ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 30, Number 2, Summer 2000

REDUCTIONS OF A GENERALIZED INCOMPLETE GAMMA FUNCTION, RELATED KAMPÉ DE FÉRIET FUNCTIONS, AND INCOMPLETE WEBER INTEGRALS

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ABSTRACT. We derive several reduction formulas for specializations of a certain generalized incomplete gamma function and its associated Kampé de Fériet function $F^{0:2;1}_{2:0;0}[x,y]$. Reductions of specializations of incomplete Weber integrals of modified Bessel functions and related functions $F^{1:0;0}_{1:0;1}[x,y]$, $F^{0:1;0}_{1:0;0}[x,y]$ heretofore also unavailable are deduced.

1. Introduction. One of a class of generalized incomplete gamma functions may be defined by

$$\Gamma(\nu,x;z) \equiv \int_x^\infty t^{\nu-1}e^{-t-z/t}\,dt, \tag{1.1}$$

where the parameters ν, x and argument z are arbitrary complex numbers. When the argument z vanishes, $\Gamma(\nu,x;z)$ reduces to the ordinary incomplete gamma function $\Gamma(\nu,x)$ of classical analysis. Although several authors (see [**1**], [**2**], [**6**], [**8**]) have studied this particular generalization, it appears that the last word concerning properties of the latter has not been said. Indeed, Veling [**8**] has recently recorded without explicit proof the reduction formula for nonnegative integers n

$$\Gamma(n,z;z^2) = z^n[K_n(2z)+e^{-2z}U_n(2z)], \tag{1.2a}$$

where

$$\begin{aligned} U_n(z) \equiv K_n(z)&\left[I_0(z)+2\sum_{j=0}^{n-2}I_{j+1}(z)\right]\\ &-(-1)^n I_n(z)\left[K_0(z)-2\sum_{j=0}^{n-2}(-1)^j K_{j+1}(z)\right], \end{aligned} \tag{1.2b}$$

Received by the editors on December 1, 1998.
AMS *Mathematics Subject Classification.* 33B99, 33C70, 33C10.
Key words and phrases. Generalizations of the incomplete gamma function, Kampé de Fériet functions, integrals of Bessel functions.

and it is understood that a summation makes no contribution when its upper limit is less than its initial index value; the functions $I_\omega(z)$ and $K_\omega(z)$ are modified Bessel functions.

In Sections 2 and 3 we shall give two not very difficult derivations of equation (1.2a) and also derive in two different ways the analogous result for nonnegative integers n

$$\Gamma(-n, z; z^2) = z^{-n}[K_n(2z) - e^{-2z}U_n(2z)]. \tag{1.2c}$$

Veling's derivation of equation (1.2a) was evidently inspired by the technique of finding independent solutions of certain recurrence relations derived from $\Gamma(\nu, x; z)$ which also produced mutatis mutandis the result (see [**1**, Eq. (6.1)])

$$\begin{aligned} z^{-\alpha}\Gamma(\alpha, x; z^2) = {} & K_\alpha(2z)\operatorname{erfc}\left(\sqrt{x} - \frac{z}{\sqrt{x}}\right) \\ & + [K_\alpha(2z) + (-1)^{\alpha-(1/2)}\pi I_\alpha(2z)]\operatorname{erfc}\left(\sqrt{x} + \frac{z}{\sqrt{x}}\right) \\ & + 2e^{-x-z^2/x}\sum_{j=0}^{\alpha-(3/2)}\left(\frac{x}{z}\right)^{j+(1/2)} \\ & \cdot [K_\alpha(2z)I_{j+(1/2)}(2z) \\ & + (-1)^{\alpha+j+(1/2)}I_\alpha(2z)K_{j+(1/2)}(2z)], \end{aligned} \tag{1.3a}$$

where $\alpha \in \{-(1/2), (1/2), (3/2), \dots\}$.

The first derivation that we shall give for equations (1.2) is similar to the one that produced equation (1.3a) and the analogous result for $z^\alpha\Gamma(-\alpha, x; z^2)$, namely (see [**6**, Eqs. (6.2)],

$$\begin{aligned} z^{\alpha}\Gamma(-\alpha, x; z^2) = {} & K_\alpha(2z)\ \operatorname{erfc}\left(\sqrt{x} - \frac{z}{\sqrt{x}}\right) \\ & - [K_\alpha(2z) + (-1)^{\alpha-(1/2)}\pi I_\alpha(2z)]\ \operatorname{erfc}\left(\sqrt{x} + \frac{z}{\sqrt{x}}\right) \\ & - 2e^{-x-z^2/x}\sum_{j=0}^{\alpha-(3/2)}\left(\frac{z}{x}\right)^{j+(1/2)} \\ & \cdot [K_\alpha(2z)I_{j+(1/2)}(2z) \\ & + (-1)^{\alpha+j+(1/2)}I_\alpha(2z)K_{j+(1/2)}(2z)], \end{aligned} \tag{1.3b}$$

where $\alpha \in \{-(1/2), (1/2), (3/2), \dots\}$.

In Section 4 we shall employ equations (1.2) and (1.3) to deduce several reduction formulas for specializations of the Kampé de Fériet function $F^{0:2;1}_{2:0;0}[x,y]$; for an introduction to the infinite series which represent these functions see, for example, [**7**, pp. 26–27]. Furthermore, as we proceed in the sections that follow, the interconnections between the functions and integrals mentioned in the abstract of this paper will become more apparent.

2. Derivation of equations (1.2). Recalling the definitions of the two incomplete Weber integrals (cf., e.g., [**5**, Eq. (1.1)])

$$K_{e^2_{\mu,\omega}}(a,z) \equiv \int_0^z e^{at^2} t^{\mu} K_{\omega}(t)\,dt, \quad \mathrm{Re}\,(1+\mu\pm\omega)>0$$

and

$$I_{e^2_{\mu,\omega}}(a,z) \equiv \int_0^z e^{at^2} t^{\mu} I_{\omega}(t)\,dt, \quad \mathrm{Re}\,(1+\mu+\omega)>0,$$

we have from [**6**, Eqs. (6.1)] upon letting $\nu = 1$, $x = z/2$,

$$\begin{aligned}
&\Gamma\left(n+2, \frac{z}{2}; \frac{z^2}{4}\right) \\
&\quad = 2\left(\frac{z}{2}\right)^{n+2}\Biggl\{K_{n+2}(z) - \frac{1}{2}(-1)^n \Gamma\left(0,\frac{z}{2}\right) I_{n+2}(z) \\
&\qquad - \frac{e^{-z/2}}{z}\biggl[K_{n+2}(z) I_{e^2_{1,0}}\left(-\frac{1}{2z}, z\right) \\
&\qquad\qquad - (-1)^n I_{n+2}(z) K_{e^2_{1,0}}\left(-\frac{1}{2z}, z\right)\biggr] \\
&\qquad + e^{-z}\sum_{j=0}^{n}[K_{n+2}(z) I_{j+1}(z) + (-1)^{n+j} I_{n+2}(z) K_{j+1}(z)]\Biggr\}
\end{aligned} \tag{2.1a}$$

and

$$\Gamma\Big(-n-2,\frac{z}{2};\frac{z^2}{4}\Big)$$
$$\begin{aligned}
&= 2\Big(\frac{2}{z}\Big)^{n+2}\bigg\{\frac{1}{2}(-1)^n\Gamma\Big(0,\frac{z}{2}\Big)I_{n+2}(z)\\
&\qquad + \frac{e^{-z/2}}{z}\bigg[K_{n+2}(z)I_{e^2_{1,0}}\Big(-\frac{1}{2z},z\Big)\\
&\qquad\qquad - (-1)^n I_{n+2}(z)K_{e^2_{1,0}}\Big(-\frac{1}{2z},z\Big)\bigg]\\
&\qquad - e^{-z}\sum_{j=0}^{n}[K_{n+2}(z)I_{j+1}(z)+(-1)^{n+j}I_{n+2}(z)K_{j+1}(z)]\bigg\},
\end{aligned} \tag{2.1b}$$

where $n=-1,0,1,\ldots$.

Next the two incomplete Weber integrals in equations (2.1) are evaluated by using [**6**, Eqs. (7.2) and (7.3)]; thus,

$$I_{e^2_{1,0}}\Big(-\frac{1}{2z},z\Big)=\frac{1}{2}ze^{(1/2)z}[1-e^{-z}I_0(z)] \tag{2.2a}$$

and

$$K_{e^2_{1,0}}\Big(-\frac{1}{2z},z\Big)=\frac{1}{2}ze^{(1/2)z}\bigg[\Gamma\Big(0,\frac{1}{2}z\Big)-e^{-z}K_0(z)\bigg]. \tag{2.2b}$$

Finally, combining equations (2.1) and (2.2), replacing z by $2z$ and n by $n-2$ yields equations (1.2) which are valid a fortiori also for $n=0$.

3. Reduction formulas for $I_{e^2_{n+1,n}}(\pm 1/2z,z)$ and $K_{e^2_{n+1,n}}(\pm 1/2z,z)$. We shall give a second derivation of equations (1.2) now utilizing heretofore unavailable reduction formulas for the incomplete Weber integrals $I_{e^2_{n+1,n}}(\pm 1/2z,z)$ and $K_{e^2_{n+1,n}}(\pm 1/2z,z)$ for nonnegative integers n which are deduced below.

It has already been shown that, in order to compute respectively the integrals $I_{e^2_{\nu+n+1,\nu+n}}(-a,z)$ and $K_{e^2_{\nu+n+1,\nu+n}}(-a,z)$ for $\mathrm{Re}\,(\nu)>-1$ and nonnegative integers n, it is essentially sufficient to know the values of $I_{e^2_{\nu+1,\nu}}(-a,z)$ and $K_{e^2_{\nu+1,\nu}}(-a,z)$ for $0 < \mathrm{Re}\,(\nu+1) \le 1$, see [**6**, Eqs. (5.4)]. Thus, setting $\nu = 0$ in the latter gives for nonnegative integers n

(3.1a)
$$I_{e^2_{n+1,n}}(-a,z) = \left(\frac{1}{2a}\right)^n \left[I_{e^2_{1,0}}(-a,z) - \frac{e^{-az^2}}{2a}\sum_{k=1}^{n}(2az)^k I_k(z)\right]$$

and

$$K_{e^2_{n+1,n}}(-a,z) = \left(-\frac{1}{2a}\right)^n \left[K_{e^2_{1,0}}(-a,z) - \frac{e^{-az^2}}{2a}\sum_{k=1}^{n}(-2az)^k K_k(z)\right]$$
$$+\frac{1}{2}\frac{e^{1/4a}}{(2a)^{n+1}}\left[n!\Gamma\left(-n,\frac{1}{4a}\right) - (-1)^n\Gamma\left(0,\frac{1}{4a}\right)\right]. \tag{3.1b}$$

Now letting $a = 1/2z$ in these results and employing equations (2.2) yields respectively for nonnegative integers n

(3.2a)
$$I_{e^2_{n+1,n}}\left(-\frac{1}{2z},z\right) = \frac{1}{2}z^{n+1}e^{-(1/2)z}\left[e^z - I_0(z) - 2\sum_{k=1}^{n} I_k(z)\right]$$

and

$$K_{e^2_{n+1,n}}\left(-\frac{1}{2z},z\right) = -\frac{1}{2}(-z)^{n+1}e^{-(1/2)z}$$
$$\cdot\left[(-1)^n n! e^z \Gamma\left(-n,\frac{1}{2}z\right)\right. \tag{3.2b}$$
$$\left. - K_0(z) - 2\sum_{k=1}^{n}(-1)^k K_k(z)\right].$$

The generalized incomplete gamma function may be written essentially in terms of two incomplete Weber integrals. Thus, setting $\nu = n$, and respectively $x = 2/z$ and $x = z/2$ in [**6**, Eqs. (4.6) and (4.7)], we arrive at

$$\Gamma\left(-n,\frac{z}{2};\frac{z^2}{4}\right) = n!\left(\frac{2}{z}\right)^n \Gamma\left(-n,\frac{z}{2}\right) I_n(z)$$
$$-2^{n+1}\frac{e^{-(1/2)z}}{z^{2n+1}}\left\{I_n(z)K_{e^2_{n+1,n}}\left(-\frac{1}{2z},z\right)\right. \tag{3.3a}$$
$$\left. - K_n(z) I_{e^2_{n+1,n}}\left(-\frac{1}{2z},z\right)\right\}$$

and

$$\Gamma\left(n,\frac{z}{2};\frac{z^2}{4}\right)=\left(\frac{z}{2}\right)^n\left[2K_n(z)-n!\Gamma\left(-n,\frac{z}{2}\right)I_n(z)\right]$$
$$+\,2^{1-n}\frac{e^{-(1/2)z}}{z}\left\{I_n(z)K_{e^2_{n+1,n}}\left(-\frac{1}{2z},z\right)\right.\tag{3.3b}$$
$$\left.-\,K_n(z)I_{e^2_{n+1,n}}\left(-\frac{1}{2z},z\right)\right\},$$

where n is a nonnegative integer.

Now substitute the results for the incomplete Weber integrals given by equations (3.2) into equations (3.3). Noticing in each result that the two terminal summation terms cancel, replacing z by $2z$, and finally adjusting each summation index gives equations (1.2). This then completes the second derivation of equations (1.2).

Evidently, once one of $\Gamma(\pm n,z;z^2)$ is obtained, the other may be deduced immediately since, from [**6**, Eq. (1.2)] or equations (3.3), upon equating the terms in braces, it is readily seen for n an integer that

$$z^{-n}\Gamma(n,z;z^2)+z^n\Gamma(-n,z;z^2)=2K_n(2z).$$

Reduction formulas for the incomplete Weber integrals of modified Bessel functions $I_{e^2_{n+1,n}}(1/2z,z)$ and $K_{e^2_{n+1,n}}(1/2z,z)$ for nonnegative integers n may also be obtained. Indeed, again employing [**6**, Eqs. (7.2) and (7.3)] we find that

$$I_{e^2_{1,0}}\left(\frac{1}{2z},z\right)=-\frac{1}{2}ze^{-(1/2)z}[1-e^zI_0(z)]$$

and

$$K_{e^2_{1,0}}\left(\frac{1}{2z},z\right)=-\frac{1}{2}ze^{-(1/2)z}\left[\Gamma\left(0,-\frac{1}{2}z\right)-e^zK_0(z)\right].$$

Now setting $a=-1/2z$ in equations (3.1) and using the above results, we arrive at

(3.4a)

$$I_{e^2_{n+1,n}}\left(\frac{1}{2z},z\right)=\frac{1}{2}(-z)^{n+1}e^{(1/2)z}\left[e^{-z}-I_0(z)-2\sum_{k=1}^{n}(-1)^kI_k(z)\right]$$

and

$$\text{(3.4b)}\quad K_{e^2_{n+1,n}}\left(\frac{1}{2z}, z\right)$$
$$= -\frac{1}{2}z^{n+1}e^{(1/2)z}\left[(-1)^n n! e^{-z}\Gamma\left(-n, -\frac{1}{2}z\right) - K_0(z) - 2\sum_{k=1}^{n} K_k(z)\right],$$

where n is a nonnegative integer.

In addition to the reduction formulas for incomplete Weber integrals given by equations (3.2) and (3.4), reduction formulas for $I_{e^2_{n+3/2,n+1/2}}(-(1/4)a, z)$, $K_{e^2_{n+3/2,n+1/2}}(-(1/4)a, z)$ are derived in [**6**]. Thus a total of six nontrivial reduction formulas for incomplete Weber integrals of modified Bessel functions now exist; and at this point it appears that a search for others will probably not prove fruitful.

The specialization $n = 0$ of equation (3.2a) or equation (2.2a) has evidently been known for some time and may be deduced from results in Luke's treatise on integrals of Bessel functions, see [**4**, p. 271 et seq.] and in particular [**4**, p. 272, Eq. (6)], where pertinent attributions are cited.

4. Reduction formulas for $F^{0:2;1}_{2:0;0}[x, y]$. Finding reduction formulas for Kampé de Fériet functions and other functions represented by generalized hypergeometric series in two and more variables is a daunting task as there is generally no a priori method, except in a few trivial and obvious cases, for determining if a particular function is even capable of reduction. However, it is sometimes the case that a particular Kampé de Fériet function may be identified in some way with a function of known reducibility. As an example of this, we make the observations below that certain specializations of $F^{0:2;1}_{2:0;0}[x, y]$ may be written in terms of the generalized incomplete gamma function $\Gamma(\nu, x; z)$; and so reductions of the latter will yield reductions of the former.

In particular, we have respectively from [**6**, Eqs. (2.7), (2.8) and (3.5)] the following:

$$\text{(4.1a)}\quad F^{0:2;1}_{2:0;0}\left[\begin{matrix} \text{——} : & 1, \nu; & 1; & \\ & & & -(1/x)z^2, z^2 \\ 1, 1+\nu : & \text{——}; & \text{——}; & \end{matrix}\right]$$
$$= x^\nu e^x\{\Gamma(1-\nu, x)\Gamma(1+\nu)z^{-\nu}I_\nu(2z) + \nu\Gamma(-\nu, x; z^2)\},$$

$$\text{(4.1b)}\quad F^{0:2;1}_{2:0;0}\left[\begin{matrix} \text{———}: & 1,\nu; & 1; & \\ & & & -(1/x)z^2, z^2 \\ 2, 1+\nu: & —; & —; & \end{matrix}\right]$$
$$= \frac{x^\nu e^x}{z^2}\{\Gamma(1-\nu,x)\Gamma(1+\nu)z^{1-\nu}I_{\nu-1}(2z) - \nu\Gamma(1-\nu,x;z^2)\},$$

and

$$\text{(4.1c)}\quad F^{0:2;1}_{2:0;0}\left[\begin{matrix} \text{———}: & 1,1; & 1; & \\ & & & -(1/x)z^2, z^2 \\ 2, n+2: & —; & —; & \end{matrix}\right]$$
$$= \frac{xe^x}{z^2}\left\{(n+1)\sum_{j=1}^{n}\frac{(-n)_j}{z^{2j}}\Gamma(j,x) + \frac{(n+1)!}{z^n}I_n(2z)\Gamma(0,x) - (-1)^n\frac{(n+1)!}{z^{2n}}\Gamma(n,x;z^2)\right\},$$

where $n = -1, 0, 1, \ldots$ and the parameter ν in equations (4.1a) and (4.1b) is not a negative integer. The Kampé de Fériet functions in equations (4.1a) and (4.1b) may also be written in terms of incomplete Weber integrals, see [**6**], and the results alluded to are specializations of much more general results, see [**6**].

Now set $x = z$ in equations (4.1) and, furthermore, $\nu = n$ in equation (4.1a) and $\nu = n+1$ in equation (4.1b); then, using the results for $\Gamma(\pm n, z; z^2)$ given by equations (1.2), we deduce for nonnegative integers n respectively:

$$\text{(4.2a)}\quad F^{0:2;1}_{2:0;0}\left[\begin{matrix} \text{———}: & 1,n; & 1; & \\ & & & -z, z^2 \\ 1, n+1: & —; & —; & \end{matrix}\right]$$
$$= -ne^{-z}U_n(2z) + e^z\{n!\Gamma(1-n,z)I_n(2z) + nK_n(2z)\},$$

$$\text{(4.2b)}\quad F^{0:2;1}_{2:0;0}\left[\begin{matrix} \text{———}: & 1,n+1; & 1; & \\ & & & -z, z^2 \\ 2, n+2: & \text{———}; & —; & \end{matrix}\right]$$
$$= (n+1)\frac{e^z}{z}\{n!\Gamma(-n,z)I_n(2z) - K_n(2z) + e^{-2z}U_n(2z)\},$$

and

$$(4.2\text{c}) \quad F^{0:2;1}_{2:0;0}\left[\begin{matrix} \text{———}: & 1,1; & 1; & \\ & & & -z, z^2 \\ 2, n+2: & \text{—}; & \text{—}; & \end{matrix}\right]$$
$$= \frac{e^z}{z}\left\{(n+1)\sum_{j=1}^{n}\frac{(-n)_j}{z^{2j}}\Gamma(j,z) + \frac{(n+1)!}{z^n}I_n(2z)\Gamma(0,z)\right\}$$
$$-(-1)^n\frac{(n+1)!}{z^{n+1}}\{e^zK_n(2z)+e^{-z}U_n(2z)\},$$

where $U_n(2z)$ is given by equation (1.2b).

Finally, letting $\alpha \in \{-(1/2), (1/2), (3/2), \dots\}$ and setting respectively $\nu = \pm\alpha$ in equation (4.1a), $\nu = 1 \pm \alpha$ in equation (4.1b), we obtain the following four reduction formulas:

$$(4.3\text{a}) \quad F^{0:2;1}_{2:0;0}\left[\begin{matrix} \text{———}: & 1,\alpha; & 1; & \\ & & & -(1/x)z^2, z^2 \\ 1, 1+\alpha: & \text{—}; & \text{—}; & \end{matrix}\right]$$
$$= \alpha(x/z)^\alpha e^x\{\Gamma(\alpha)\Gamma(1-\alpha,x)I_\alpha(2z) + z^\alpha\Gamma(-\alpha,x;z^2)\},$$

$$(4.3\text{b}) \quad F^{0:2;1}_{2:0;0}\left[\begin{matrix} \text{———}: & 1,-\alpha; & 1; & \\ & & & -(1/x)z^2, z^2 \\ 1, 1-\alpha: & \text{——}; & \text{—}; & \end{matrix}\right]$$
$$= -\alpha(z/x)^\alpha e^x\{\Gamma(-\alpha)\Gamma(1+\alpha,x)I_{-\alpha}(2z) + z^{-\alpha}\Gamma(\alpha,x;z^2)\},$$

$$(4.4\text{a}) \quad F^{0:2;1}_{2:0;0}\left[\begin{matrix} \text{———}: & 1,1+\alpha; & 1; & \\ & & & -(1/x)z^2, z^2 \\ 2, 2+\alpha: & \text{———}; & \text{—}; & \end{matrix}\right]$$
$$= \frac{1+\alpha}{z}\left(\frac{x}{z}\right)^{1+\alpha} e^x\{\Gamma(1+\alpha)\Gamma(-\alpha,x)I_\alpha(2z) - z^\alpha\Gamma(-\alpha,x;z^2)\},$$

$$(4.4\text{b}) \quad F^{0:2;1}_{2:0;0}\left[\begin{matrix} \text{———}: & 1,1-\alpha; & 1; & \\ & & & -(1/x)z^2, z^2 \\ 2, 2-\alpha: & \text{———}; & \text{—}; & \end{matrix}\right]$$
$$= \frac{1-\alpha}{z}\left(\frac{x}{z}\right)^{1-\alpha} e^x\{\Gamma(1-\alpha)\Gamma(\alpha,x)I_{-\alpha}(2z) - z^{-\alpha}\Gamma(\alpha,x;z^2)\},$$

where $\alpha \in \{-(1/2), (1/2), (3/2), \dots\}$ and where $z^{\alpha}\Gamma(-\alpha, x; z^2)$ and $z^{-\alpha}\Gamma(\alpha, x; z^2)$ are given respectively by the right side of equations (1.3a) and (1.3b).

Other reduction formulas for the Kampé de Fériet function $F^{0:2;1}_{2:0;0}[x,y]$ may be obtained by noting the easily shown identity for nonnegative integers p:

$$
\begin{aligned}
&F^{0:2;1}_{2:0;0}\left[\begin{matrix} \text{———} : & 1, \nu+p; & 1; & \\ & & & x, y \\ 1+p, 1+\nu+p : & \text{——};& \text{—}; & \end{matrix}\right] \\
&\qquad = \frac{\nu+p}{\nu}\frac{p!}{x^p} F^{0:2;1}_{2:0;0}\left[\begin{matrix} \text{——} : & 1, \nu; & 1; & \\ & & & x, y \\ 1, 1+\nu : & \text{—}; & \text{—}; & \end{matrix}\right] \\
&\qquad\quad - \sum_{j=0}^{p-1} \frac{\nu+p}{\nu+j}\frac{x^{j-p}}{j!}\, {}_1F_2[1; 1+j, 1+\nu+j; y].
\end{aligned}
\tag{4.5}
$$

Thus, for example, in equation (4.5) replace x by $-x^{-1}z^2$, y by z^2, and set $p = 1$. Then ${}_1F_2[1; 1, 1+\nu; z^2]$ reduces to ${}_0F_1[-; 1+\nu; z^2]$ which is proportional to $I_\nu(2z)$. Now noting the well-known functional relation $\Gamma(\nu+1, z) = \nu\Gamma(\nu, z) + z^\nu \exp(-z)$, we see that equation (4.2b), with $\nu = n$, and equations (4.4a) and (4.4b), with $\nu = \pm\alpha$, may be obtained respectively from equation (4.2a) and equations (4.3a), (4.3b) when the latter are used together with equation (4.5).

5. Reduction formulas for $F^{1:0;0}_{1:0;1}[x,y]$ and $F^{0:1;0}_{1:0;0}[x,y]$. There is a connection between the incomplete Weber integral $I_{e^2_{\mu,\nu}}(a,z)$ and the Kampé de Fériet function $F^{1:0;0}_{1:0;1}[az^2, z^2/4]$ which may be exploited to obtain reduction formulas for the latter. Thus, from [**5**, Eq. (3.4)], we have for $\operatorname{Re}(1+\mu+\nu) > 0$

$$
\begin{aligned}
&F^{1:0;0}_{1:0;1}\left[\begin{matrix} (1+\mu+\nu)/2 : & \text{——}; & \text{——}; & \\ & & & az^2, z^2/4 \\ (3+\mu+\nu)/2 : & \text{——}; & 1+\nu; & \end{matrix}\right] \\
&\qquad = \frac{2^\nu(1+\mu+\nu)\Gamma(1+\nu)}{z^{1+\mu+\nu}} I_{e^2_{\mu,\nu}}(a,z).
\end{aligned}
\tag{5.1}
$$

Now substituting $\mu = n+1$, $\nu = n$, $n = 0,1,2,\dots$, $a = \pm 1/2z$ in equation (5.1), utilizing respectively equations (3.4a), (3.2a) and then replacing z by $2z$ in the result, we deduce for nonnegative integers n

$$F^{1:0;0}_{1:0;1}\left[\begin{array}{lll} n+1: & \text{——}; & \text{——}; \\ & & & z, z^2 \\ n+2: & \text{——}; & n+1; \end{array}\right]$$

(5.2a)

$$= \frac{1}{2}(n+1)!\left(-\frac{1}{z}\right)^{n+1} e^{z}\left\{e^{-2z} - I_0(2z) - 2\sum_{k=1}^{n}(-1)^k I_k(2z)\right\}$$

and

$$F^{1:0;0}_{1:0;1}\left[\begin{array}{lll} n+1: & \text{——}; & \text{——}; \\ & & & -z, z^2 \\ n+2: & \text{——}; & n+1; \end{array}\right]$$

(5.2b)

$$= \frac{1}{2}(n+1)!\left(\frac{1}{z}\right)^{n+1} e^{-z}\left\{e^{2z} - I_0(2z) - 2\sum_{k=1}^{n} I_k(2z)\right\}.$$

Furthermore, if in equation (5.1) we substitute $\mu = n+3/2$, $\nu = n+1/2$ for $n = -1,0,1,\dots$ and replace a by $-a/4$, we may then utilize the previously mentioned reduction formula for $I_{e^2_{n+3/2,n+1/2}}(-a/4, z)$ given in [**6**, Eq. (5.9a)] to obtain, upon replacing z by $2z$ in the result,

$$F^{1:0;0}_{1:0;1}\left[\begin{array}{lll} n+3/2: & \text{——}; & \text{———}; \\ & & & -az^2, z^2 \\ n+5/2: & \text{——}; & n+3/2; \end{array}\right]$$

(5.2c)

$$= \frac{\Gamma(n+5/2)}{2z^2}\left(\frac{1}{az^2}\right)^{n+(1/2)} \cdot \left\{\frac{e^{1/a}}{a}\left[\operatorname{erf}\left(z\sqrt{a} + \frac{1}{\sqrt{a}}\right) + \operatorname{erf}\left(z\sqrt{a} - \frac{1}{\sqrt{a}}\right)\right] - 2\sqrt{\frac{z}{a}}e^{-az^2}\sum_{k=0}^{n}(az)^k I_{k+(1/2)}(2z)\right\},$$

where $n = -1,0,1,\dots$.

Three analogous reduction formulas for $F^{0:1;0}_{1:0;0}[x,y]$ may also be obtained by using equations (5.2) and the Kummer-type transformation

$$F^{0:1;0}_{1:0;0}\left[\begin{matrix} \text{——} : & 1; & \text{—}; & \\ & & & -x, y \\ \alpha+1 : & \text{—}; & \text{—}; & \end{matrix}\right] = e^{-x} F^{1:0;0}_{1:0;1}\left[\begin{matrix} \alpha : & \text{——}; & \text{—}; & \\ & & & x, y \\ \alpha+1 : & \text{——}; & \alpha; & \end{matrix}\right]$$

which is a specialization of [**5**, Eq. (3.6)].

In conclusion, we mention that although reduction formulas for Kampé de Fériet functions occur sparsely in the literature, enough are known so that attempts have been and are being made to collect them in tabular form; see [**7**, p. 28 et seq.] and [**3**]. Thus, the reduction formulas for $F^{0:2;1}_{2:0;0}[x,y]$, $F^{1:0;0}_{1:0;1}[x,y]$ and $F^{0:1;0}_{1:0;0}[x,y]$ obtained herein should, for example, be of value in the compilation of such tables.

Acknowledgment. The author thanks the referee for a careful reading of the manuscript and several helpful suggestions.

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