exists $\overline{S} \in C(Z, X)$ that extends S. So,* *from now on when using Definition* 2.1*, we can restrict ourselves to normed spaces* Y, Z*.*

Proof. By Lemma 2.2, for every $S \in C(Y, X)$, there exist $p \in \mathcal{P}_Y$ and $S_p \in C(Y_p, X)$ such that $S = S_p \circ \pi_p$. We can assume that p is the restriction to $Y, q \mid Y$ for some $q \in \mathcal{P}_Z$. Consider the linear isometry $i : Y_p \to Z_q$, $\pi_p(y) \mapsto \pi_q(y)$, $y \in Y$, and set $\overline{Y}_p := i(Y_p)$ and $\overline{S}_p := S_p \circ \bar{i}^{-1}$ where $\bar{i} : Y_p \to \overline{Y}_p$ is the restriction of the map i to its image. Then $\overline{S}_p \in C(\overline{Y}_p, X)$ and, by assumption, there is an $\overline{\overline{S}}_p \in C(Z_q, X)$ extending $\overline{S}_p$. It follows that $\overline{S} := \overline{\overline{S}}_p \circ \pi_q$ meets the requirements.

It follows directly from Definition 2.1 and the properties of compactoid sets (see [**6**]) that

Proposition 2.4. (i) *If* X *has the* CEP*, then every locally convex space linearly homeomorphic to* X *and every complemented subspace of* X *has the* CEP*.*

(ii) *If* $\{X_i\}_{i \in I}$ *is a family of locally convex spaces having the* CEP*, then* $\prod_{i \in I} X_i$ *endowed with the product topology has the* CEP*.*

Later (in Examples of Section 4) we will see that not every subspace of a space with the CEP has the CEP.

3. The nonspherically complete case. The first goal of this section is to prove that when $\mathbf{K}$ is not spherically complete, there are no nontrivial locally convex spaces over $\mathbf{K}$ with the CEP (Theorem 3.3). For that we need some preliminary results.

Lemma 3.1. *Let* X, Y *be locally convex spaces with* $X \neq \{0\}$*. Then* $C(Y, X) = \{0\}$ *if and only if* $Y' = \{0\}$*.*

Proof. We only have to prove the "if." Assume that $Y' = \{0\}$. Let $T : Y \to X$ be a compact linear operator and take $q \in \mathcal{P}_X$. Then $\pi_q \circ T \in C(Y, X_q)$ and, by Lemma 2.2 there exist $p \in \mathcal{P}_Y$ and $T_p \in C(Y_p, X_q)$ such that $\pi_q \circ T = T_p \circ \pi_p$. Now, as $Y' = \{0\}$, we have that $(Y_p)' = \{0\}$ and hence, by [**13**, Theorem 2.3], $T_p = 0$. Therefore,

$\pi_q \circ T = 0$ for all $q \in \mathcal{P}_X$. Since X is Hausdorff, it follows that $T = 0$.

Lemma 3.2. *Suppose* $\mathbf{K}$ *is not spherically complete. Let* X, Y *be locally convex spaces over* $\mathbf{K}$ *with* $X \neq \{0\}$. *Then the following are equivalent.*

(i) *For every locally convex space* Z *containing a copy* Y_1 *of* Y *and every* $T \in C(Y_1, X)$, *there exists a* $\overline{T} \in C(Z, X)$ *that extends* T.

(ii) $Y' = \{0\}$.

Proof. The implication (ii) $\Rightarrow$ (i) is a direct consequence of Lemma 3.1.

(i) $\Rightarrow$ (ii). Since Y is Hausdorff, Y is the linearly homeomorphic to a subspace Y_1 of the locally convex space $Z := \prod_{p \in \mathcal{P}_Y} \breve{Y}_p$ where, for each $p \in \mathcal{P}_Y$, $\breve{Y}_p$ is the spherical completion of Y_p. Note that ([**23**, Corollary 4.3]) $(\breve{Y}_p)' = \{0\}$ for all $p \in \mathcal{P}_Y$ and so $Z' = \{0\}$. By (i) and Lemma 3.1, it follows that $C(Y_1, X) = \{0\}$ and, again applying Lemma 3.1, we conclude that $Y' = \{0\}$.

Theorem 3.3. *If* $\mathbf{K}$ *is not spherically complete, then no locally convex space* $X \neq \{0\}$ *over* $\mathbf{K}$ *has the* CEP*.*

Proof. Suppose X has the CEP. Then, apply Lemma 3.2 for $Y = \mathbf{K}$. This gives $\mathbf{K}' = \{0\}$, a contradiction.

One could weaken the CEP by making smaller the category of locally convex spaces in which it is defined, considering only normed or Banach spaces. The new goal would be to find out whether for these weaker conditions we obtain nontrivial spaces satisfying them. But, as it is shown in Proposition 2.3, this does not have any effect at all.

We now consider the CEP in the category of polar spaces (we call it p-CEP). Then again for $\mathbf{K}$ nonspherically complete, there are no nontrivial examples of spaces with the p-CEP. We have indeed:

Theorem 3.4. *If there exists a polar space* $X \neq \{0\}$ *over* $\mathbf{K}$ *with the* p-CEP*, then* $\mathbf{K}$ *is spherically complete.*

Proof. Let $X \neq \{0\}$ be as above, and take $x \in X \setminus \{0\}$. By [**23**, Theorem 4.15], we have to prove that every $S \in L(c_0, [x])$ has an extension $\overline{S} \in L(l^\infty, [x])$.

So take $S \in L(c_0, [x])$. If $i : [x] \to X$ is the canonical inclusion from $[x]$ into X, then $T := i \circ S$ is a compact linear operator from c_0 to X. By assumption, T has an extension $\overline{T} \in C(l^\infty, X)$. Also, by polarity of X and [**11**, Lemma 2.2], there exists a continuous linear projection $P : X \to [x]$. Then $\overline{S} := P \circ \overline{T}$ satisfies the required conditions.

Hence, a polar space X has the p-CEP if and only if X has the CEP.

Also observe that in this case (using the polar counterpart of Proposition 2.3) we can, in the definition of the p-CEP, restrict ourselves to Y, Z polar normed spaces.

On the other hand, in the polar case, the local behavior for the CEP stated in Lemma 3.2 does not hold anymore as it is shown in the next example.

Example. Suppose $\mathbf{K}$ is not spherically complete. Let X be any locally convex space (over $\mathbf{K}$) and take $Y = l^\infty$. Clearly Y is a polar space with nontrivial dual.

Let Z be a polar locally convex space containing a copy Y_1 of Y, and let $T : Y_1 \to X$ be a compact linear operator. By [**19**, Lemma 4.6], Y_1 is complemented in Z and so T admits a compact linear extension $\overline{T} : Z \to X$.

4. The spherically complete case. In this section we assume that $\mathbf{K}$ is spherically complete.

Theorem 4.1. *Every metrizable locally convex space X over $\mathbf{K}$ has the* CEP.

Proof. First assume that X is a normed space, and let Y, Z be normed spaces with $Y \subset Z$ and $T \in C(Y, X)$. There exists a sequence $(f_n)_n$ in Y' and a t-orthogonal sequence $(x_n)_n$ in $X (t \in (0, 1))$ such that

$\|f_n\|\|x_n\|$ tends to zero, and

$$T(y) = \sum_{n=1}^{\infty} f_n(y)x_n, \quad y \in Y$$

([**13**, Theorem 2.3]). Since **K** is spherically complete, every f_n has an extension $g_n \in Z'$ with $\|g_n\| = \|f_n\|$. Then $\overline{T} : Z \to X$, $z \in Z \mapsto \sum_{n=1}^{\infty} g_n(z)x_n \in X$ is, again by [**13**, Theorem 2.3], a compact linear operator which clearly extends T.

Now assume that X is a metrizable locally convex space, and let Y, Z and $T \in C(Y, X)$ be as above. There exists an absolutely convex bounded subset B of X such that $T(B_Y) \subset B$ and the topologies induced by X and X_B on $T(B_Y)$ coincide ([**2**, p. 121]), which implies that $T(B_Y)$ is compactoid in X_B. Hence, $T_B : Y \to X_B$, $y \in Y \mapsto T(y) \in X_B$ is a compact linear operator. Since the normed space X_B has the CEP, we derive the existence of $\overline{T}_B \in C(Z, X_B)$ that extends T_B. Set $\overline{T} := i_B \circ \overline{T}_B$, where i_B is the canonical inclusion $X_B \to X$. Then $\overline{T}$ is a compact linear operator from Z to X that extends T.

Therefore, in the category of normed spaces, we have that if **K** is spherically complete, then every normed space over **K** has the CEP (Theorem 4.1) and that when **K** is not spherically complete there are no nontrivial normed spaces over **K** with the CEP (Theorem 3.3). These facts show that the non-Archimedean situation is in sharp contrast with the classical one for Banach spaces over **R** or **C**. Indeed, it was proved in [**16**, Theorem 4.1] that a real or complex Banach space has the CEP in the category of Banach spaces if and only if it is an $\mathcal{L}_\infty$-space.

Another interesting class of locally convex spaces with the CEP is formed by the (weakly) sequentially complete locally convex spaces with an "orthogonal" basis (Theorem 4.2).

Recall ([**3**]) that a sequence $(x_n)_n$ in a locally convex space X is a *Schauder basis* for X if every $x \in X$ can be written uniquely as $x = \sum_{n=1}^{\infty} \lambda_n x_n$ where the coefficient functionals $f_n : x \in X \mapsto \lambda_n \in \mathbf{K}$ are continuous. If in addition the topology of X is defined by a family $\mathcal{P}_X$ of non-Archimedean seminorms satisfying the condition

$$\text{if } x = \sum_{n=1}^{\infty} \lambda_n x_n, \quad \text{then } p(x) = \max_n p(\lambda_n x_n) \quad \text{for all } p \in \mathcal{P}_X,$$

then $(x_n)_n$ is said to be an "*orthogonal*" *basis* in X. Every Fréchet space with a Schauder basis is (weakly) sequentially complete and its Schauder basis is "orthogonal." For this last class of spaces we have

Theorem 4.2. *Every* (*weakly*) *sequentially complete locally convex space* X *over* $\mathbf{K}$ *with an* "orthogonal" *basis has the* CEP.

Proof. Let $(x_n)_n$ be an "orthogonal" basis of X with associated coefficient functionals $f_n \in X'$. Let Y, Z be normed spaces with $Y \subset Z$ and $T \in C(Y, X)$. By using Theorem 5.2 of [**7**], we deduce the existence of an $a \in X$ such that $T(B_Y) \subset \hat{a}$ where $\hat{a} = \{x \in X : |f_n(x)| \leq |f_n(a)| \text{ for all } n\}$.

For each $n \in \mathbf{N}$, let $g_n := f_n \circ T \in Y'$. Then $\|g_n\| \leq |\pi|\,|f_n(a)|$ for all n, where $\pi \in \mathbf{K}$ with $|\pi| > 1$. Also, for $y \in Y$, $T(y) = \sum_n g_n(y) x_n$. Since $\mathbf{K}$ is spherically complete, every g_n has an extension $h_n \in Z'$ with $\|h_n\| = \|g_n\|$. Then $\overline{T} : Z \to X$, $z \in Z \to \sum_{n=1}^{\infty} h_n(z) x_n \in X$ is a well-defined (by sequential completeness of X) linear extension of T satisfying

$$\overline{T}(B_Z) \subset \{x \in X : |f_n(x)| \leq |\pi|\,|f_n(a)| \text{ for all } n\},$$

from which ([**7**, Theorem 5.2]) it follows that $\overline{T}(B_Z)$ is compactoid in X and hence that $\overline{T}$ is compact.

It follows directly that every perfect sequence space endowed with the normal topology has the CEP. Even more,

Proposition 4.3. *Let* X *be a locally convex space with a Schauder basis. If* X *is quasicomplete and* $(X', \sigma(X', X))$ *is sequentially complete, then* X *has the* CEP.

Proof. If X satisfies the above conditions, then X can be algebraically identified with a perfect sequence space Λ while $X' = \Lambda^*$. Also, if τ_Λ denotes the topology on Λ associated with the original topology on X, we have $\sigma(\Lambda, \Lambda^*) \leq \tau_A \leq n(\Lambda, \Lambda^*)$ (see [**18**, Section 3]). Now it is enough to prove that if Y is a normed space over $\mathbf{K}$ and $T : Y \to \Lambda$ is a linear operator, then T is τ_Λ-compact if and only if T is $n(\Lambda, \Lambda^*)$-compact. To see this, observe that since $\tau_\Lambda \leq n(\Lambda, \Lambda^*)$ it is clear that T $n(\Lambda,\Lambda^*)$-compact implies T τ_Λ-compact. To prove the converse, assume that T is τ_Λ-compact. By τ_Λ-quasicompleteness of Λ, $\overline{T(B_Y)}^{\tau_\Lambda}$ is τ_Λ-c-compact. Since τ_Λ and $n(\Lambda,\Lambda^*)$ have the same topological dual, it follows from [**1**, Proposition 3] that $\overline{T(B_Y)}^{\tau_\Lambda}$ is c-compact and bounded in $(\Lambda, n(\Lambda,\Lambda^*))$ which implies that $T : Y \to \Lambda$ is $n(\Lambda,\Lambda^*)$-compact.

Remark. In Proposition 4.3 the quasicompleteness of X cannot be weakened by considering only (weakly) sequential completeness for X. As an example, take $X = (c_0, \sigma(c_0, l^\infty))$ as in the set of examples at the end of this section.

To obtain more spaces with the CEP we introduce the following continuous version of the CEP.

Definition 4.4. We say that X is *weakly injective* if for every pair of locally convex spaces Y, Z with $Y \subset Z$ and every $T \in L(Y,X)$ there exists $\overline{T} \in L(Z,X)$ that extends T.

The next result shows that this definition is an extension to locally convex spaces of the concept of weakly injective normed space given in [**23**, p. 106].

Proposition 4.5 (Compare with Proposition 2.3). *Let X be a normed space. If, for every pair of normed spaces E, F with $E \subset F$ and every $T \in L(E,X)$, there exists $\overline{T} \in L(F,X)$ that extends T, then for every pair of locally convex spaces Y, Z with $Y \subset Z$ and every $S \in L(Y,X)$, there exists $\overline{S} \in L(Z,X)$ that extends S.*

Proof. Following the same proof as in Lemma 2.2, changing "compactoid" into "bounded," for every $S \in L(Y,X)$, there exist $p \in \mathcal{P}_Y$ and $S_p \in L(Y_p, X)$ such that $S = S_p \circ \pi_p$. The rest follows as in the proof of Proposition 2.3.

Also, among the different characterizations of weakly injective normed spaces given in [**23**, Theorem 4.12], the only one making sense for arbitrary locally convex spaces also works in this more general context, as the following result shows.

Proposition 4.6. *X is weakly injective if and only if for every locally convex space Y that contains a copy X_1 of X, X_1 is complemented in Y. In particular, every weakly injective space is complete.*

Proof. To prove the "if," observe that since X is Hausdorff, X is linearly homeomorphic to a subspace X_1 of the locally convex space $Y := \prod_{p \in \mathcal{P}_X} \check{X}_p$, where for each $p \in \mathcal{P}_X$, $\check{X}_p$ is the spherical completion of X_p. By assumption, X_1 is complemented in Y. On the other hand, it follows from [**23**, Theorem 4.12.ii)] and Proposition 4.5 that each $\check{X}_p$ is weakly injective, implying that Y is also weakly injective and so the same is true for X_1 (and hence for X) by complementation of X_1 in Y.

Conversely, suppose X is weakly injective. Let Y be a locally convex space that contains a copy X_1 of X. Since X_1 is weakly injective, the identity map on X_1 extends to a continuous linear projection from Y onto X_1.

In contrast to Proposition 2.3, we have that Proposition 4.5 is not true in general for an arbitrary locally convex space X. This will be a consequence of the following.

Proposition 4.7. *$(X, \sigma(X, X'))$ is complete if and only if $X = (X')^*$. In particular, if X is a normed space, then $(X, \sigma(X, X'))$ is weakly injective if and only if $\dim X < \infty$.*

Proof. First observe that, since $\mathbf{K}$ is spherically complete, the canonical map $J_X : X \to (X')^*$ is injective. In this case we identify X and $J_X(X)$ and, applying [**15**, 20.9.(20)], we obtain that the completion of $(X, \sigma(X, X'))$ is $((X')^*, \sigma((X')^*, X'))$ from which the conclusion follows.

Corollary 4.8. *Let G be an infinite dimensional weakly injective normed space over $\mathbf{K}$ (e.g., take $G = l^\infty$). Then, $X := (G, \sigma(G, G'))$ is a nonweakly injective locally convex space satisfying that for every pair of normed spaces E, F with $E \subset F$, every $T \in L(E, X)$ has an extension $\overline{T} \in L(F, X)$.*

Proof. By Proposition 4.7, we clearly have that X is not weakly injective. For the rest of the proof, observe that since $\mathbf{K}$ is spherically complete, G and X have the same bounded sets ([**21**, Theorem 7.5]) and so $L(E, X) = L(E, G)$ for every normed space E over $\mathbf{K}$.

Theorem 4.9. *Every weakly injective locally convex space* X *over* $\mathbf{K}$ *has the* CEP.

Proof. Since X is Hausdorff, X is linearly homeomorphic to a subspace X_1 of the product space $E := \prod_{p \in \mathcal{P}_X} X_p$ which, by Theorem 4.1 and Proposition 2.4 (ii), has the CEP. Also, weakly injectivity of X together with Proposition 4.6 imply that X_1 is complemented in E. By Proposition 2.4 (i) we deduce that X has the CEP.

The converse is not true in general. Take any non-complete metrizable space and apply Theorem 4.1 and Proposition 4.6.

Remark. Other classes of locally convex spaces with the CEP. 1. *Every compactoid regular inductive limit of a sequence of metrizable spaces has the* CEP. Indeed, let X be a compactoid regular inductive limit of a sequence $(X_n)_n$ of metrizable spaces (i.e., $X = \cup_n X_n$, X is endowed with the finest locally convex topology on X making all the inclusions $X_n \to X$ continuous and, for every compactoid subset A of X, there is an n such that $A \subset X_n$ and A is compactoid in X_n, see [**8**]). Let Y, Z be normed spaces with $Y \subset Z$, and let $T \in C(Y, X)$. There is an $n \in \mathbf{N}$ such that $T(B_Y) \subset X_n$ and is compactoid in X_n. It follows that $T(Y) \subset X_n$, and the map $T_n : Y \to X_n$, $y \in Y \mapsto Ty \in X_n$ is a compact linear operator. By hypothesis and Theorem 4.1, T_n has a compact linear extension $\overline{T}_n : Z \to X_n$ and, composing $\overline{T}_n$ with the canonical inclusion $X_n \to X$, we obtain $\overline{T} : Z \to X$, a compact linear extension of T.

This fact in conjunction with Theorem 3.1.7 of [**8**] imply that if X is the semicompact inductive limit of a sequence $(X_n)_n$ of Banach spaces (i.e., for each $n = 1, 2, \dots$ the inclusion $X_n \to X_{n+1}$ is compact), then X has the CEP. In particular, the space of germs of analytic functions at zero endowed with the usual inductive limit topology (see [**8**]) has the CEP. These spaces play a central role for the definition of a non-Archimedean Laplace transform in [**9**] and [**14**] and for the definition of a non-Archimedean Fourier transform in [**10**].

Now let W be a Hausdorff locally compact, σ-compact and zero-dimensional topological space. Consider in the space $C_c(W)$ of all the continuous functions $W \to \mathbf{K}$ having compact support, the canonical inductive limit topology (see [**8**]). Then the associated inductive sequence is a (non-semicompact) compactoid regular one. Hence, the space $C_c(W)$ has the CEP. In the classical theory, the elements of the complex space $C_c(W)'$ are the well-known Radon measures. In the non-Archimedean case, the elements of $C_c(W)'$ are precisely the integrals defined by Monna and Springer in [**17**].

2. *Every locally convex space X in which all the compactoid sets are finite dimensional, in particular every $\mathbf{K}$-vector space equipped with the finest locally convex topology, has the* CEP. Indeed, if $T : Y \to X$ is a compact linear operator from a locally convex space Y to X, then its range $R(T)$ is finite dimensional. Then apply Theorem 4.1.

But not every locally convex space over $\mathbf{K}$ has the CEP, as we are going to show now.

Proposition 4.10. *Let $X \neq \{0\}$ be a normed space over $\mathbf{K}$. Then $X_\sigma := (X, \sigma(X, X'))$ has the* CEP *if and only if X is weakly injective.*

Proof. One has just to apply Proposition 4.5 and to observe that, if $\mathbf{K}$ is spherically complete, the weakly bounded and the bounded sets of X coincide ([**21**, Theorem 7.5]) and hence $L(Y, X) = C(Y, X_\sigma)$ for every normed space Y over $\mathbf{K}$.

As an application we obtain

Examples (of locally convex spaces without the CEP). 1. Suppose the valuation of $\mathbf{K}$ is dense. It follows from [**23**, Corollary 5.19] that c_0 is not weakly injective. So, by using Proposition 4.10, we obtain that the CEP is not satisfied for $X := (c_0, \sigma(c_0, l^\infty))$.

2. Let c_{00} be the linear subspace of c_0 consisting of all finitely nonnull sequences in $\mathbf{K}$. This space is not complete, and hence it is not weakly injective, so we again can apply Proposition 4.10 to conclude that $X := (c_{00}, \sigma(c_{00}, l^\infty))$ does not have the CEP.

Observe that these spaces are linearly homeomorphic to a subspace of

$\mathbf{K}^I$ for some set I ([**22**, Theorem 2.(δ)]). Also, by Proposition 2.4 (ii), we have that $\mathbf{K}^I$ has the CEP. Hence, these examples also show that a subspace of a space with the CEP does not always have the CEP.

To finish this section, we give some applications of the previous results to characterize spherical completeness of a field $\mathbf{K}$.

Theorem 4.11. *For a non-Archimedean valued field* $\mathbf{K}$*, the following properties are equivalent.*

(i) $\mathbf{K}$ *is spherically complete.*

(ii) *For every pair of normed spaces* Y, Z *with* $Y \subset Z$ *and every sequence* $(f_n)_n$ *in* Y' *with* $\lim_n \|f_n\| = 0$*, there exists a sequence* $(g_n)_n$ *in* Z' *with* $\lim_n \|g_n\| = 0$ *and* $g_n|Y = f_n$ *for all* n.

(iii) *For every pair of normed spaces* Y, Z *with* $Y \subset Z$ *and every compactoid sequence* $(f_n)_n$ *in* Y'*, there exists a compactoid sequence* $(g_n)_n$ *in* Z' *such that* $g_n \mid Y = f_n$ *for all* n.

(iv) *For every pair of normed spaces* Y, Z *with* $Y \subset Z$ *and every bounded sequence* $(f_n)_n$ *in* Y'*, there exists a bounded sequence* $(g_n)_n$ *in* Z' *such that* $g_n \mid Y = f_n$ *for all* n.

Proof. By Proposition 2.3 and [**4**, Lemma 2], respectively [**12**, Lemma 3.2], property (ii), respectively (iii), means that c_0, respectively l^∞, has the CEP which, by Theorems 3.3 and 4.1, is equivalent to spherical completeness of $\mathbf{K}$.

Analogously, by Proposition 4.5 and [**12**, Lemma 3.2], property (iv) means that l^∞ is weakly injective, which happens if and only if $\mathbf{K}$ is spherically complete [**23**, 4.A].

Remark. Observe that the equivalences (i) $\Leftrightarrow$ (ii) and (i) $\Leftrightarrow$ (iii) of Theorem 4.11 can also be proved directly by using the Hahn-Banach theorem for normed spaces over spherically complete fields (see [**23**]).

The special case when the valuation of $\mathbf{K}$ is discrete will be studied in the next section.

5. The case when the valuation of K is discrete. Recall [**8**] that a locally convex space X is locally complete if and only if, for every absolutely convex closed bounded set A in X, the normed space X_A is complete.

Theorem 5.1. *Every locally complete, hence every (weakly) sequentially complete, locally convex space X over a discretely valued field $\mathbf{K}$ has the* CEP.

Proof. Let Y, Z be normed spaces with $Y \subset Z$, and let $T \in C(Y, X)$. Then $T(Y) \subset X_A$, where $A = \overline{T(B_Y)}$. Local completeness of X implies that X_A is a Banach space and so it is weakly injective because the valuation of $\mathbf{K}$ is discrete ([**23**, p. 181]). Then the continuous linear operator $S : Y \to X_A$, $x \mapsto Tx$ can be extended to a continuous linear operator $\overline{S} : Z \to X_A$. Now composing $\overline{S}$ with the compact inclusion $X_A \to X$, we obtain the desired compact extension $\overline{T}$ of T.

Lots of examples of locally complete, respectively (weakly) sequentially complete, spaces can be found in [**8**] and [**21**].

Remark. Note that if the valuation of $\mathbf{K}$ is discrete and X is a (weakly) sequentially complete space over $\mathbf{K}$, then X and $X_\sigma := (X, \sigma(X, X'))$ have the CEP. The converse is true if X is a normed space (apply Propositions 4.6 and 4.10). However, this converse is not always true as we show in the next example. For that, we need the following, easily proved, general result.

Lemma 5.2. *Let X be a $\mathbf{K}$-vector space and τ_1, τ_2 two locally convex topologies on X such that (X, τ_1) and (X, τ_2) have the same compactoid sets. Then*

$$(X, \tau_1) \quad \text{has the CEP} \iff (X, \tau_2) \quad \text{has the CEP.}$$

Example. Let (E, τ) be a Köthe sequence space in which every bounded set is compactoid (examples of such spaces can be found in [**8**]). Take for X a proper dense subspace of E. Clearly X is not (weakly) sequentially complete. By metrizability of X and Theorem 4.1, X has the CEP. Now apply the previous lemma for $\tau_1 = \sigma(X, X')$ and $\tau_2 = \tau$ to have the CEP for X_σ.

To finish, we state some characterizations of discretely valued fields.

Theorem 5.3. *For a non-Archimedean valued field* $\mathbf{K}$*, the following are equivalent.*

(i) *The valuation of* $\mathbf{K}$ *is discrete.*

(ii) *Every locally complete space of* $\mathbf{K}$ *has the* CEP.

(iii) *Every* (*weakly*) *sequentially complete space over* $\mathbf{K}$ *has the* CEP.

(iv) $(c_0, \sigma(c_0, l^\infty))$ *has the* CEP.

(v) *For every pair of normed spaces* Y, Z *with* $Y \subset Z$ *and every bounded sequence* $(f_n)_n$ *in* Y' *with* $\lim_n f_n = 0$ *in* $\sigma(Y', Y)$*, there exists a bounded sequence* $(g_n)_n$ *in* Z' *with* $\lim_n g_n = 0$ *in* $\sigma(Z', Z)$ *and* $g_n \mid Y = f_n$ *for all* n*.*

Proof. For (i) ⇒ (ii) ⇒ (iii), see Theorem 5.1.

(iii) ⇒ (iv) because c_0 is a perfect sequence space and so it is (weakly) sequentially complete.

Also, for (iv) ⇒ (i), see Theorem 3.3 and the last examples of the previous section.

Finally, property (v) is the formulation in terms of sequences of the fact that c_0 is weakly injective ([**4**, Lemma 2]), which happens if and only if the valuation of $\mathbf{K}$ is discrete (see [**23**]).

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