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AN ALTERNATIVE FOR THE SPECTRAL RADIUS OF POSITIVE INTEGRAL OPERATORS —A FUNCTIONAL ANALYTIC APPROACH

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ABSTRACT. In a former paper, the author has investigated the solution of Fredholm integral equations of the second kind with positive integral operators in weighted function spaces. These results can be obtained more easily by using a functional analytic approach. We demonstrate this for an alternative theorem concerning the spectral radius of positive integral operators. To this end, first some refinements of results on positive operators in abstract Banach spaces have to be derived.

0. Introduction. In [**5**], among other things, alternative theorems for positive integral operators are derived in a closed subspace $C_{\sigma^{-1}}(\Omega) \subset C_{\sigma^{-1}}(\Omega_N)$, respectively in $L_\sigma(\Omega)$, where σ is a continuous weight function, σ^{-1} means $1/\sigma$ and where the weighted spaces fulfill the inclusions $C_{\sigma^{-1}}(\Omega) \subset C(\Omega)$ and $L_\sigma(\Omega) \supset L(\Omega)$ with $\Omega = [a,b]$. The cases are handled there separately.

In this paper we show that both cases can be treated in a unified manner by using a functional analytic approach and the duality relation $[L_\sigma(\Omega)]^* = L_{\infty,\sigma^{-1}}(\Omega)$ where $L_\sigma(\Omega)$, respectively $L_{\infty,\sigma^{-1}}(\Omega)$, is a generalization of $L(\Omega)$, respectively $L_\infty(\Omega)$.

The first three sections form the functional analytic part and the last section the application part.

The paper is structured as follows. In Section 1 some preliminaries and notations are given. Section 2 derives the relation $\rho(B) = \|B\|_\kappa$ under weaker conditions than known so far, where κ is a positive eigenvector of the positive operator B. In Section 3 an abstract alternative theorem for the spectral radius is proven. Finally, in Section 4 the alternative theorem is applied to integral operators. The conditions imposed on the integral kernel are usually fulfilled with

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Green's-function kernels. The application in the space $C_{\sigma^{-1}}(\Omega_N) \subset L_{\infty,\sigma^{-1}}(\Omega)$ is an immediate consequence of the general theorem (cf. 4.1), and that in the space $L_\sigma(\Omega)$ follows when B is replaced by the operator B^T where B^T has the kernel which is adjoint to that of B (cf. 4.2).

1. Preliminaries and notations. The definitions concerning cones and positive operators in a Banach space vary somewhat in the literature. Here, definitions on this subject are taken from [**7**], which is also the most frequently used reference source.

Throughout this paper, V is a real Banach space, K a cone in V, and B a bounded linear operator in V which is also written as $B \in \mathcal{B}(V)$. The spectral radius of B is denoted by $\rho(B)$. Further, V^* is the adjoint of V, that is, the Banach space of linear functionals on V with the usual norm. K^* means the wedge of all nonnegative functionals of V^*, and B^T is the adjoint of B. K^* is a cone, if and only if $\overline{K-K} = V$. Assume $\sigma \in K$, $\sigma \neq 0$. Then, the set of σ-measurable elements of V endowed with the σ-norm $\|u\|_\sigma := \inf\{t \geq 0 \mid -t\sigma \leq u \leq t\sigma\}$ is denoted by V_σ. When $B \in \mathcal{B}(V)$, the spectral radius of B considered as an operator $B : V_\sigma \to V_\sigma$ is denoted by $\rho_\sigma(B)$. For $\sigma \in K, \sigma \neq 0$, we define the operator norm $\|B\|_\sigma := \sup_{0 \neq u \in V_\sigma} \|Bu\|_\sigma / \|u\|_\sigma$.

Let $\Omega := [a, b] \subset \mathbf{R}$ be a bounded closed interval of the real line. By $L(\Omega)$ we mean the space of real functions on Ω which are measurable and summable, and by $L_\infty(\Omega)$ the space of real functions on Ω which are measurable and bounded almost everywhere.

2. The relation $\rho(B) = \|B\|_\kappa$. Under appropriate conditions, for the operator norm $\|\cdot\|_\sigma$ with $\sigma = \kappa$ we show that $\rho(B) = \|B\|_\kappa$ where κ is an eigenvector in K corresponding to $\rho(B)$. This is proven in [**12**] for $\kappa \in \operatorname{int}(K)$. The relation $\rho(B) = \|B\|_\kappa$ shows that the Jacobi method (i.e., the method of successive approximations) with iteration operator B is optimal for the choice $\sigma = \kappa$.

A statement on the existence of a positive eigenvalue can be found, for example, in [**9**, Theorem 6.1]. This theorem reads as follows. If K is quasi-reproducing (i.e., $\overline{K-K} = V$) and $B \in \mathcal{B}(V)$ is positive and completely continuous with $\rho(B) > 0$, then $\rho(B)$ is an eigenvalue of both B and B^T with corresponding eigenvectors $\kappa \in K$, respectively

$l_\chi \in K^*$.

The following theorem gives a more precise result. Since it is often difficult to prove that B is σ-bounded from below, it is convenient to have only to verify that B is σ-bounded from above, which is simple to verify, as a rule. Therefore, the following theorem is proven for this case.

First, we need a definition.

According to [**7**, p. 43], we call two elements x, y of a cone K *equivalent* (written $x \sim y$), if $x \leq \alpha y$ and $y \leq \alpha x$ for some number $\alpha \geq 0$.

Theorem 2.1. *Let the following conditions be satisfied*:

(i) *the cone $K \subset V$ is normal and reproducing*

(ii) *the operator $B \in \mathcal{B}(V)$ is positive*

(iii) *B has an eigenvector $\kappa \in K$ corresponding to the eigenvalue $\rho(B) > 0$*

or

(iii′) *B is completely continuous and $\rho(B) > 0$*

(iv) *B is σ-bounded from above where $\sigma \sim \kappa$.*

Then,

$$\rho(B) = \rho_\sigma(B) = \rho_\kappa(B) = \|B\|_\kappa.$$

Proof. Let the conditions (i)–(iv) be fulfilled.

As $\sigma \sim \kappa$, B is κ-bounded from above. Now, [**7**, p. 91] shows that there is a constant $\gamma = \gamma(B) > 0$ such that $-\gamma\|u\|\kappa \leq Bu \leq \gamma\|u\|\kappa$, $u \in V$. Further, K is normal so that a constant $R > 0$ exists such that $\|u\| \leq R\|u\|_\kappa$, $u \in V_\kappa$. From $B\kappa = \rho(B)\kappa$, one then infers

$$-\frac{\gamma R\|u\|_\kappa}{\rho(B)}\rho(B)^n\kappa \leq B^n u \leq \frac{\gamma R\|u\|_\kappa}{\rho(B)}\rho(B)^n\kappa, \quad u \in V_\kappa,$$

so that $\rho_\kappa(B) \leq \rho(B)$. On the other hand, the inequality $\rho(B)^n \leq \|B^n\|_\kappa$ holds. Therefore $\rho(B) \leq \rho_\kappa(B)$. Hence, one has $\rho_\kappa(B) = \rho(B)$.

Finally, $\rho_\kappa(B) = \|B\|_\kappa$. To see this, we first observe that

$$\rho_\kappa(B) \le \|B\|_\kappa := \sup_{0 \ne u \in V_\kappa} \frac{\|Bu\|_\kappa}{\|u\|_\kappa}.$$

The inequality $\rho_\kappa(B) \ge \|B\|_\kappa$ is proven as follows. One has $-\|u\|_\kappa \kappa \le u \le \|u\|_\kappa \kappa$, $u \in V_\kappa$, and hence $-\|u\|_\kappa B\kappa \le Bu \le \|u\|_\kappa B\kappa$, $u \in V_\kappa$. Now, $B\kappa = \lambda_\kappa \kappa$ and therefore $-\|u\|_\kappa \lambda_\kappa \kappa \le Bu \le \|u\|_\kappa \lambda_\kappa \kappa, u \in V_\kappa$, which means that $\|Bu\|_\kappa \le \lambda_\kappa \|u\|_\kappa$, $u \in V_\kappa$. This entails $\|B\|_\kappa \le \lambda_\kappa \le \rho_\kappa(B)$. So, the proof is complete.

3. An alternative for the spectral radius. We first mention that in applications it is important that the wedge K^* can be replaced by any *total set* $L^* \subset K^*$ (cf. [**7**, p. 22]). For example, $L^* = \{f_x \in (C[a,b])^*, x \in [a,b] \mid f_x(u) = u(x),\ u \in C[a,b]\}$ is evidently total in K^* with $K \subset C[a,b]$ defined by $K = \{u \in C[a,b] \mid u(x) \ge 0, x \in [a,b]\}$.

Again, the condition that B has to be σ-bounded can be weakened, as is shown in the following theorem. For the sake of easy reference, we first formulate the following conditions:

(a) The operator B is completely continuous, and σ is a quasi-interior point of K.

(b) The cone K is normal and solid, and σ is an interior point of K.

(c) The cone K is reproducing and normal, and the operator B is σ-bounded from above.

(d) The cone K is reproducing and normal, the operator B is φ-bounded from above, and σ is a quasi-interior point of K; here, φ may be different from σ.

Next, we need a definition.

According to [**7**, p. 110], an operator A is called *irreducible* if $Ax \le \alpha x$ (with $x \in K$ and $x \ne 0$) implies that x is a quasi-interior point of the cone.

Herewith, one has

Theorem 3.1. *Assume the following conditions*:

(i) $B \in \mathcal{B}(V)$ *is positive with respect to the cone* $K \subset V$

(ii) *B has an eigenvector $\kappa \in K$ corresponding to $\rho(B) > 0$*

or

(ii′) *K is quasi-reproducing, B is completely continuous and possesses a nonzero eigenvalue.*

(iii) *One of the conditions* (a)–(d) *is fulfilled.*

(iv) *B is irreducible.*

Then, for all $\varphi \in P_\kappa = \{\varphi \in K \mid \varphi \sim \kappa\}$ one has the alternative:

(a) *either*

$$f(B\varphi) = \rho(B) f(\varphi), \quad f \in K^*$$

or

$$\inf_{\substack{f\in K^* \\ f(\varphi)\neq 0}} \frac{f(B\varphi)}{f(\varphi)} < \rho(B) < \sup_{\substack{f\in K^* \\ f(\varphi)\neq 0}} \frac{f(B\varphi)}{f(\varphi)}.$$

(b) *In addition,*

$$\sup_{\varphi\in P_\kappa} \inf_{\substack{f\in K^* \\ f(\varphi)\neq 0}} \frac{f(B\varphi)}{f(\varphi)} = \rho(B) = \inf_{\varphi\in P_\kappa} \sup_{\substack{f\in K^* \\ f(\varphi)\neq 0}} \frac{f(B\varphi)}{f(\varphi)}.$$

Proof. First we remark that, according to [**7**, p. 87], condition (ii′) entails (ii).

Now, let the conditions (i)–(iv) be satisfied. Then $\rho(B) > 0$ because of (ii). Assume $\varphi \in P_\kappa$.

First, we prove the alternative:

either

$$B\varphi = \rho(B)\varphi \tag{3.1}$$

or

$$\rho(B)\varphi \not\leq B\varphi \not\leq \rho(B)\varphi. \tag{3.2}$$

To this end, let (3.1) be false, i.e., let $B\varphi \neq \rho(B)\varphi$. Assume $B\varphi \leq \rho(B)\varphi$. Now, under the conditions (i)–(iii), a functional $l_\chi \in K^*$,

$l_\chi \neq 0$, exists such that $B^* l_\chi = \rho(B) l_\chi$ (cf. [**7**, p. 170]). Further, from $\rho(B)\varphi - B\varphi \geq 0$, $\rho(B)\varphi - B\varphi \neq 0$, and because B is irreducible by assumption, there is an $n \in \mathbf{N}$ such that $l_\chi(B^n(\rho(B)\varphi - B\varphi)) > 0$ (cf. [**7**, p. 113]). On the other hand, $l_\chi[B^n(\rho(B)\varphi - B\varphi)] = 0$ so that we get a contradiction.

Hence, the righthand side of (3.2) is proven.

The proof of the lefthand side of (3.2) is obtained by replacing $\rho(B)\varphi - B\varphi$ by $B\varphi - \rho(B)\varphi$ and using the same arguments as before.

The rest of the proof is as follows. Equation (3.1) is equivalent to $f(B\varphi) = \rho(B) f(\varphi)$, $f \in K^*$. Relation (3.2) is equivalent to $B\varphi - \rho(B)\varphi \notin K$ and $B\varphi - \rho(B)\varphi \notin (-K)$. From [**7**, p. 21], one has: there are functionals $f_0, g_0 \in K^*, f_0, g_0 \neq 0$, such that $f_0(B\varphi - \rho(B)\varphi) = -1 < 0$ and $g_0(B\varphi - \rho(B)\varphi) = 1 > 0$. Therefore, one has $f_0(B\varphi) < \rho(B) f_0(\varphi)$. As $f_0(B\varphi) \geq 0$, it follows that $f_0(\varphi) > 0$ and hence

$$\frac{f_0(B\varphi)}{f_0(\varphi)} < \rho(B).$$

Further, one concludes $g_0(B\varphi) > \rho(B) g_0(\varphi)$ and therefore $g_0(B\varphi) > 0$. As $\varphi \in P_\kappa$, one has $\varphi \neq 0$, and there are numbers $\alpha(\varphi) > 0$ and $\beta(\varphi) > 0$ such that $\alpha(\varphi)\kappa \leq \varphi \leq \beta(\varphi)\kappa$. Hence,

$$\varphi \geq \alpha(\varphi)\kappa = \frac{\alpha(\varphi)}{\rho(B)} \rho(B)\kappa = \frac{\alpha(\varphi)}{\rho(B)} B\kappa \geq \frac{\alpha(\varphi)}{\rho(B)\beta(\varphi)} B\varphi$$

so that

$$g_0(\varphi) \geq \frac{\alpha(\varphi)}{\rho(B)\beta(\varphi)} g_0(B\varphi) > 0.$$

Therefore, one has

$$\rho(B) < \frac{g_0(B\varphi)}{g_0(\varphi)}.$$

Altogether, it follows that

$$\inf_{\substack{f \in K^* \\ f(\varphi) \neq 0}} \frac{f(B\varphi)}{f(\varphi)} < \rho(B) < \sup_{\substack{f \in K^* \\ f(\varphi) \neq 0}} \frac{f(B\varphi)}{f(\varphi)}.$$

Now, Part (b) is proven. From the last relation, it follows that

$$\sup_{\varphi \in P_\kappa} \inf_{\substack{f \in K^* \\ f(\varphi) \neq 0}} \frac{f(B\varphi)}{f(\varphi)} \leq \rho(B) \leq \inf_{\varphi \in P_\kappa} \sup_{\substack{f \in K^* \\ f(\varphi) \neq 0}} \frac{f(B\varphi)}{f(\varphi)}.$$

Further, the supremum on the lefthand side and the infimum on the righthand side are attained for $\varphi = \kappa$ so that the equality sign follows, respectively. $\square$

Theorem 3.1 is an important result in that it allows us to obtain the corresponding alternative for a wide class of integral operators. In contrast to this, the stronger condition of σ-boundedness of the operator B usually can only be verified in special cases, for integral operators.

4. Applications to integral operators. Let $\Omega = [a,b] \subset \mathbf{R}$ be a closed interval, let $\sigma \in C(\Omega)$ and $N = N_\sigma \subset \partial\Omega$ as well as $\Omega_N = \Omega_{N_\sigma} = \Omega \setminus N = \Omega \setminus N_\sigma$. We suppose that σ satisfies $\sigma(x) > 0$, $x \in \Omega_N$, $\sigma(x) = 0$, $x \in N = N_\sigma$, where $N = \varnothing$ is possible.

In this section, the results of Section 3 are applied to integral operators in the spaces $C_{\sigma^{-1}}(\Omega_N) \subset L_{\infty,\sigma^{-1}}(\Omega)$ and $L_\sigma(\Omega)$ (cf. subsections 4.1 and 4.2). The results in the space $L_\sigma(\Omega)$ are simply obtained by just considering the adjoint $B^T : L_{\infty,\sigma^{-1}}(\Omega) \to L_{\infty,\sigma^{-1}}(\Omega)$ of $B : L_\sigma(\Omega) \to L_\sigma(\Omega)$. This is because $[L_\sigma(\Omega)]^* = L_{\infty,\sigma^{-1}}(\Omega)$ holds true.

4.1 Application in $C_{\sigma^{-1}}(\Omega_N) \subset L_{\infty,\sigma^{-1}}(\Omega)$. Let $V = C(\Omega)$, the usual norm being defined by $\|u\|_\infty = \max_{x\in\Omega} |u(x)|$, $u \in V$. This is a complete subspace of $L_\infty(\Omega)$. Assume $K_\infty = \{u \in C(\Omega) \mid u(x) \ge 0,$ $x \in \Omega\}$. Then K_∞ is a cone which is normal and reproducing (and also solid). We remark that the set of nonnegative functions in $L_\infty(\Omega)$ also forms a normal and reproducing cone (which is, however, not solid). The σ-norm is defined by $\|u\|_\sigma := \inf\{t \ge 0 \mid -t\,\sigma(x) \le u(x) \le t\,\sigma(x),\ x \in \Omega\}$, and we have

$$\begin{aligned}\|u\|_\sigma &= \inf\{t \ge 0 \mid -t\,\sigma(x) \le u(x) \le t\,\sigma(x),\ x \in \Omega\} \\ &= \sup_{x\in\Omega_N} \frac{|u(x)|}{\sigma(x)} =: \|u\|_{\infty,\sigma^{-1}}, \quad u \in V_\sigma,\end{aligned} \tag{4.1}$$

where the space V_σ is defined by

$$V_\sigma := \left\{ u \in C(\Omega) \mid \sup_{x\in\Omega_N} \frac{|u(x)|}{\sigma(x)} < \infty \right\} =: C_{\sigma^{-1}}(\Omega_N). \tag{4.2}$$

Since $K = K_\infty$ is normal, $V_\sigma = C_{\sigma^{-1}}(\Omega_N)$ is complete with respect to the norm $\|\cdot\|_\sigma = \|\cdot\|_{\infty,\sigma^{-1}}$. Evidently,

$$(4.3) \qquad u(x) = 0, \quad x \in N = N_\sigma, \quad \text{for } u \in C_{\sigma^{-1}}(\Omega_N).$$

Let $B : V \to V$ be the integral operator defined by

$$(4.4) \qquad (Bu)(x) = \int_\Omega K(x,s)u(s)\,ds, \quad x \in \Omega, u \in V,$$

where

$$(4.5) \qquad K(\cdot,\cdot) \in C(\Omega \times \Omega)$$

$$(4.6) \qquad K(x,s) \geq 0, \quad x, s \in \Omega.$$

Then $B \in \mathcal{B}(V)$, B is completely continuous, and B is positive (with respect to K_∞). We remark that the kernel in (4.4) may also have, e.g., single logarithmic discontinuities, which sometimes occurs with Green's functions. Further, for $V = L_\infty(\Omega)$, the space V_σ is given by $V_\sigma = L_{\infty,\sigma^{-1}}(\Omega) = \{u \in L_\infty(\Omega) \mid \|u\|_\sigma := \|u\|_{\infty,\sigma^{-1}} = \operatorname{ess\,sup}_{x\in\Omega} |u(x)|/\sigma(x) < \infty\}$. This is a complete subspace of $L_\infty(\Omega)$ in the norm $\|\cdot\|_\sigma = \|\cdot\|_{\infty,\sigma^{-1}}$.

For the application of Theorem 3.1, a condition stronger than (4.6) is needed. For this, let $N = N_K \subset \partial\Omega$ and $\Omega_{N_K} = \Omega \setminus N_K$. Further, assume

$$(4.7) \qquad \begin{gathered} K(x,s) > 0, \quad x, s \in \Omega_{N_K} \\ K(x,s) = 0 \begin{cases} x \in N_K & s \in \Omega_{N_K} \\ s \in N_K & x \in \Omega_{N_K} \\ x \in N_K & s \in N_K \end{cases} \end{gathered}$$

For example, for $\Omega = [0,l]$ and the Green's functions $K(x,s) = G(x,s)$ with

$$(4.8) \qquad G(x,s) = \begin{cases} (1 - x/l)s & 0 \leq s \leq x \leq l \\ (1 - s/l)x & 0 \leq x \leq s \leq l \end{cases}$$

and

$$(4.9) \qquad G(x,s) = \begin{cases} s & 0 \leq s \leq x \leq l \\ x & 0 \leq x \leq s \leq l \end{cases},$$

condition (4.7) is fulfilled with $N_K = \{0, l\}$, respectively $N_K = \{0\}$, and one has $\kappa(x) = \sin((\pi/l)x)$ for (4.8), respectively $\kappa(x) = \sin((\pi/(2l))x)$ for (4.9).

In the following application the advantage of the weak conditions of Theorem 3.1 becomes evident.

Theorem 4.1. *Let the conditions* (4.5) *and* (4.7) *be fulfilled. Then one has the following alternative. For every* $\varphi \in P_\kappa := \{\varphi \in K_\infty \mid, \varphi \sim \kappa\}$,

(a) *either*

$$\int_\Omega K(x,s)\varphi(s)\,ds = \rho_\infty(B)\varphi(x), \quad x \in \Omega$$

or

$$\inf_{x\in\Omega_N} \frac{1}{\varphi(x)} \int_\Omega K(x,s)\varphi(s)\,ds < \rho_\infty(B) < \sup_{x\in\Omega_N} \frac{1}{\varphi(x)} \int_\Omega K(x,s)\varphi(s)\,ds.$$

(b) *In addition,*

$$\begin{aligned}\sup_{\varphi\in P_\kappa} \inf_{x\in\Omega_N} \frac{1}{\varphi(x)} \int_\Omega K(x,s)\varphi(s)\,ds &= \rho_\infty(B) = \rho_{\infty,\kappa^{-1}}(B) = \|B\|_{\infty,\kappa^{-1}} \\ &= \inf_{\varphi\in P_\kappa} \sup_{x\in\Omega_N} \frac{1}{\varphi(x)} \int_\Omega K(x,s)\varphi(s)\,ds.\end{aligned}$$

Proof. The conditions of Theorem 3.1 are satisfied.

(i) is clearly true. Further, (ii′) holds under the conditions (4.5) and (4.7). Moreover, condition (a) in (iii) is valid. Finally, B is irreducible in the space $V = C_N(\Omega) = \{u \in C(\Omega) \mid u(x) = 0,\, x \in N\}$. So (iv) is fulfilled. □

We remark that condition (4.7) can be made more general; for this, see, e.g., [**2**, pp. 527–529].

4.2 Application in $L_\sigma(\Omega)$. Let

$$\|u\|_{1,\sigma} := \int_\Omega |u(x)|\sigma(x)\,dx, \quad u \in L(\Omega). \tag{4.10}$$

The norms $\|\cdot\|_{1,\sigma}$ and $\|\cdot\|_1$ are equivalent if and only if $N = N_\sigma = \varnothing$. In any case, the inequality

$$\|u\|_{1,\sigma} \le \sigma_{max}\|u\|_1, \quad u \in L(\Omega), \tag{4.11}$$

holds with $\sigma_{\max} := \max_{x\in\Omega}\sigma(x)$. Let

$$L_\sigma(\Omega) := \left\{ u \mid u \text{ is measurable on } \Omega, \text{ and } \int_\Omega |u(x)|\sigma(x)\,dx < \infty \right\}. \tag{4.12}$$

Then, $L(\Omega) \subset L_\sigma(\Omega)$, the embedding being continuous due to (4.11). In case $N_\sigma = \varnothing$, one has $L(\Omega) = L_\sigma(\Omega)$, otherwise $L(\Omega) \not\supseteq L_\sigma(\Omega)$. Now, let $K(\cdot,\cdot) \in C(\Omega \times \Omega)$ and B be given by (4.4). Further, let $(B^T u)(x) = \int_\Omega K(s,x)u(s)\,ds$, $x \in \Omega$, be the operator associated with the kernel $K^T(x,s) := K(s,x)$ for appropriate functions u. In this setting, we have

Theorem 4.2. *Let the conditions* (4.5) *and* (4.7) *be fulfilled. Then one has the alternative: For every $\varphi \in P_\chi := \{\varphi \in K_\infty \mid \varphi \sim \chi\}$ where $B^T\chi = \rho_\infty(B^T)\chi$,*

(a) *either*

$$\int_\Omega \varphi(s)K(s,x)\,ds = \rho_\infty(B^T)\varphi(x), \quad x \in \Omega$$

or

$$\inf_{x\in\Omega_N} \frac{1}{\varphi(x)} \int_\Omega \varphi(s)K(s,x)\,ds < \rho_\infty(B^T) = \rho_1(B) < \sup_{x\in\Omega_N} \frac{1}{\varphi(x)} \int_\Omega \varphi(s)K(s,x)\,ds.$$

(b) *In addition,*

$$\begin{aligned}\sup_{\varphi\in P_\chi} \inf_{x\in\Omega_N} \frac{1}{\varphi(x)} \int_\Omega \varphi(s)K(s,x)\,ds &= \rho_\infty(B^T) = \rho_1(B) = \rho_{1,\chi}(B) = \|B\|_{1,\chi} \\ &= \inf_{\varphi\in P_\chi} \sup_{x\in\Omega_N} \frac{1}{\varphi(x)} \int_\Omega \varphi(s)K(x,s)\,ds.\end{aligned}$$

Proof. One applies Theorem 3.1 with B replaced by the adjoint $B^T : L_{\infty,\sigma^{-1}}(\Omega) \to L_{\infty,\sigma^{-1}}(\Omega)$ since $[L_\sigma(\Omega)]^* = L_{\infty,\sigma^{-1}}(\Omega)$. Here, $B^T : L_{\infty,\sigma^{-1}}(\Omega) \to C_{\sigma^{-1}}(\Omega_N)$. Further, K^* may be replaced by $L^* = \{f = f_x \mid f_x(u) = u(x),\, x \in \Omega_N\}$. Then the assertion follows with $\rho_\infty(B^T)$. Finally, one takes into account $\rho_\infty(B^T) = \rho_1(B) = \rho_{1,\chi}(B) = \|B\|_{1,\chi}$. $\square$

The list of following references is, of course, by no means exhaustive. Only those references are given which were used.

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