ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 30, Number 2, Summer 2000

COEXISTENCE OF CENTERS AND LIMIT CYCLES IN POLYNOMIAL SYSTEMS

ROBERT E. KOOIJ AND ANDRÉ ZEGELING

ABSTRACT. In this paper we consider polynomial systems on the plane with coexisting centers and limit cycles. We prove that cubic systems which are symmetric with respect to a line which is not invariant have either zero or exactly two limit cycles. The same result is proved for rationally reversible cubic systems. New configurations of coexisting centers and limit cycles in cubic systems are presented. Also an example of an integrable quartic system with a unique limit cycle and a center is given. As a bonus we construct a septic system with 57 limit cycles.

1. Introduction. It is a well-known result that a quadratic system with a center has no limit cycles, see Ye [**20**]. However, Borukhov [**3**] and Dolov [**6**] gave examples of quartic and cubic systems with both centers and limit cycles. In this paper we give new configurations of coexisting centers and limit cycles in polynomial systems. The existence of centers in the examples for cubic systems is proved by using a symmetry principle. In Section 3 we derive some results for cubic systems symmetric with respect to a line, also known as time-reversible systems. The main result is that these systems, if the line is not invariant, have either zero or two limit cycles (Theorem 3.1). Using the results of Section 3, we present in Section 4 an example of a cubic system with three centers and two limit cycles where each limit cycle surrounds exactly one singularity. In a similar fashion in Section 5 an example is given of a cubic system with one center and two limit cycles where each limit cycle surrounds three singularities. In Section 6 we show that all rationally reversible cubic systems have either zero or two limit cycles (Theorem 6.1). This generalizes the result of Section 3. In Section 7 we give an example of a quartic system with a unique limit cycle and center. Here the existence of the center is not proved using a symmetry principle but by calculating the first integral

Received by the editors on November 4, 1997, and in revised form on August 6, 1998.
AMS *Mathematics Subject Classification.* 34C05, 58F14.

of the system explicitly. We conclude the paper with some remarks about systems symmetric with respect to two lines. It is indicated how to construct a septic system with 44 limit cycles, five centers and one periodic annulus. We show that for an appropriate perturbation of this system, 13 additional limit cycles appear, bifurcating from the closed orbits surrounding the five centers. The number 57 thus obtained is the largest lower bound for the number of limit cycles in septic systems found so far.

2. The example of Dolov. The cubic system considered by Dolov in [**6**] reads

$$\begin{aligned} \dot{x} &= x^2(-1+\varepsilon+y) - y + y^2 = P, \\ \dot{y} &= x - x^3 = Q, \end{aligned} \tag{2.1}$$

where $\varepsilon \in \mathbf{R}$.

Note that the phase portrait of system (2.1) is symmetric with respect to the y-axis because $P(-x,y) = P(x,y)$ and $Q(-x,y) = -Q(x,y)$. Systems like system (2.1) which are invariant with respect to reflection with respect to a straight line and reversion of time, are known as time-reversible systems, see Sevryuk [**18**]. Probably the first example of such systems was given by Poincaré [**15**].

For $\varepsilon = 0$, system (2.1) has the following singularities for $x \geq 0$: $O(0,0)$ a center, $A(1,-1)$ a saddle, $B(1,1)$ a first order stable weak focus and $C(0,1)$ a saddle. The fact that $B(1,1)$ is a first order stable weak focus follows from the first Poincaré-Lyapunov constant, see [**1**], being negative.

Notice that O is a center because of the symmetry principle, see, for instance, Ye [**20**]. By using the direction of the vector field on the line segment AC and the Dulac function $D(x,y) = \exp((y-1)/(1-\sqrt{2}))|2y+(1-\sqrt{2})x^2+1+\sqrt{2}|^{2/(\sqrt{2}-1)}$ to prove that there are no limit cycles for $x > 0$, we can construct the phase portrait of system (2.1) with $\varepsilon = 0$, see Figure 2.1.

If in system (2.1) we take $0 < \varepsilon \ll 1$, then the singularity $\overline{B}(1, \sqrt{1-\varepsilon})$ is unstable, hence system (2.1) has a unique small-amplitude limit cycle around $\overline{B}$ by the Andronov-Hopf theorem. If we assume that $\overline{B}$ is surrounded globally by at most one limit cycle, then the corresponding

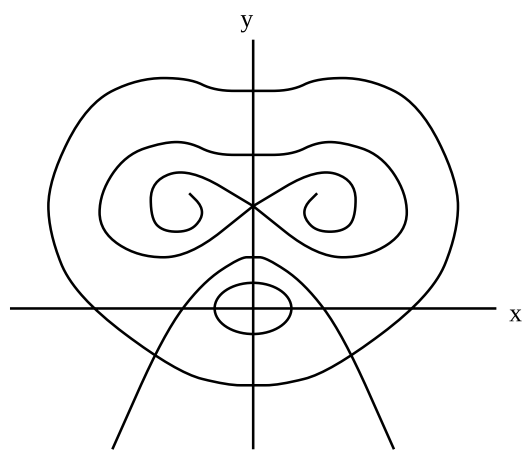

FIGURE 2.1. System (2.1), $\varepsilon = 0$.

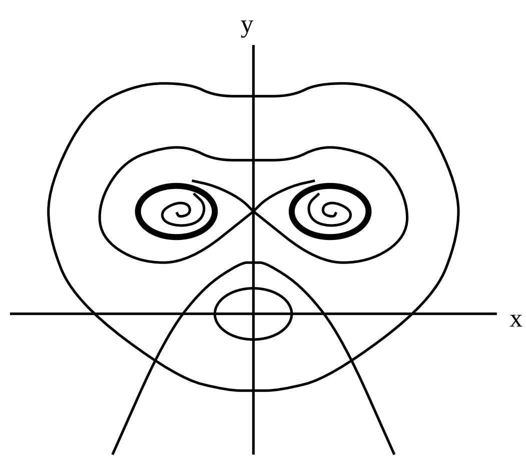

FIGURE 2.2. System (2.1), $0 < \varepsilon \ll 1$.

phase portrait is as drawn in Figure 2.2. In the next section it is shown (Theorem 3.1) that the above assumption on the number of limit cycles surrounding $\overline{B}$ is correct.

3. Cubic systems with line symmetry. Imitating Dolov [**6**] we consider cubic systems which are symmetric with respect to a line. Without loss of generality, we can assume that the line of symmetry is $x = 0$. If the system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ is symmetric with respect to $x = 0$, then there are two possibilities:

(i) $P(-x, y) = P(x, y), \qquad Q(-x, y) = -Q(x, y),$ or

(ii) $P(-x, y) = -P(x, y), \qquad Q(-x, y) = Q(x, y).$

For case (ii) $x = 0$ is a straight line solution of the system and we do not consider this case because we want to use the symmetry to prove the existence of centers on the line of symmetry. Obviously a center cannot be part of a real straight line solution.

The general cubic system that satisfies the symmetry conditions (i) reads

$$\begin{aligned} \dot{x} &= (a_{00} + a_{01}y + a_{02}y^2 + a_{03}y^3 + x^2(a_{20} + a_{21}y))/2, \\ \dot{y} &= x(b_{10} + b_{11}y + b_{12}y^2 + b_{30}x^2). \end{aligned} \tag{3.1}$$

In order to study the closed orbits of system (3.1) in the half plane $x > 0$, we can apply the transformation $x^2 = u$, $(dt/d\tau) = (1/x)$, reducing system (3.1) to

$$\begin{aligned} \frac{du}{d\tau} &= a_{00} + a_{01}y + a_{02}y^2 + a_{03}y^3 + u(a_{20} + a_{21}y), \\ \frac{dy}{d\tau} &= b_{10} + b_{11}y + b_{12}y^2 + b_{30}u. \end{aligned} \tag{3.2}$$

It can be assumed that $b_{30} \neq 0$ because for $b_{30} = 0$ all singularities of system (3.2) are located on a straight line solution and then system (3.2) has no closed orbits. Under this assumption we can apply the change of variables $b_{10} + b_{11}y + b_{12}y^2 + b_{30}u = \xi$, which reduces system (3.2) to

$$\begin{aligned} \frac{dy}{d\tau} &= \xi, \\ \frac{d\xi}{d\tau} &= c_{00} + c_{10}y + c_{20}y^2 + c_{30}y^3 + \xi(c_{01} + c_{11}y). \end{aligned} \tag{3.3}$$

In system (3.3) we recognize a cubic Liénard system with linear damping, a system extensively studied by Dumortier and Rousseau [**8**] and Dumortier and Li [**7**].

The main result of [**7**], [**8**] is the following:

Lemma 3.1. *The cubic Liénard system with linear damping has at most one limit cycle. If it exists it is hyperbolic.*

Remark 3.1. Limit cycles in the cubic Liénard system with linear damping surround either one, two or three singularities.

Using Lemma 3.1 we can obtain the following result.

Theorem 3.1. *If a cubic system is symmetric with respect to a line that is not invariant, then the system has either zero or exactly two limit cycles. If the limit cycles exist they are hyperbolic.*

Proof. Because system (3.1) is equivalent to system (3.3) for $x > 0$ it follows from Lemma 3.1 that system (3.1) has at most one limit cycle for $x > 0$. The proof is completed by using the symmetry of system (3.1).

4. An example of a cubic system with three centers and two limit cycles. Consider the following system

$$\begin{aligned} \dot{x} &= y(1 - 2x^2 - (1/4)y^2), \\ \dot{y} &= x(-1 + (1/4)x^2 + y^2). \end{aligned} \tag{4.1}$$

It is easy to draw the phase portrait of system (4.1), see Figure 4.1.

Notice that system (4.1) is symmetric with respect to both the x- and the y-axis and hence system (4.1) has at least five centers. Next consider the system

$$\begin{aligned} \dot{x} &= y(1 - 2x^2 - (1/4)y^2) + \lambda y^2 = P, \\ \dot{y} &= x(-1 + (1/4)x^2 + y^2) = Q, \end{aligned} \tag{4.2}$$

with $0 < \lambda \ll 1$.

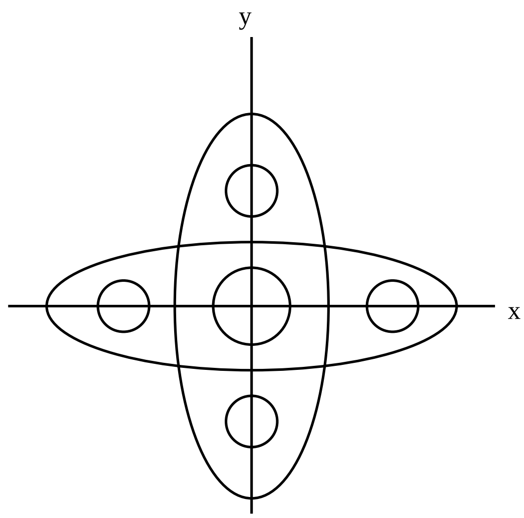

FIGURE 4.1. System (4.1).

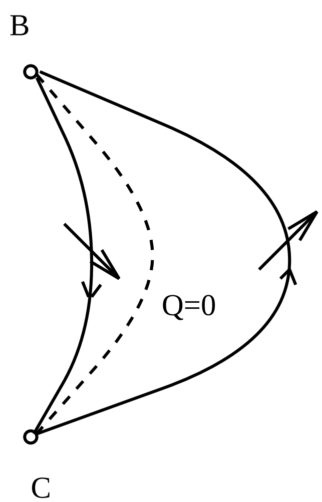

FIGURE 4.2. The flow on BC and CB.

System (4.2) is still symmetric with respect to the y-axis but the symmetry with respect to the x-axis has been broken. In fact, it is easy to show that $A(2,0)$ is a first order stable weak focus of system (4.2).

To study the relative positions of the separatrices of the saddles of system (4.2), we note that $(P^2+Q^2)(\partial\theta/\partial\lambda) = -xy^2(-1+(1/4)x^2+y^2)$, where $\theta = \arctan(Q/P)$. Denote the intersections of P and Q in the first and the fourth quadrant as B and C, respectively. We can use the properties of semi-complete families of rotated vector fields, see Perko [**14**], to follow the movement of the separatrices of B and C, as λ increases from zero. For the saddle connection BC, λ rotates the vector field (4.2) in a counterclockwise direction whereas for the saddle connection CB, λ rotates the vector field (4.2) in a clockwise direction, see Figure 4.2.

If we transform system (4.2) to a cubic Liénard system by putting $x^2 = u$, $(dt/d\tau) = (1/x)$, then the results of Dumortier and Rousseau [**8**] can be used to show that system (4.2) has no closed orbits for $x > 0$. It follows that the phase portrait of system (4.2) is as drawn in Figure 4.3.

Finally consider the system

$$\begin{aligned} \dot{x} &= y(1 - 2x^2 - (1/4)y^2) + \lambda y^2, \\ \dot{y} &= x(-1 + (1/4)x^2 + y^2) + \mu xy, \end{aligned} \tag{4.3}$$

with $0 < \mu \ll \lambda \ll 1$.

Because for system (4.3) the focus $A(2,0)$ has changed its stability, there will be a unique limit cycle in the vicinity of $A(2,0)$ by the Andronov-Hopf theorem. For sufficiently small μ the relative positions of the separatrices of system (4.2) will not be affected. The phase portrait of system (4.3) is given in Figure 4.4. It is easy to check that, due to symmetry of system (4.3) with respect to the y-axis, apart from the three centers on the y-axis, system (4.3) also has a family of closed orbits surrounding five singularities and a family of closed orbits surrounding all nine singularities.

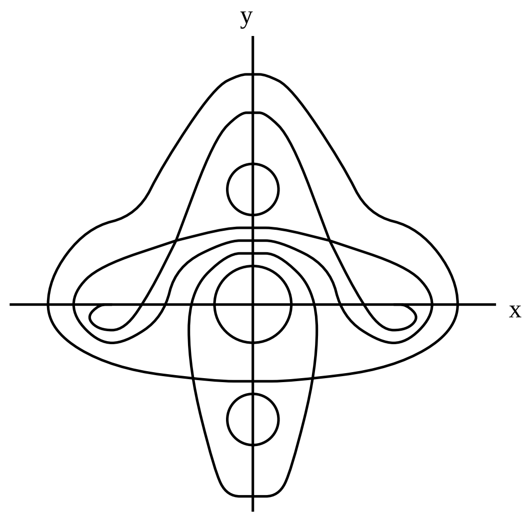

FIGURE 4.3. System (4.2).

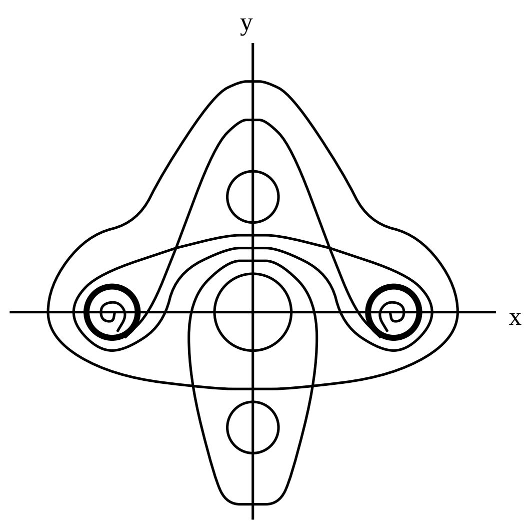

FIGURE 4.4. System (4.3).

5. An example of a cubic system with one center and limit cycles surrounding three singularities. Consider the cubic system

$$\begin{aligned} \dot{x} &= (1/2)y(y^2-1),\\ \dot{y} &= x(10-10x^2-y^2). \end{aligned} \tag{5.1}$$

It is easy to draw the phase portrait of system (5.1), see Figure 5.1. Notice that the phase portrait of system (5.1) is symmetric with respect to both the x- and the y-axis. It is important to notice that the homoclinic orbit passing through the saddle $A(1,0)$ is completely situated in the half plane $x > 0$. This may be checked by computing the first integral of system (5.1) or by integrating system (5.1) numerically.

Next consider the system

$$\begin{aligned} \dot{x} &= (1/2)y(y^2-1)-(1/2)\lambda,\\ \dot{y} &= x(10-10x^2-y^2+\mu y), \end{aligned} \tag{5.2}$$

where $\lambda, \mu \in \mathbf{R}$.

In order to study system (5.2) for $x > 0$ we make the transformation $x^2 = u$, $(dt/d\tau) = (1/x)$ to obtain

$$\begin{aligned} \frac{du}{d\tau} &= y^3-y-\lambda,\\ \frac{dy}{d\tau} &= 10-10u-y^2+\mu y. \end{aligned} \tag{5.3}$$

System (5.3) is equivalent to a system studied by Dumortier and Rousseau [**8**]. In fact, for $\lambda = \mu = 0$, system (5.3) corresponds to the phase portrait that belongs to the origin of the bifurcation diagram in [**8**], Figure 8. It follows from the results of [**7**], [**8**] that there exists λ and μ with $0 < \lambda^2 + \mu^2 \ll 1$ such that system (5.3) has a unique hyperbolic limit cycle surrounding three singularities. Furthermore, λ and μ can be chosen in such a way that the limit cycle is arbitrarily close to the homoclinic orbit situated in the half plane $u > 0$ that exists for $\lambda = \mu = 0$. Consequently, because system (5.2) is symmetric with respect to the y-axis, there exist values of λ and μ such that system (5.2) has one center and exactly two limit cycles. Furthermore, each limit cycle surrounds three singularities and is hyperbolic. The problem that remains concerns the relative positions of the separatrices of the

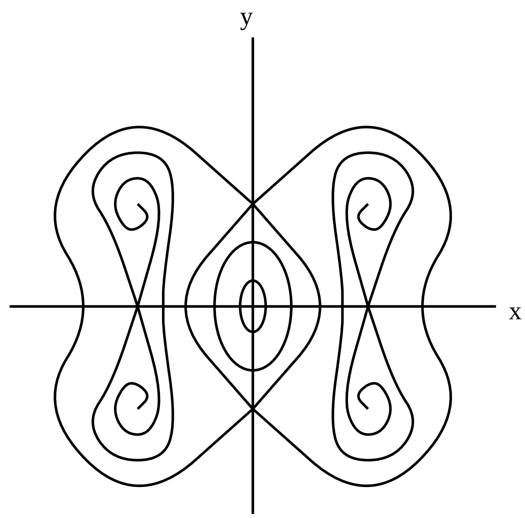

FIGURE 5.1. System (5.1).

saddles on the y-axis. If we choose to bifurcate an unstable limit cycle in system (5.3), then three generic positions for the separatrices exist, see Figure 5.2.

For given parameter values λ and μ we have to use numerical methods to determine which of the three generic positions of the separatrices occurs. The results of a numerical study of system (5.2) with $\lambda = .15$ and $\mu = .25$ reveal that the relative positions of the separatrices for these parameter values are as given in Figure 5.2c.

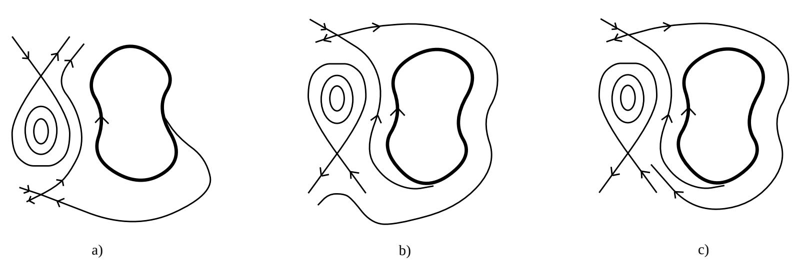

FIGURE 5.2. Possible separatrix configurations for system (5.2).

6. Rationally reversible cubic systems. We have seen that all the systems in the previous sections are time-reversible, i.e., these systems are invariant under reflection with respect to a straight line and reversion of time. The notion of time-reversible systems can be generalized to rationally reversible systems, see Żołądek [**21**].

A rationally reversible system V at a center O admits some rational noninvertible map $\Phi : \mathbf{R}^2 \to \mathbf{R}^2$ and a polynomial vector field V' on $\mathbf{R}^2$ such that

(i) $\Phi_* V$ and $V' \circ \Phi$ are collinear,

(ii) the curve of noninvertibility Γ_Φ of Φ passes through O and there is a neighborhood $U \subset \mathbf{R}^2$ of O such that the boundary of $\Phi(U)$ contains a part of the curve $\Gamma' = \Phi(\Gamma_\Phi)$, the vector field V' is tangent to Γ' at $\Phi(O)$ from the outside of $\Phi(U)$ and $V'(\Phi(O)) \neq 0$.

Under such conditions the point O must be a center because the real trajectories of V are the preimages of compact pieces of trajectories of V' lying in $\Phi(U)$.

The following lemma, proved by Żołądek, shows that for cubic systems there are 17 classes of rationally reversible systems.

Lemma 6.1. *Any rationally reversible cubic vector field $V(x, y)$ is reversible by means of one of the following 17 pairs (Φ, V'), $\Phi = (X, Y)$, where we choose some special coordinates (x, y), $T_1 = x + y + f$, $T_2 = ax^2 + bxy + cy^2 + dx + ey + 1$ and $\eta = xy - ay^2 + 2x + 2(1+a)y + 1 - a$, denoted in Table* 6.1.

Remark 6.1. In class 3 there are additional restrictions on the parameters, see [**21**], but we will not use them here.

It is the aim of this section to prove the following generalization of Theorem 3.1.

Theorem 6.1. *If a cubic system is rationally reversible, then the system either has zero or two limit cycles. If the limit cycles exist they are hyperbolic.*

TABLE 6.1. Cases which lead to rationally reversible cubic systems.

Class	(X,Y)	$(dX/dt)/(dY/dt)$
1	(x^2,y)	$(k+lX+mY+nY^2+pXY+qY^3)/(r+sX+tY+uY^2)$
2	$(x,(y^2/(x+y)))$	$(k+lX+mX^2)/(Y(n+pX)+qY^2)$
3	$(x,(y^2/(xy+ax^2+bx+1)))$	$(k+lX+mX^2)/(n+pX+(q+rX)Y+sXY^2)$
4	$(T_1x,(T_1/y))$	$(kX+lY+mXY)/(Y(n+pY+qY^2))$
5	$(T_1x,(T_1/y))$	$X(kX+lY+mXY)/(Y(kX+lY+nXY+pY^2+qXY^2))$
6	$(T_1x,(T_1/y))$	$(X(k+lY+mY^2)+nY^2)/(Y(k+pY+qY^2+rY^3))$
7	$(T_1x,(T_1/y))$	$X(k+lX)/(Y(k+mX+nY+pXY))$
8	$(T_1x,(T_1^2/y))$	$X(k+lX)/(2Y(k+mX+nY))$
9	$(T_1x,(T_1^2/y))$	$(kX+lY+mXY)/(2Y(k+pY))$
10	$(T_1x,(T_1^3/y))$	$X(k+lX)/(3Y(k+mX+nY))$
11	$((T_1^2/x),(T_1/y))$	$2X(kY+lX+mXY)/(Y(kY+nX+pY^2+qXY+rXY^2))$
12	$((T_1^2/x),(T_1/y))$	$2X(kY+lX+mY^2+nXY+pXY^2)/(Y^2(k+qY+rY^2))$
13	$((T_1^3/x),(T_1^2/y))$	$3X(kY+lX)/(2Y(kY+mX+nY^2))$
14	$((T_1^3/x),(T_1^2/y))$	$3X^2(kY+lX+mY^2)/(2Y^2(kX+nXY+pY^2))$
15	$((T_1^4/x),(T_1^2/y))$	$X(kX+lY^2)/(Y^3(m+nY))$
16	$((T^2/x),(T_2/y))$	$(X(kX+lY)+X^2(mX+nY))/(Y(kX+lY)+Y^2(pX+qY))$
17	$((x^3/y),(x^2/\eta))$	$3kX^2(1+3Y)/(kY(2X+3XY-9Y^2))$

It follows from the definition that for every class of rationally reversible systems the curve of noninvertibility splits the plane into two regions whose phase portraits have the same topology. Therefore we can prove Theorem 6.1 by showing that all the systems in the third column of Table 6.1 have either no, or at most one, hyperbolic limit cycle. For convenience's sake we will denote by S_i the system in the third column of Table 6.1 for class i. Note that class 1 corresponds to time-reversible systems studied in the previous sections. In fact, S_1 is equivalent to system (3.2).

Proof of Theorem 6.1. First we will show that the systems S_2, S_3, S_4, S_6, S_7, S_8, S_9, S_{10}, S_{12} and S_{15} have no limit cycles. All these systems have in common that if they possess real singularities then they belong to a real straight line solution. Obviously such singularities cannot be surrounded by limit cycles.

Because closed orbits of system S_5 cannot intersect the X- and Y-axes, we may apply the transformation $U = X$, $V = (X/Y)$, $(dt/d\tau) = (V/U^2)$, reducing S_5 to the linear system

$$\frac{dU}{d\tau} = kV + l + mU, \qquad \frac{dV}{d\tau} = (m-n)V - p - qU.$$

Obviously, a linear system has no limit cycles.

The transformation $U = X$, $V = (X/Y)$, $(dt/d\tau) = (V/U)$ reduces both system S_{11} as well as S_{13} to a quadratic system with a straight line solution $U = 0$. It is well known that such systems admit at most one hyperbolic limit cycle, see Rychkov [**17**] and Coppel [**5**]. In fact, it can be shown that system S_{13} has no limit cycles at all by applying Dulac's criterion with Dulac function $|X|^{-1/3}|Y|^{-3}$. We can also use Dulac's criterion with Dulac function $|X|^{-2/3}|Y|^{-3}$ to prove that system S_{17} has no limit cycles.

By the change of variables $U = (1/X)$, $V = (1/Y)$, $(dt/d\tau) = -V^2U$, system S_{14} reduces to

$$\frac{dU}{d\tau} = 3mU + 3kUV + 3lV^2, \qquad \frac{dV}{d\tau} = 2pU + 2nV + 2kV^2.$$

This is a quadratic system whose second order terms $3kUV + 3lV^2$ and $2kV^2$ have a common factor $V = 0$. It was proved by Coppel [**5**] that quadratic systems satisfying this property have at most one hyperbolic limit cycle. System S_{16} can be transformed to the same type of quadratic system by the change of variables $U = X$, $V = (X/Y)$, $(dt/d\tau) = (V/U^2)$.

Because system S_1 has already been dealt with in Section 3, the proof is finished.

7. An example of an integrable quartic system with one center and one limit cycle. In Sections 2–5 we have proved the existence of centers in the given examples by using the symmetry principle. In this section an integrating factor is used to prove the existence of a center.

This idea has been used by Dolov [**6**] as well. He gave an example of an integrable quintic system with coexisting center and limit cycle.

Here we will show that such coexistence can also occur for quartic systems.

Consider the following system:

$$\begin{aligned}
\dot{x} &= -2y(x^2+y^2)(x-2)+(x-y)(x^2+2y^2-1)(x-2)=P,\\
\dot{y} &= x(x^2+y^2)(x-2)+(x+y)(x^2+2y^2-1)(x-2)\\
&\quad -\frac{7}{10}(x^2+2y^2-1)(x^2+y^2)=Q.
\end{aligned} \tag{7.1}$$

It may be checked that system (7.1) has the following algebraic invariant curves: $C_1=0$, $C_2=0$, $C_3=0$, $C_4=0$, where $C_1=x^2+2y^2-1$, $C_2=x+iy$, $C_3=x-iy$, $C_4=x-2$. It is easy to verify that system (7.1) has an integrating factor $\mu(x,y)=(1/(C_1C_2C_3C_4))$. For more details on the relation between integrating factors and the existence of algebraic invariant curves we refer to [**11**]. A tedious calculation shows that the only finite singularities of system (7.1) are $A(0,0)$ and $B(3,1)$. A simple calculation reveals that A is an unstable focus and B is a center because in a neighborhood of B the integrating factor $\mu(x,y)$ is a regular function. Using the integrating factor $\mu(x,y)$ we can find the first integral $H^*(x,y)$ of system (7.1):

$$H^*(x,y)=\begin{cases} H(x,y)\exp(\operatorname{sgn}(y)\pi) & \text{if } x\geq 0,\\ H(x,y)\exp(-\operatorname{sgn}(y)\pi) & \text{if } x<0,\end{cases}$$

where $H(x,y)=(x^2+2y^2-1)(x-2)^{-7/5}(x^2+y^2)\exp(-2\arctan(y/x))$.

From the fact that $\mu(x,y)$ is an integrating factor, it can be proved that $C_1=0$ is the unique limit cycle of system (7.1). Suppose on the contrary that there is another limit cycle γ. Then in the neighborhood of γ there is a closed curve C, consisting of the part of a trajectory of system (7.1) between two consecutive intersection points with a transversal cut out by this part of the trajectory. Now apply Green's theorem to the annulus D between C and γ to obtain $\oint_\gamma(\mu P,\mu Q)\cdot n\,ds+\oint_C(\mu P,\mu Q)\cdot n\,ds+\iint_D \operatorname{div}(\mu P,\mu Q)\,dx\,dy$, where n is the unit outward normal to the region. This leads to a contradiction since the first and last integrals vanish and the second does not.

In addition it can be shown that the characteristic exponent of $C_1=0$, given by $h=\oint_{C_1=0}\operatorname{div}(P,Q)\,dt$ satisfies $h<0$. This shows that $C_1=0$ is hyperbolic and stable, see for instance [**1**].

8. A septic system with 57 limit cycles. A number of papers have appeared which focused on the estimation of a lower bound for the number of limit cycles for polynomial systems of degree n, i.e., systems of the following form:

$$\frac{dx}{dt} = \sum_{i+j=0}^{n} a_{ij} x^i y^j, \qquad \frac{dy}{dt} = \sum_{i+j=0}^{n} b_{ij} x^i y^j. \tag{8.1}$$

This is of course part of the sixteenth problem in the list posed by Hilbert [**9**]. The Hilbert number $H(n)$ is defined as the supremum of the number of limit cycles in (8.1), as the coefficients in the righthand sides vary.

The following estimations of $H(n)$ are known:

$$-H(n) \geq \frac{1}{2}(n^2 + 5n - 20 - 6 \cdot (-1)^n) : \text{ Otrokov [13]}$$

$$-H(n) \geq \frac{1}{2}(n^2 + n - 2) : \text{ Il'yashenko [10]}$$

$$-H(n) \geq \text{ integer part of } (1/4)(n+2)(n-1) : \text{ Basarab-Horwath and Lloyd [2]}$$

$$-H(2^k - 1) \geq 4^{k-1}(2k - (35/6)) + 3 \cdot 2^k - (5/3) : \text{ Christopher and Lloyd [4].}$$

It should be noted that the relative positions for the limit cycles for the above estimations are all different.

Christopher and Lloyd [**4**] exploited the idea of using systems symmetric with respect to both x- and y-axis, i.e., systems of the following form:

$$\dot{x} = y/2 \sum_{i+j=0}^{n} a_{ij} x^{2i} y^{2j}, \qquad \dot{y} = x/2 \sum_{i+j=0}^{n} b_{ij} x^{2i} y^{2j}. \tag{8.2}$$

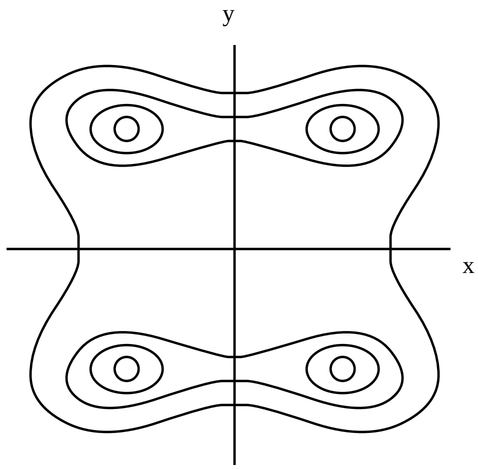

FIGURE 8.1. System (8.3).

Starting with a cubic system with three limit cycles, they obtain a septic system which is symmetric with respect to both coordinate axes and with three limit cycles in each quadrant. Next they perturb this system, staying within the class of septic systems, such that from the closed orbits surrounding the five centers a total of 13 limit cycles appear. Hence, they obtain 25 limit cycles. By repeating this procedure they obtain the lower bound mentioned above. We will now show that if we start with a cubic system with 11 limit cycles, then the same method can be applied. This will lead us to the following result.

Theorem 8.1. *Let $H(7)$ denote the maximum possible number of limit cycles for septic systems. Then $H(7) \geq 57$.*

Proof. Li Jibin and Huang Qiming [**12**] gave the following example of a cubic system with 11 limit cycles:

$$\begin{aligned} \dot{x} &= y(1-y^2) + \mu x(x^2 - 3y^2 - \lambda), \\ \dot{y} &= -x(1-2x^2) + \mu y(x^2 - 3y^2 - \lambda), \end{aligned} \tag{8.3}$$

$0 < \mu \ll 1$, $\lambda \approx -4.8$, see Figure 8.1.

The 11 limit cycles lie within the region $|x| < k$, $|y| < k$, where $k > 1$ is sufficiently large.

Through the change of variables $x = u^2 - k$, $y = v^2 - k$, $(dt/d\tau) = 2uv$, system (8.3) becomes

$$\begin{aligned}\frac{du}{d\tau} &= v(v^2-k)(1-(v^2-k)^2)\\ &\quad + \mu v(u^2-k)((u^2-k)^2 - 3(v^2-k)^2 - \lambda)\\ \frac{dv}{d\tau} &= -u(u^2-k)(1-2(u^2-k)^2)\\ &\quad + \mu u(v^2-k)((u^2-k)^2 - 3(v^2-k)^2 - \lambda).\end{aligned} \tag{8.4}$$

Since $(du/d\tau) = vP(u^2, v^2)$, $(dv/d\tau) = uQ(u^2, v^2)$, (8.4) is symmetric with respect to both $u = 0$ and $v = 0$.

Therefore, all 11 limit cycles in (8.3) are mapped onto each quadrant in (8.4) and hence (8.4) has 44 limit cycles. In addition, anti-saddles situated at $u = 0$ or at $v = 0$ are centers. Therefore, (8.4) has centers at $(0, 0)$, $(0, \pm\sqrt{k})$, $(\pm\sqrt{k}, 0)$.

Now we add a perturbation to (8.4); we will add the term $\mu\eta v R_1(u)$ in $(du/d\tau)$, where $\eta \ll \mu$, such that the 44 limit cycles in (8.4) persist, and $R_1(u) = a_1 u + a_3 u^3 + u^5$:

$$\begin{aligned}\frac{du}{d\tau} &= v(v^2-k)(1-(v^2-k)^2)\\ &\quad + \mu v(u^2-k)((u^2-k)^2 - 3(v^2-k)^2 - \lambda) + \mu\eta v R_1(u),\\ \frac{dv}{d\tau} &= -u(u^2-k)(1-2(u^2-k)^2)\\ &\quad + \mu u(v^2-k)((u^2-k)^2 - 3(v^2-k)^2 - \lambda).\end{aligned} \tag{8.5}$$

Note that (8.5) is still symmetric with respect to $v = 0$ because $du/d\tau = v\tilde{P}(u, v^2)$, $dv/d\tau = \tilde{Q}(u, v^2)$.

It is also important to note that for $\mu = 0$ system (8.5) is Hamiltonian. Therefore limit cycles bifurcating from the closed orbits in system (8.5) with $\mu = 0$ surrounding the center $(0, \sqrt{k})$ correspond with zeros of the following Pontryagin-integral, see [**16**]:

$$\begin{aligned}I(h) &= \oint_{\gamma h} v(u^2-k)((u^2-k)^2 - 3(v^2-k)^2 - \lambda)\, dv\\ &\quad + \eta \oint_{\gamma_h} v(a_1 u + a_3 u^3 + u^5)\, dv\\ &= I_1(h) + \eta I_2(h),\end{aligned}$$

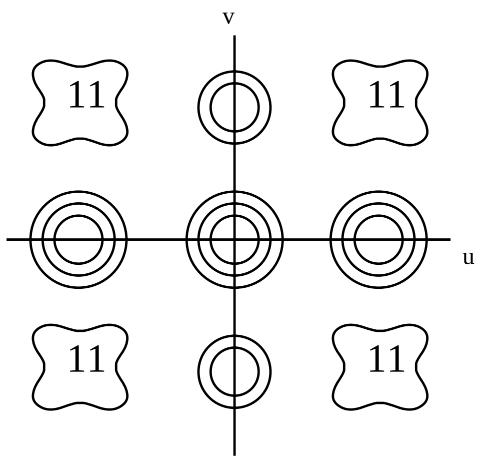

FIGURE 8.2. System (8.6).

where γ_h denotes a closed orbit of the unperturbed system, corresponding to a compact component of the Hamiltonian.

Notice that, due to symmetry $I_1(h) \equiv 0$. By applying Stokes' Theorem it follows that $I_2(h) = \iint_{V_h} v(a_1 + 3a_3u^2 + 5u^4)\,du\,dv$, where V_h is the region enclosed by γ_h. Notice that the closed orbits γ_h under consideration do not intersect $v = 0$.

Therefore it is easy to see that it is possible to choose a_1, a_3, $0 < a_1 \ll -a_3 \ll 1$, such that $I_2(h)$ has two simple zeros, see also [**4**]. Then (8.5) has two limit cycles around $(0, \sqrt{k})$. By symmetry there will also be two limit cycles around $(0, -\sqrt{k})$. Therefore system (8.5) has, at least, $44 + 4 = 48$ limit cycles.

Now we add another perturbation to (8.5); we will add the term $\mu\eta\rho R_2(v)$ in $(dv/d\tau)$, where $\rho \ll \eta$ such that the 48 limit cycles in (8.5) persist, and $R_2(v) = b_1v + b_3v^3 + b_5v^5 + v^7$:

$$
\begin{aligned}
\frac{du}{d\tau} &= v(v^2-k)(1-(v^2-k)^2) \\
&\qquad + \mu v(u^2-k)((u^2-k)^2 - 3(v^2-k)^2 - \lambda) + \mu\eta v R_1(u), \\
\frac{dv}{d\tau} &= -u(u^2-k)(1-2(u^2-k)^2) \\
&\qquad + \mu u(v^2-k)((u^2-k)^2 - 3(v^2-k)^2 - \lambda) + \mu\eta\rho R_2(v).
\end{aligned}
\tag{8.6}
$$

Limit cycles bifurcating from the closed orbits surrounding the centers at $v = 0$ correspond with zeros of the following integral:

$$\begin{aligned} J(h) &= \oint_{\gamma_h} v(u^2 - k)((u^2 - k)^2 - 3(v^2 - k)^2 - \lambda)\, dv \\ &\quad + \eta\left(\oint_{\gamma_h} vR_1(u)\, dv + \rho \oint_{\gamma_h} R_2(v)\, du\right) \\ &= J_1(h) + \eta J_2(h) + \eta\rho J_3(h). \end{aligned}$$

Notice that $J_1(h) \equiv 0$, $J_2(h) \equiv 0$, due to symmetry. Furthermore, $J_3(h) = -\iint_{V_h} (b_1 + 3b_3 v^2 + 5b_5 v^4 + 7v^6)\, du\, dv$.

It is possible to choose b_1, b_3, b_5, $0 < -b_1 \ll b_3 \ll -b_5 \ll 1$, such that $J_3(h)$ has three zeros for each family of closed orbits surrounding the centers on the u-axis. Then system (8.6) has, at least, $48 + 9 = 57$ limit cycles, see Figure 8.2.

REFERENCES

1. A.A. Andronov et al., *Theory of bifurcation of dynamic systems on a plane*, trans. by John Wiley and Sons, New York, 1973.

2. P. Basarab-Horwath and N.G. Lloyd, *Co-existing fine foci and bifurcating limit cycles*, Nieuw Archief voor Wiskunde **3** (1988), 295–302.

3. V.T. Borukhov, *Qualitative behaviour of trajectories of a system of differential equations*, Differntsial'nye Uravneniya **8** (1972), 1682–1683; Differential Equations **8** (1972), 1296–1297.

4. C.J. Christopher and N.G. Lloyd, *Polynomial systems*: *A lower bound for the Hilbert numbers*, Proc. Roy. Soc. London Ser. A **450** (1995), 219–224.

5. W.A. Coppel, *Some quadratic systems with at most one limit cycle*, in *Dynamics reported*, Vol. 2 (U. Kirchgraber and H.O. Walther, eds.), John Wiley and Sons, Ltd. and B.G. Teubner, 1989, 61–88.

6. M.V. Dolov, *Limit cycles in the case of a center*, Differentsial'nye Uravneniya **8** (1972), 1691–1692; Differential Equations **8** (1972), 1304–1305.

7. F. Dumortier and Li Chengzhi, *On the uniqueness of limit cycles surrounding one or more singularities for Liénard equations*, Nonlinearity **9** (1996), 1489–1499.

8. F. Dumortier and C. Rousseau, *Cubic Liénard equations with linear damping*, Nonlinearity **3** (1990), 1015–1040.

9. D. Hilbert, *Mathematical problems*, Bull. Amer. Math. Soc. **8** (1902), 437–479.

10. Yu.S. Il'yashenko, *The origin of limit cycles under perturbations of the equation $dw/dz = -R_z/R_w$ where $R(z,w)$ is a polynomial*, Math. USSR-Sb. **7**, 1969, 353–364.

11. R.E. Kooij and C.J. Christopher, *Algebraic invariant curves and the integrability of polynomial systems*, Appl. Math. Letters **6** (1993), 51–53.

12. Li Jibin and Huang Qiming, *Bifurcations of limit cycles forming compound eyes in the cubic system*, Chin. Ann. Math. Ser. B **8** (1987), 391–403.

13. N.F. Otrokov, *On the number of limit cycles of a differential equation in a neighbourhood of a singular point*, Mat. Sb. **34** (1954), 127–144.

14. L.M. Perko, *Rotated vector fields and the global behaviour of limit cycles for a class of quadratic systems in the plane*, J. Differential Equations **18** (1975), 63–86.

15. H. Poincaré, *Mémoire sur les courbes défines par les équations différentielles* I–VI, Oeuvre I, Gauthier-Villar, Paris, 1880–1890.

16. L.S. Pontryagin, *On dynamical systems close to Hamiltonian ones*, Zh. Eksper. Teoret. Fiz. (1934), 883–885 (in Russian).

17. G.S. Rychkov, *On the limit cycles of the equation* $u(x+1)\,du = (-x + ax^2 + bxu + cu + du^2)\,dx$, Differentsial'nye Uravneniya **8** (1972), 2257–2259; Differential Equations **8** (1972), 1748–1750.

18. M.B. Sevryuk, *Reversible systems*, Lecture Notes in Math. **1211**, Springer-Verlag, Berlin, 1986.

19. Shi Songling, *A concrete example of the existence of four limit cycles for plane quadratic systems*, Scientia Sinica **23** (1980), 153–158.

20. Ye Yenqian et al., *Theory of limit cycles*, Transl. Math. Monographs **66** (1986), Rhode Island.

21. H. Żołądek, *The classification of reversible cubic systems with center*, Topol. Meth. Nonlin. Anal. **4** (1994), 79–136.

22. ———, *Remarks on "The classification of reversible cubic systems with center,"* Topol. Meth. Nonlin. Anal. **8** (1996), 335–342.

Delft University of Technology, Information Technology and Systems, Department of Applied Mathematical Analysis, Mekelweg 4, 2628 CD Delft, The Netherlands

Current address: KPN Research, 2260 AK Leidschendam, The Netherlands
E-mail address: R.E.Kooij@kpn.com

Singel 37, 3984 NV, Odijk, The Netherlands
E-mail address: h.zegeling@cc.ruu.nl