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STABILITY OF DIFFEOMORPHISMS ALONG ONE PARAMETER

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ABSTRACT. The structural stability theorem, proved by Robbin [**6**] and Robinson [**7**], says that for an Axiom A diffeomorphism f with the strong transversality condition, there exists a sufficiently small neighborhood U of f in the set of C^1 diffeomorphisms such that if $g \in U$ then there is a homeomorphism h near the identity map such that f is conjugate to g, i.e., $hf = gh$.

In this paper we further investigate the size of the neighborhood U and the distance of the homeomorphism h with the identity map. We show that if $\{f_\varepsilon\}$ is a one-parameter family of C^3 diffeomorphisms, f_0 satisfies Axiom A and the strong transversality condition, and f_ε is C^0 $O(\varepsilon^3)$-close and C^1 $O(\varepsilon^2)$-close to f_0, then for all small $|\varepsilon|$, there is a homeomorphism h_ε with C^0 $O(\varepsilon^2)$ near the identity map, such that $h_\varepsilon f_0 = f_\varepsilon h_\varepsilon$.

1. Definitions and the main theorem. First of all, we introduce notations and basic definitions.

Throughout this paper, let M denote a smooth compact manifold with a distance d induced from the Riemannian metric, d_{C^0} denote a distance in the set of continuous maps on M with the standard C^0-topology, and d_{C^1} denote a distance in the set of C^1 diffeomorphisms on M with the strong C^1-topology. For $r = 0$ or 1, $p \in \mathbf{N}$, we say that f is C^r $O(\varepsilon^p)$ to g if the ratio $|d_{C^r}(f,g)/\varepsilon^p|$ is bounded as $\varepsilon \to 0$.

A compact invariant set Λ for a diffeomorphism f on M has a *hyperbolic structure* if $TM|_\Lambda$, the restriction of the tangent bundle TM of M to Λ has two subbundles $\mathbf{E}^s$ and $\mathbf{E}^u$ such that $TM|_\Lambda = (\mathbf{E}^s \oplus \mathbf{E}^u)|_\Lambda$ where $\oplus$ is the Whitney sum of two subbundles, and if there exist $C > 0$ and $0 < \mu < 1$ such that, for any $x \in M$ and for all

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$n \geq 0$,

$$\begin{aligned} Df_x^n \mathbf{E}^\sigma(x) &= \mathbf{E}^\sigma(f^n(x)) \quad \text{for } \sigma = s, u, \\ |Df_x^n v^s| &\leq C\mu^n |v^s| \quad \text{for } v^s \in \mathbf{E}^s(x), \quad \text{and} \\ |Df_x^{-n} v^u| &\leq C\mu^n |v^u| \quad \text{for } v^u \in \mathbf{E}^u(x). \end{aligned}$$

A point x is *nonwandering* for f if for every neighborhood U of x there is an integer $n > 0$ such that $U \cap f^n(U) \neq \varnothing$. A point x is *periodic* for f if $f^n(x) = x$ for some $n > 0$. The *stable manifold* of x for f is the set $W^s(x) = \{y \in M : d(f^n(x), f^n(y)) \to 0 \text{ as } n \to \infty\}$. The *unstable manifold* of x for f is the set $W^u(x) = \{y \in M : d(f^n(x), f^n(y)) \to 0$ as $n \to -\infty\}$.

A diffeomorphism f satisfies Axiom A if the nonwandering set has a hyperbolic structure and the periodic points of f are dense in the nonwandering set. If f satisfies Axiom A, then $W^s(x)$ and $W^u(x)$ are injectively immersed submanifolds for all points $x \in M$ (see [**1**]). Such a diffeomorphism satisfies the *strong transversality condition* if $T_x W^s(x) + T_x W^u(x) = T_x M$ for all $x \in M$.

We are now in a position to state the result.

Main theorem. *Let M be a smooth compact manifold, $\{f_\varepsilon\}$ a one-parameter family of C^3 diffeomorphisms on M, and f_0 satisfies Axiom* A *and the strong transversality condition. Let f_ε be C^0 $O(\varepsilon^3)$-close and C^1 $O(\varepsilon^2)$-close to f_0. Then for all small $|\varepsilon|$, there is a homeomorphism h_ε on M, with C^0 $O(\varepsilon^2)$ near the identity map, such that $h_\varepsilon f_0 = f_\varepsilon h_\varepsilon$.*

In [**5**], Murdock considered a one-parameter of vector fields $\{X_\varepsilon\}$ on M with a gradient-like Morse-Smale vector field X_0 (when $\varepsilon = 0$) and showed that a constant $c > 0$ exists such that, for all small ε, every solution $p(t)$ of X_0 is shadowed by a solution $q_\varepsilon(t)$ of X_ε in the sense that $d(p(t), q_\varepsilon(t)) \leq c\varepsilon$ for all $t \in \mathbf{R}$. Avoiding the difficulty of establishing a homeomorphism carrying one to the other, he proved the result by constructing shadowing orbits directly.

In the proof of the main theorem, we shall construct the homeomorphism h_ε. The way of the construction is based on the proof of Robbin [**6**] and Robinson [**7**] for the structural stability theorem. Some crucial estimates are summarized in the key lemma.

In order to prove that the function h_ε is one-to-one, we need the definitions of d_{f_0}-Lipschitz vector fields and subbundles, due to Robbin [**6**]. For $x, y \in M$, define $d_{f_0}(x,y) = \sup\{d(f_0^n(x), f_0^n(y)) : n \in \mathbf{Z}\}$. Then d_{f_0} is a metric on the manifold M. Let $\mathcal{X}^0(M)$ be the set of continuous vector fields on M with a norm $\|\cdot\|_0$. A vector field $v \in \mathcal{X}^0(M)$ is d_{f_0}-*Lipschitz* if there is a least positive constant $\Lambda(v)$ such that $|v(x) - v(y)| \le \Lambda(v) d_{f_0}(x,y)$ for all $x, y \in M$. Here, in order to subtract $v(x)$ and $v(y)$, we think of $TM \subset M \times \mathbf{R}^{2m}$ for some Euclidean space. Let $\mathcal{X}^{f_0}(M)$ be the set of d_{f_0}-Lipschitz vector fields on M and $\|v\|_{f_0} = \max\{\|v\|_0, \Lambda(v)\}$. Then $\|\cdot\|_{f_0}$ is a norm as shown in [**6**]. A subbundle $E \subset TM$ is d_{f_0}-*Lipschitz* if there is a least positive constant $\Lambda(E)$ such that $|E(x) - E(y)| \le \Lambda(E) d_{f_0}(x,y)$, where $|E(x) - E(y)|$ is an appropriate distance function between Euclidean spaces.

2. Proof of the main theorem. We briefly sketch the proof as follows.

First, for $v \in \mathcal{X}^0(M)$, let

$$\begin{aligned} Q_\varepsilon(v) &= \exp^{-1} \circ f_\varepsilon^{-1} \circ \exp \circ v \circ f_0 - Tf_0^{-1} \circ v \circ f_0, \\ L(v) &= v - Tf_0^{-1} \circ v \circ f_0. \end{aligned}$$

We shall construct a right inverse J of L, i.e., $LJ(v) = v$. Second, define $\Theta_\varepsilon : \mathcal{X}^0(M) \to \mathcal{X}^0(M)$ by

$$\Theta_\varepsilon(v) = JQ_\varepsilon(v).$$

We will prove that Θ_ε is a contraction and apply the contraction mapping theorem to Θ_ε so that it has a fixed point $\tilde{v}_\varepsilon$. We show that this fixed point $\tilde{v}_\varepsilon$ is a solution of the equation $Q_\varepsilon(\tilde{v}_\varepsilon) = L(\tilde{v}_\varepsilon)$ as follows:

$$L(\tilde{v}_\varepsilon) = L\Theta_\varepsilon(\tilde{v}_\varepsilon) = LJQ_\varepsilon(\tilde{v}_\varepsilon) = Q_\varepsilon(\tilde{v}_\varepsilon).$$

From the definitions of L and Q_ε, we get that $\exp_x^{-1} \circ f_\varepsilon^{-1} \circ \exp_{f_0(x)} \circ \tilde{v}_\varepsilon \circ f_0(x) = \tilde{v}_\varepsilon(x)$, and so $\exp \tilde{v}_\varepsilon \circ f_0(x) = f_\varepsilon \circ \exp \tilde{v}_\varepsilon(x)$. Define $h_\varepsilon(x) = \exp \tilde{v}_\varepsilon(x)$ for $x \in M$. Therefore, $h_\varepsilon \circ f_0(x) = f_\varepsilon \circ h_\varepsilon(x)$. Finally we will prove that $\tilde{v}_\varepsilon$ is d_{f_0}-Lipschitz small and conclude that h_ε is one-to-one.

To start with, we recall some classical properties in [**8**]. Because f_0 satisfies Axiom A, there is a *spectral decomposition* of the nonwandering set $\Omega(f_0) = \Omega_1 \cup \cdots \cup \Omega_m$ where the Ω_i are pairwise disjoint and each Ω_i is closed, invariant and topologically transitive. Since f_0 satisfies the strong transversality condition, there is a partial ordering among these sets Ω_i defined by $\Omega_i \leq \Omega_j$ if and only if $W^u(\Omega_i) \cap W^s(\Omega_j) \neq \varnothing$. We can extend this partial ordering to a total ordering and reindex the sets such that if $W^u(\Omega_i) \cap W^s(\Omega_j) \neq \varnothing$, then $i \leq j$. Let $TM|_{\Omega(f_0)} = (\mathbf{E}^u \oplus \mathbf{E}^s)|_{\Omega(f_0)}$ be the hyperbolic invariant splitting for the diffeomorphism f_0 on $\Omega(f_0)$.

As in [**6**] and [**7**], there are neighborhoods U_i of Ω_i, $i = 1, \ldots, m$, and compatible families of stable and unstable subbundles $\{E_i^\sigma(x) \subset T_xM : x \in O(U_i)\}$, $\sigma = s, u$, where $O(U_i) = \{f^n(x) \in M : x \in U_i, n \in \mathbf{Z}\}$. That is, for $i, j = 1, \ldots, m$,

1. $U_i \cap U_j = \varnothing$ for $i \neq j$.
2. $E_i^u|_{\Omega_i} = \mathbf{E}^u|_{\Omega_i}$ and $E_i^s|_{\Omega_i} = \mathbf{E}^s|_{\Omega_i}$.
3. $TM|_{O(U_i)} = (E_i^u + E_i^s)|_{O(U_i)}$.
4. E_i^u and E_i^s are $T\ f_0$-invariant.
5. $E_i^u(x) \supset E_j^u(x)$ and $E_i^s(x) \subset E_j^s(x)$ if $1 \leq i < j$ and $x \in O^+(U_i) \cap O^-(U_j)$. Here $O^+(U_i) = \{f^n(x) \in M : x \in U_i, n \geq 0\}$ and $O^-(U_i) = \{f^n(x) \in M : x \in U_i, n \leq 0\}$.
6. (Hyperbolic estimate) There is a Riemannian metric and a constant $0 < \mu < 1$ such that $\|Tf_0^{-1} \circ v^u\|_0 \leq \mu \|v^u\|_0$ and $\|Tf_0 \circ v^s\|_0 \leq \mu \|v^s\|_0$ for $v^u \in E_i^u|_{U_i}$, $v^s \in E_i^s|_{U_i}$.
7. E_i^u and E_i^s are d_{f_0}-Lipschitz.

Choose a partition of unity $\theta_1, \ldots, \theta_m$ subordinate to the cover $O(U_1), \ldots, O(U_m)$ of M, i.e., for every i, $\theta_i : M \to [0, \infty)$ is a smooth function such that $\operatorname{supp}(\theta_i) \subset O(U_i)$ and $\sum_{i=1}^m \theta_i(x) = 1$ for all $x \in M$. For $v \in \mathcal{X}^0(M)$, we write $\theta_i v = v_i^u + v_i^s$ with $v_i^\sigma(x) \in E_i^\sigma(x)$ for $x \in O(U_i)$ and $\sigma = s, u$. Hence $\operatorname{supp}(v_i^\sigma) \subseteq \operatorname{supp}(\theta_i) \subset O(U_i)$ for $\sigma = s, u$. Define $J : \mathcal{X}^0(M) \to \mathcal{X}^0(M)$ by

$$J(v) = \sum_{i=1}^m \left(\sum_{n=1}^\infty Tf_0^n \circ v_i^s \circ f_0^{-n} - \sum_{n=0}^\infty Tf_0^{-n} \circ v_i^u \circ f_0^n \right).$$

Then J is a well-defined continuous linear map on $\mathcal{X}^0(M)$, see [**6**], and clearly $LJ(v) = v$.

The following lemma gives all estimates on J and Q_ε which we shall need to show that $\Theta_\varepsilon = JQ_\varepsilon$ is a contraction. Refer to [**2**] and [**3**] for similar estimates.

Key lemma. $K_1 > 0$ *exists such that*

$$\|J\|_0 \le K_1(1-\mu)^{-1}, \tag{1}$$

$$\Lambda(J(v)) \le K_1(1-\mu)^{-1}(\Lambda(v) + \|v\|_0), \tag{2}$$

and, moreover, $\delta > 0$ *exists such that, for all* $\|v\|_0, \|w\|_0 < \delta$,

$$\|Q_\varepsilon(0)\|_0 \le d_{C^0}(f_\varepsilon, f_0), \tag{3}$$

$$\|Q_\varepsilon(v) - Q_\varepsilon(w)\|_0 \le (K_1 \max\{\|v\|_0, \|w\|_0\} + d_{C^1}(f_\varepsilon, f_0))\|v - w\|_0, \tag{4}$$

$$\|Q_\varepsilon(v)\|_0 \le K_1\|v\|_0\|v\|_0 + d_{C^1}(f_\varepsilon, f_0)\|v\|_0 + d_{C^0}(f_\varepsilon, f_0), \tag{5}$$

$$\Lambda(Q_\varepsilon(v)) \le (K_1\|v\|_0 + d_{C^1}(f_\varepsilon, f_0))(1 + \Lambda(v)). \tag{6}$$

We defer the proof of the key lemma to the end of this section.

From the key lemma, we have the following estimates on Θ_ε.

$$\begin{aligned}
\|\Theta_\varepsilon(v) - \Theta_\varepsilon(w)\|_0 &\le \|J\|_0\|Q_\varepsilon(v) - Q_\varepsilon(w)\|_0 \\
&\le K_1(1-\mu)^{-1}(K_1 \max\{\|v\|_0, \|w\|_0\} \\
&\qquad + d_{C^1}(f_\varepsilon, f_0))\|v - w\|_0 \\
\|\Theta_\varepsilon(v)\|_0 &\le \|J\|_0\|Q_\varepsilon(v)\|_0 \\
&\le K_1(1-\mu)^{-1}(K_1\|v\|_0\|v\|_0 + d_{C^1}(f_\varepsilon, f_0)\|v\|_0 \\
&\qquad + d_{C^0}(f_\varepsilon, f_0)) \\
\Lambda(\Theta_\varepsilon(v)) &\le K_1(1-\mu)^{-1}\{(K_1\|v\|_0 + d_{C^1}(f_\varepsilon, f_0))(1 + \Lambda(v)) \\
&\qquad + K_1\|v\|_0\|v\|_0 \\
&\qquad + d_{C^1}(f_\varepsilon, f_0)\|v\|_0 + d_{C^0}(f_\varepsilon, f_0)\}.
\end{aligned}$$

Without loss of generality, we assume that the parameter $\varepsilon > 0$. From the assumptions, $d_{C^0}(f_0, f_\varepsilon) < K_2\varepsilon^3$ and $d_{C^1}(f_0, f_\varepsilon) < K_3\varepsilon^2$ for some constants $K_2, K_3 > 0$.

In order to find the subspace of $\mathcal{X}^0(M)$ in which Θ_ε preserves and is a contraction, we choose a suitable $K > 0$, such that for all sufficiently small ε with $0 < \varepsilon < 1 - \mu$,

$$\begin{gathered} K\varepsilon^2 < \delta, \\ K_1(1-\mu)^{-1}(K_1K\varepsilon^2 + K_3\varepsilon^2) < \frac{1}{2}, \\ K_1(1-\mu)^{-1}(K_1K\varepsilon^2K\varepsilon^2 + K_3\varepsilon^2K\varepsilon^2 + K_2\varepsilon^3) \le K\varepsilon^2, \\ \begin{aligned} K_1(1-\mu)^{-1}\{&(K_1K\varepsilon^2 + K_3\varepsilon^2)(1+K\varepsilon) \\ &+K_1K\varepsilon^2K\varepsilon^2 + K_3\varepsilon^2K\varepsilon^2 + K_2\varepsilon^3\} \le K\varepsilon. \end{aligned} \end{gathered}$$

Thus, for all $v, w \in \mathcal{X}^0(M)$ with $\|v\|, \|w\| < K\varepsilon^2$ and every Lipschitz vector field $u \in \mathcal{X}^0(M)$ with $\Lambda(u) < K\varepsilon$, we have that

$$\begin{gathered} \|\Theta_\varepsilon(v) - \Theta_\varepsilon(w)\|_0 < \frac{1}{2}\|v - w\|_0, \\ \|\Theta_\varepsilon(v)\|_0 \le K\varepsilon^2, \\ \Lambda(\Theta_\varepsilon(u)) \le K\varepsilon. \end{gathered}$$

Therefore Θ_ε preserves and is a contraction on the space $\{v \in \mathcal{X}^0(M) : \|v\| \le K\varepsilon^2\}$ and Θ_ε also preserve the subspace $\{v \in \mathcal{X}^0(M) : \|v\| \le K\varepsilon^2, \Lambda(v) \le K\varepsilon\}$. So Θ_ε has a unique fixed point $\tilde{v}_\varepsilon$ with $\|\tilde{v}_\varepsilon\| \le K\varepsilon^2$ and $\Lambda(\tilde{v}_\varepsilon) \le K\varepsilon$. Define $h_\varepsilon(x) = \exp(\tilde{v}_\varepsilon(x))$ for all $x \in M$, then $h_\varepsilon \circ f_0(x) = f_\varepsilon \circ h_\varepsilon(x)$. Since $\tilde{v}_\varepsilon$ is continuous, h_ε is continuous. Because h_ε is homotopic to the identity, h_ε is of degree one and hence onto (see [**4**]). Moreover, $d_{C^0}(h_\varepsilon, id_M) = d_{C^0}(\exp(\tilde{v}_\varepsilon), id_M) = \|\tilde{v}_\varepsilon\|_0 \le K\varepsilon^2$. Finally, we have to prove that h_ε is one to one.

Suppose $h_\varepsilon(x) = h_\varepsilon(y)$. By the conjugacy, we have $h_\varepsilon(f_0^k(x)) = f_\varepsilon^k(h_\varepsilon(x)) = f_\varepsilon^k(h_\varepsilon(y)) = h_\varepsilon(f_0^k(y))$ for all $k \in \mathbf{Z}$. There exists $n \in \mathbf{R}$ such that $d_{f_0}(x, y) \le 2d(f_0^n(x), f_0^n(y))$. Let $p = f_0^n(x)$ and $q = f_0^n(y)$, then $h_\varepsilon(p) = h_\varepsilon(q)$ and $d_{f_0}(p, q) = d_{f_0}(x, y) \le 2d(f_0^n(x), f_0^n(y)) = 2d(p, q)$. As in Lemma 2.3 of [**6**], $\alpha > 0$ exists such that $\alpha d(p, q) - d(h_\varepsilon(p), h_\varepsilon(q)) \le |\tilde{v}_\varepsilon(p) - \tilde{v}_\varepsilon(q)|$. Because $h_\varepsilon(p) = h_\varepsilon(q)$ and $\Lambda(\tilde{v}_\varepsilon) \le K\varepsilon$,

$$\alpha d(p, q) \le |\tilde{v}_\varepsilon(p) - \tilde{v}_\varepsilon(q)| \le K\varepsilon d_{f_0}(p, q) \le 2K\varepsilon d(p, q).$$

Consider ε small enough such that $\alpha - 2K\varepsilon > 0$, then $d(p, q) = 0$ and $p = q$. Thus, $x = f_0^{-n}(p) = f_0^{-n}(q) = y$, and hence h_ε is one to one.

We now turn to prove the key lemma and so complete the proof of the main theorem.

Proof of the key lemma. The six inequalities are proved in (1)–(6)'s order.

(1) By using hyperbolic estimates, it can be shown that $C > 1$ and $0 < \mu < 1$ exist such that $\|Tf_0^n \circ v_i^s \circ f_0^{-n}\|_0 \leq C\mu^n \|v_i^s\|_0$ and $\|Tf_0^{-n} \circ v_i^u \circ f_0^n\|_0 \leq C\mu^n \|v_i^u\|_0$ for all $n \geq 0$ and all i. Thus

$$\|J\|_0 \leq \sum_{i=1}^m 2 \sum_{n=0}^\infty C\mu^n = \sum_{i=1}^m 2C(1-\mu)^{-1} \leq K_1(1-\mu)^{-1}$$
for some $K_1 > 0$.

(2) In [**6**] (see also [**7**]) Robbin showed that for $\sigma = u, s, \Lambda(Tf_0^{-n} \circ v_i^\sigma \circ f_0^n) \leq C\mu^n \Lambda(v_i^\sigma) + bCn\mu^{n-1}\|v_i^\sigma\|_0$, here b is a bound on the second derivatives of f_0 in local coordinates. Therefore,

$$\begin{aligned}\sum_{n=0}^\infty \Lambda(Tf_0^{-n} \circ v_i^\sigma \circ f_0^n) &\leq \sum_{n=0}^\infty (C\mu^n \Lambda(v_i^\sigma) + bCn\mu^{n-1}\|v_i^\sigma\|_0) \\ &\leq C(1-\mu)^{-1}\Lambda(v_i^\sigma) + bC(1-\mu)^{-2}\|v_i^\sigma\|_0,\end{aligned}$$

and

$$\begin{aligned}\Lambda(J(v)) \leq \sum_{i=0}^m \left(\sum_{n=1}^\infty \Lambda(Tf_0^n \circ v_i^s \circ f_0^{-n}) + \sum_{n=0}^\infty \Lambda(Tf_0^{-n} \circ v_i^u \circ f_0^n) \right) \\ \leq K_1(1-\mu)^{-1}(\Lambda(v) + \|v\|_0), \quad \text{for some } K_1 > 0.\end{aligned}$$

(3) Clearly,

$$\|Q_\varepsilon(0)\|_0 = \|\exp_x^{-1} \circ f_\varepsilon^{-1} \circ f_0(x)\|_0 = d_{C^0}(f_\varepsilon, f_0).$$

(4) Let $G_\varepsilon(v_{f_0(x)}) = Tf_0^{-1}(v_{f_0(x)}) - \exp_x^{-1}(f_\varepsilon^{-1}(\exp_{f_0(x)}(v_{f_0(x)})))$. Since f_0 and f_ε are C^3, G_ε is C^2 and so $K_1 > 0$ and $\delta > 0$ exist such that $\|D^2 G_\varepsilon(v)\|_0 \leq K_1$ for all $\|v\|_0 < \delta$. By the mean value

theorem, we have for all $\|v\|_0, \|w\|_0 < \delta$,

$$\begin{aligned}\|Q_\varepsilon(v)-Q_\varepsilon(w)\|_0 &= \sup_{x\in M} |G_\varepsilon(v_{f_0(x)})-G_\varepsilon(w_{f_0(x)})| \\ &= \sup_{y\in M} |G_\varepsilon(v_y)-G_\varepsilon(w_y)| \\ &= \sup_{y\in M} \left| \int_0^1 DG_\varepsilon(w_y+s(v_y-w_y))(v_y-w_y)\, ds\right| \\ &\le \sup_{\substack{y\in M \\ |v_y^*|\le\|v\|_0,\|w\|_0}} |DG_\varepsilon(v_y^*)|\cdot|v_y-w_y| \\ &= \sup_{\substack{y\in M \\ |v_y^*|\le\|v\|_0,\|w\|_0}} \left\{\left|\int_0^1 D^2G_\varepsilon(sv_y^*)v_y^*\, ds\right| \right. \\ &\qquad\qquad \left. +\|DG_\varepsilon(0)\|_0\right\}\|v-w\|_0 \\ &\le (K_1 \max\{\|v\|_0, \|w\|_0\} + d_{C^1}(f_\varepsilon, f_0))\|v-w\|_0.\end{aligned}$$

(5) Taking $w = 0$ in the inequality (4), we get

$$\|Q_\varepsilon(v)\|_0 \le K_1\|v\|_0\|v\|_0 + d_{C^1}(f_\varepsilon, f_0)\|v\|_0 + d_{C^0}(f_\varepsilon, f_0)$$

(6) Using the mean value theorem again, we have

$$\begin{aligned}|Q_\varepsilon(v_x) - Q_\varepsilon(v_y)| &\le \|DG_\varepsilon(v^*)\|_0 d(v\circ f_0^{-1}(x), v\circ f_0^{-1}(y)) \\ &\le (K_1\|v\|_0 + d_{C^1}(f_\varepsilon, f_0))(d(f_0^{-1}(x), f_0^{-1}(y)) \\ &\quad + |v\circ f_0^{-1}(x) - v\circ f_0^{-1}(y)|) \\ &\le (K_1\|v\|_0 + d_{C^1}(f_\varepsilon, f_0))(d_{f_0}(x,y) + \Lambda(v)d_{f_0}(x,y)) \\ &= (K_1\|v\|_0 + d_{C^1}(f_\varepsilon, f_0))(1+\Lambda(v))d_{f_0}(x,y).\end{aligned}$$

So $\Lambda(Q_\varepsilon(v)) \le (K_1\|v\|_0 + d_{C^1}(f_\varepsilon, f_0))(1+\Lambda(v))$. ◻

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