ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 30, Number 2, Summer 2000

THE STRUCTURE OF SYMMETRY GROUPS OF ALMOST PERFECT ONE FACTORIZATIONS

DELLA DUNCAN AND EDWIN IHRIG

ABSTRACT. The concept of j-perfection of a one factorization is introduced. A one factorization is perfect if and only if it is 2-perfect. We call 3-perfect one factorizations almost perfect one-factorizations (or APOFs). First we give some general results concerning the automorphism groups of j-perfect one factorizations, and then we classify all APOFs which have more than one automorphism of order two with fixed points. Several structure theorems for the automorphism groups of APOFs are also given.

0. Introduction. Let Γ be an undirected simple connected graph with vertex set V. A one factor (or perfect matching) of Γ is a subgraph of Γ in which each vertex in V has degree one. A one factorization $\mathcal{F}$ of Γ is a collection of one factors of Γ so that each edge of Γ occurs in exactly one of the one factors in $\mathcal{F}$. Associated with any one factorization $\mathcal{F}$ is its group of automorphisms, $\mathrm{Aut}(\mathcal{F})$, which is the collection of all the permutations of V which transforms any one factor in $\mathcal{F}$ in to another one factor in $\mathcal{F}$.

Given any finite group G, Cameron has shown there is a one factorization of K_{2n} (the complete graph on $2n$ vertices) which has G as its automorphism group. This result is a consequence of an analogous result by Mendelson given for Steiner triple systems (see [**7**]). We provide the details here in Section 4 for easy reference. This result means that there are unlikely to be any strong structure results that apply to the automorphism groups of a general one factorization.

However, in contrast to this situation, it has been discovered that one factorizations with extra structure can have automorphism groups with a very restricted form. An example of this are perfect one factorizations. A perfect one factorization (POF) is a one factorization in which the union of any two distinct one factors is connected. There are a number of very restrictive theorems governing the nature of the automorphism groups of POFs. See [**8**] for an overview. For example,

Received by the editors on July 8, 1998, and in revised form on April 8, 1999.

any nonsolvable subgroup of an automorphism group of a POF must have cardinality that divides $|V|-2$ evenly and can have at most three elements of order 2 (see [**3**] and [**4**]). Also, one of its elements of order two must be central. So, in any case, the radical of this group is nontrivial. This means, for example, that no semi-simple group can occur as the subgroup of the automorphism group of a perfect one factorization.

Another illustrative result is the following: Hartman and Rosa [**2**] showed that for each n, K_{2n} has a one factorization with an automorphism group which acts transitively on the vertices. A POF can have a vertex transitive group if and only if n is prime (see [**5**]). Moreover, given any group, G, of odd order, one can construct a one factorization on $K_{2|G|}$ which has an automorphism group that acts transitively on the vertex set and has G as a subgroup (with $G \oplus \mathbf{Z}_2$ acting simply transitively on the vertex set). In contrast to this, a POF on K_{2n} with a vertex transitive automorphism group must have $\mathrm{Aut}(\mathcal{F})$ isomorphic to $[\mathbf{Z}_{2p}]\mathbf{Z}_k$ where $\mathbf{Z}_k$ is a subgroup of the cyclic group $\mathbf{Z}_p^*$, the group of multiplicative units in $\mathbf{Z}_p$. In particular any subgroup G of $\mathrm{Aut}(\mathcal{F})$ must be two step solvable, i.e., $[G,G]$ must be abelian.

The natural question which arises is whether there is some way to relax the perfect condition of a POF without losing all the structure which is imposed by this condition. One can also ask whether there is some way to preserve some of the structure while still looking at one factorizations on graphs which are more general than K_{2n}. The first step is to try to identify a useful, but weaker, condition than the perfect condition. It turns out that there is a series of conditions stretching from the POF condition down to no extra condition at all. We call this condition j-perfect. A one factorization is called j-perfect if the union of any j distinct one factors is connected. A POF is 2-perfect. Any one factorization is $|V|/2$-perfect. j-perfection describes a simple graph theoretical property of the one factorization which makes itself felt in the structure of the automorphism group. The main purpose of this paper is to initiate the study of how this property affects the structure of $\mathrm{Aut}(\mathcal{F})$.

There is another property which gives direct control over the structure of the automorphism group in a more direct way. Its disadvantage is that, unlike j-perfection, it does not appear to have a simple graph theoretical interpretation. However, we are able to show that j-

perfection gives information about this property, so it serves the role of being an intermediate property in the analysis of the structure of $\mathrm{Aut}(\mathcal{F})$. We call this property j-irreducibility, and it relies on the concept of subspace introduced by Cameron [**1**]. Let S be a subset of the vertex set. We call S a subspace if any one factor that has a single edge connecting two vertices of S has all its edges connecting either two vertices in S or two vertices in $V - S$. We call a one factorization j-irreducible if any subspace with more than j vertices must be the whole vertex set. One of the things that makes this a useful concept is that the set of fixed points of an automorphism is a subspace. Thus j-irreducibility gives control over the fixed point structure of the automorphism group. As with j-perfection, we have a spectrum of conditions. Any POF on K_{2n} is 2-irreducible, and, in fact, a j-perfect one factorization on K_{2n} is j-irreducible. Every one factorization is n-irreducible where $n = |V|/2$, and the relationship between j-perfection and j-irreducibility is that a j-perfect one factorization is $(j + |V| - |\mathcal{F}| - 1)$-irreducible. See Proposition 1.1.

In Section 1 we will explore some of the basic relationships between j-perfection, j-irreducibility and the cycle structure of the elements of $\mathrm{Aut}(\mathcal{F})$. We also explore some basic properties of automorphisms that fix every one factor of $\mathcal{F}$. Knowledge of these automorphisms enables one to gain information by contrasting the action of the automorphism group on the vertices with its action on the one factors.

In Section 2 we study some special automorphisms that are crucial to the understanding of $\mathrm{Aut}(\mathcal{F})$. We call these automorphisms modified one factor symmetries. They are automorphisms of order two with exactly two fixed points. The key idea is that if you connect an edge between the fixed points, and connect every other vertex to its image under the automorphism, then you will obtain a one factor of $\mathcal{F}$. It is then possible to construct parts of $\mathcal{F}$ using knowledge about $\mathrm{Aut}(\mathcal{F})$ alone. We finish this section by giving some applications of these basic results. We specialize to 3-perfect one factorizations, which we call almost perfect one factorizations, or APOFs. Theorem 2.3 gives a detailed description of APOFs with a one factor transitive $\mathrm{Aut}(\mathcal{F})$ which has a modified one factor symmetry. This result does not assume the graph is K_{2n} but only assumes it satisfies a condition called involution completeness. It is interesting that under these circumstances that the APOF is forced to be a POF, and, in fact, very specific POFs on very specific graphs. Also in this section we are able to give some solvability results (Corollary 2.6 and Theorem 2.4).

Section 3 concerns another type of automorphism of order two. These automorphisms have no fixed points, and they satisfy the property that if every vertex is connected to its image by an edge, the resulting one factor is in $\mathcal{F}$. These are called one factor symmetries. Section 4 gives the details of Cameron's result mentioned in the beginning of this introduction.

1. Preliminaries. We start with some notation, definitions and basic results that will be needed to prove the results in the later sections. Let $n \geq 3$ be an integer.

1.1. *Notation.* (a) For any set S, $|S|$ is the cardinality of S.

(b) $\Gamma = (V, E)$ denotes an undirected simple connected graph with vertex set V and edge set E. $|V|$ will always be even and Γ regular, and we assume Γ has a one factorization. Define n and r as follows

$$|V| = 2n$$

and

$$\deg(\Gamma) = r.$$

(c) If X is a set, then $S_X = \{f : X \to X \mid f \text{ is one to one and onto}\}$. We use $\hat{1}$ to denote the identity function in S_X.

(d) If $T \subset S_V$, then $\langle T \rangle$ denotes the subgroup generated by T. If $\sigma_1, \ldots, \sigma_n$ are elements of S_V, then $\langle \sigma_1, \ldots, \sigma_n \rangle$ denotes the subgroup generated by these elements.

(e) Let $T \subset S_V$. Then $F_T = \{x \in V \mid \tau(x) = x \ \forall\ \tau \in T\}$. If $\tau \in S_V$, let $F_\tau = F_{\{\tau\}}$.

(f) Let $G \subset S_V$ where G is a subgroup. Then $G_v = \{g \in G | g(v) = v\}$.

(g) Let $G \subset S_V$. If $g \in G$ then $o(g)$ is the order of g.

Next we need the definitions of one factor along with j-perfect and j-irreducible one factorizations.

Definition 1.2. (a) $\sigma \in S_V$ is called a one factor of Γ if $o(\sigma) = 2$, $F_\sigma = 0$ and $\{v, \sigma(v)\}$ is an edge of Γ for each $v \in V$.

(b) $\mathcal{F}$ is called a one factorization of Γ if $\mathcal{F} \subset S_V$ and

(i) Each element of $\mathcal{F}$ is a one factor of Γ.

(ii) If $\sigma, \sigma' \in \mathcal{F}$ and $\sigma \neq \sigma'$ then $F_{\sigma\sigma'} = 0$, i.e., two distinct one factors have no edges in common.

(iii) $\cup\mathcal{F} = E$ where $\cup\mathcal{F} = \{\{v, \sigma(v)\} : v \in V \text{ and } \sigma \in \mathcal{F}\}$.

Thus every edge of Γ is in some one factor.

(c) A one factorization $\mathcal{F}$ of Γ is called j-perfect if, whenever $S \subset \mathcal{F}$ with $|S| \geq j$, then $\langle S \rangle$ acts transitively on V. This is equivalent to saying the union of any j distinct one factors is connected.

(d) $\mathcal{F}$ is called perfect (POF) if $\mathcal{F}$ is 2-perfect. The union of any two distinct one factors forms a Hamiltonian circuit. $\mathcal{F}$ is called almost perfect (an APOF) if $\mathcal{F}$ is 3-perfect.

(e) Let $\mathcal{F}$ be a one factorization of Γ. Let $W \subset V$. W is called a subspace of V if whenever $\sigma \in \mathcal{F}$ and $\sigma(W) \cap W \neq \varnothing$ then $\sigma(W) = W$. See [**1**, p. 5].

(f) $\mathcal{F}$ is j-irreducible if whenever W is a subspace of V with $|W| > j$ then $W = V$.

The following result relates reducibility to j-perfect.

Proposition 1.3. (a) *If $\mathcal{F}$ is j-perfect on Γ with* $\deg(\Gamma) = r$*, then it is $(j+2n\text{-}1\text{-}r)$-irreducible, and if $j \equiv r \bmod 2$, then it is $(j+2n-2-r)$ irreducible.*

(b) *Any one factorization is both n-perfect and n-irreducible.*

(c) *No one factorization is either 1-perfect or 1-irreducible.*

(d) *If $\chi(\Gamma)$ denotes the vertex chromatic number of Γ and $\mathcal{F}$ is j-irreducible, then $j \geq 2n/\chi(\Gamma)$.*

Proof. We start with (a). Let W be a subspace with $|W| > j + 2n - 1 - r$. We will show this forces $W = V$. Let $x_0 \in W$, and let $N = \{x \in V : \{x_0, x\} \text{ is an edge of } \Gamma\}$. Then $|N| = r$ so

$$|N \cap W| = |N| + |W| - |N \cup W| > j - 1.$$

Let $x_1, \ldots, x_j$ be distinct vertices in $N \cap W$. Let σ_i, $i = 1, \ldots, j$ be one factors in $\mathcal{F}$ so that $\sigma_i(x_0) = x_i$. These one factors exist since $x_i \in N$. Since W is a subspace, we have $\sigma_i(W) = W$ for each i. Let $G = \langle \sigma_1, \ldots, \sigma_j \rangle$. We have $GW = W$. Since $\mathcal{F}$ is j-perfect, G acts transitively on V so $W = V$.

To finish (a) consider when $j \equiv r \bmod 2$. Let $k = (j + 2n - 1 - r)$ which is odd. We have shown $\mathcal{F}$ is k-irreducible. Now let W be a subspace with $|W| > k - 1$. Since no one factorization is 1-perfect, with $|V| > 2$, we have $j \geq 2$. Hence

$$|W| + r \geq k + r = j + 2n - 1 \geq 2n + 1.$$

Let $x \in W$, and let $N = \{y \mid \{x, y\} \text{ is an edge of } \Gamma\}$. Then the above equation shows $N \cap W \neq \varnothing$. Let $y \in W \cap N$. Since $y \in N$ there is a one factor σ so that $\sigma(x) = y$. $y \in W$ and W is a subspace so $\sigma(W) = W$. Hence $|W|$ is even. Now $|W| > k - 1$ which is even. Thus $|W| > k$, and $W = V$ because $\mathcal{F}$ is k-irreducible.

Next we consider (b). Note that [**1**, p. 25, Theorem 2.2] shows any one factorization on K_{2n} is n-irreducible. The following proof for both n-perfect and n-irreducible is similar to the one presented there. We first show n-perfect. Let S be a collection of n one factors. Then $\langle S \rangle \{v \mid \text{for any } v \in V\}$ has at least $n + 1$ vertices. Thus $\langle S \rangle$ can have only one orbit since $|V| = 2n$ and distinct orbits have empty intersections.

To show $\mathcal{F}$ is n-irreducible, assume that W is a subspace with $|W| > n$. Let $\sigma \in \mathcal{F}$. W and $\sigma(W)$ must intersect since their cardinalities are larger than n. Hence $\sigma(W) = W$. This is true for all $\sigma \in \mathcal{F}$; therefore $\langle \mathcal{F} \rangle W = W$. But $\langle \mathcal{F} \rangle$ acts transitively on V since the union of all the one factors is the entire graph which is assumed to be connected.

To show (c), observe that since no one factor is connected when $n > 1$, and every subset W on V with two elements is a subspace. Thus, $\mathcal{F}$ cannot be either 1-perfect or 1-irreducible.

We finish with (d). Vertex color V with $\chi(\Gamma)$ colors so that no two adjacent vertices have the same color. Let S_i denote the collection of vertices with color i. For at least one i we must have $|S_i| \geq 2n/\chi(\Gamma)$. $S_i \cap \sigma(S_i) = \varnothing$ for all $\sigma \in \mathcal{F}$. Thus S_i is a subspace for all i. So either $j \geq |S_i|$ or $S_i = V$. If $S_i = V$ then $\chi(\Gamma) = 1$ which is not possible since Γ has an edge.

Corollary 1.4. (a) *If* Γ *is* K_{2n}, *then any* j*-perfect* $\mathcal{F}$ *is* j*-irreducible.*

(b) *An* APOF *on* K_{2n} *is* 2*-irreducible.*

Proof. To show (a) use Proposition 1.3(a) with $r = 2n - 1$.

In (b) let $\mathcal{F}$ be an APOF on K_{2n}. Then $\mathcal{F}$ is 3-perfect and $r = 2n-1$ so $3 \equiv 2n - 1 \bmod 2$. Therefore, by Proposition 1.3(b) we have $\mathcal{F}$ is 2-irreducible. $\square$

The following example illustrates that even perfect one factorizations need not be $n-1$ irreducible if the graph is not K_{2n}.

Example. Let Γ be the cycle on $2n$ vertices; that is, let $V = \mathbf{Z}_{2n}$ and $E = \{\{x, x+1\} : x \in \mathbf{Z}_{2n}\}$. Then the set of even vertices W is a subspace of V for any one factorization of Γ since $\sigma(W) \cap W$ is empty for any one factor.

Definition 1.5. Let $\mathcal{F}$ be a one factorization of Γ.

(a) $\mathrm{Aut}(\mathcal{F}) = \{\tau \in S_V \mid \tau^{-1}\sigma\tau \in \mathcal{F} \ \forall \ \sigma \in \mathcal{F}\}$.

(b) $\iota : \mathrm{Aut}(\mathcal{F}) \to S_{\mathcal{F}}$ is defined by

$$\iota(\tau)(\sigma) = \tau^{-1}\sigma\tau$$

where $\tau \in \mathrm{Aut}(\mathcal{F})$ and $\sigma \in \mathcal{F}$.

The structure of $\ker(\iota)$ is of fundamental importance in the study of the symmetries of one factorizations. Cameron calls elements of $\ker(\iota)$ strict automorphisms [**1**, p. 11] and shows that $\ker(\iota)$ is isomorphic to $\mathbf{Z}_2^k$ for some k when $\Gamma = K_{2n}$ (see [**1**, p. 11, Theorem 1.4]). In [**1**] it is also shown that $\ker \iota$ is trivial for POFs of K_{2n}. The situation for a general graph is more complicated. For example, $\ker \iota = \mathbf{D}_n$ for the unique POF on the $2n$ cycle. The following result can be used for more general graphs.

Theorem 1.6. *Let $K = \ker(\iota)$.*

(a) *Every nonidentity automorphism in K is fixed-point free.*

(b) *If $\tau \in K$ and $o(\tau)$ is odd, then $o(\tau) \leq 2n/\chi(\Gamma)$ where $\chi(\Gamma)$ is the chromatic number of Γ.*

(c) *Let $\mathcal{F}$ be j-irreducible with $j < n$.*

(i) *If $\tau \in K$, then $o(\tau) \leq j$.*

(ii) *If K does not act transitively on V, then $|K| \leq j$.*

(iii) *If K' is a proper subgroup of K, then $|K'| \leq j$.*

(iv) *If K is nilpotent, then $|K| \leq j$.*

(d) *If $\mathcal{F}$ is 2-irreducible and $K \neq \{\hat{1}\}$, then n must be odd,* $\deg(\Gamma)$ *must be n or $n-1$, and $K \simeq \mathbf{Z}_2$.*

Proof. Parts (a) and (b) are proved for K_{2n} in [**1**, p. 10, Theorem 1.3(i)]. Consider (a) in the more general case. Let $\tau \in K$ and $\tau(x) = x$. Since $\tau^{-1}\sigma\tau = \sigma$ for all σ in $\mathcal{F}$ we have that τ commutes with every element of $\langle \mathcal{F} \rangle$. Since Γ is connected $\langle \mathcal{F} \rangle$ acts transitively on V. So for any $y \in V$ there exists $g \in \langle \mathcal{F} \rangle$ such that $gx = y$. Thus $\tau(y) = \tau(gx) = g\tau(x) = gx = y$ which implies $\tau = \hat{1}$.

Next assume $\tau \in K$ with $o(\tau)$ odd. Since K consists of fixed point free automorphisms, each orbit of $\langle \tau \rangle$ has cardinality $o(\tau)$. Thus we have $2n/o(\tau)$ orbits. We claim no edge in Γ connects two vertices in the same $\langle \tau \rangle$ orbit. Suppose we have an edge from x to y where x and y are in the same orbit. Let σ be the one factor such that $\sigma(x) = y$. x and y are in the same $\langle \tau \rangle$ orbit so $\tau^k(x) = y$. Then

$$\begin{aligned} \tau^{2k}(x) &= \tau^k(y) \\ &= \tau^k(\sigma(x)) \\ &= \sigma\tau^k(x) \\ &= \sigma(y) \\ &= x. \end{aligned}$$

Therefore τ^{2k} has x as a fixed point. Hence $\tau^{2k} = \hat{1}$. Note that $\tau^k \neq \hat{1}$ since $\tau^k(x) = y$ and $x \neq y$. So τ^k has order two. But $o(\tau)$ is odd which forces $o(\tau^{2k})$ to be odd which is a contradiction. Define a vertex coloring of Γ with $2n/o(\tau)$ colors by using the same color for all vertices in any one orbit of $\langle\tau\rangle$, but using different colors for each orbit. Since no two vertices within an orbit are connected by an edge, this gives a vertex coloring of Γ, and $\chi(\Gamma) \leq 2n/o(\tau)$ completing (b).

We now show (c) starting with parts (ii) and (iii). Let K' be any subgroup of K, not necessarily proper. Let W be a K' orbit. We claim that W is a subspace. By (a) we have $|K'| = |W|$. Let $\sigma \in \mathcal{F}$ and $x \in W$ be such that $\sigma(x) \in W$. Now let $k \in K'$. The $\sigma(kx) = k\sigma(x)$, therefore $\sigma(kx) \in W$ for all $k \in K'$. But W is the K' orbit of x so $\sigma(W) = W$. Hence if $|K'| > j$, then $|K'| = 2n$. Both (ii) and (iii) are clear since $|K'| = 2n$ implies K' acts transitively on V. No proper subgroup K' of K can act transitively on V since K has trivial isotropy subgroup so $|K'| < |K| \leq |V|$.

Next consider (iv). We can assume $|K| = 2n$ since if $|K| < 2n$ then by (ii) $|K| \leq j$. Let G be the Sylow 2-subgroup of K and H the subgroup of index 2 in G. Since every Sylow subgroup of a nilpotent group is normal, let K' be the product of H with all of the other Sylow p-subgroups of K where p is an odd prime. K' is a subgroup of index 2 in K. By (iii) $n = |K'| \leq j$ which is a contradiction completing (iv). For (i) let $\tau \in K$. If $\langle\tau\rangle$ is a proper subgroup of K, then (iii) implies $o(\tau) = |\langle\tau\rangle| \leq j$. If $\langle\tau\rangle = K$, then K is nilpotent and (iv) may be applied.

We now consider (d). First we use (c) to show $K \simeq \mathbf{Z}_2$. From (c)(i) we know that every element of K must have order 2 or less. So K is an elementary abelian 2-group and therefore nilpotent. Then by (c) (iv) we have $K \simeq \mathbf{Z}_2$. Now let $\tau \neq \hat{1}$ be an element of K. Then $o(\tau) = 2$. We will show that τ has at most one edge in common with any one factor $\sigma \in \mathcal{F}$. Suppose not. Then there are vertices x and y with $\sigma(x) = \tau(x)$, $\sigma(y) = \tau(y)$ and $|W| = 4$ where

$$W = \{x, \tau(x), y, \tau(y)\}.$$

We claim W is a subspace which will contradict the 2-irreducibility of Γ. Let $\sigma' \in \mathcal{F}$ so $\sigma'(W) \cap W = W' \neq \varnothing$. τ leaves W invariant and τ commutes with σ', so τ leaves W' invariant. σ' leaves W' invariant since σ' has order two. Thus $\langle \tau, \sigma' \rangle$ leaves W' invariant. We may assume that $|\langle \tau, \sigma' \rangle| = 4$ for if not then $\tau = \sigma'$ and clearly $\sigma'(W) = W$.

Neither τ nor σ' have fixed points. We claim that $\tau\sigma'$ has no fixed points in W. If $z \in W$ and $\tau\sigma'(z) = z$ then $\sigma'(z) = \tau(z)$ but $\sigma = \tau$ on W and this contradicts $\sigma \neq \sigma'$. Therefore, $\langle \tau, \sigma' \rangle$ has a trivial isotropy subgroup when it acts on W. Thus 4 divides $|W'|$ which shows $W = W'$.

Next we show that all of the edges of τ, with one possible exception, must be edges of Γ. Let $\{x, \tau(x)\}$ and $\{y, \tau(y)\}$ be distinct edges on τ which are not in Γ. Let $W = \{x, y, \tau(x), \tau(y)\}$. Let $\sigma \in \mathcal{F}$ and $a \in W$ be such that $\sigma(a) \in W$. Again, let

$$W' = \{a, \sigma(a), \tau(a), \tau\sigma(a)\}.$$

We have $W' \subset W$ and W' is invariant under σ. Either $|W'| = 4$ in which case $\sigma(W) = W$ and W is a subspace or $|W'| = 2$. In the latter case as before we find $\tau(a) = \sigma(a)$ which shows $\tau_{|W}$ has the edge $\{a, \sigma(a)\}$. But τ has only two edges $\{x, \tau(x)\}$ and $\{y, \tau(y)\}$ in V neither of which are edges of Γ by assumption. Thus $|W'| = 2$ is not possible.

We can now show n is odd. Assume n is even. By the above, τ has at least $n-1$ edges from Γ. Let σ be a one factor containing one of these edges. Since τ commutes with each one factor in $\mathcal{F}$ it permutes the edges of σ under conjugation. τ fixes an edge of σ if and only if τ shares that edge in common with σ. If n is even, then τ permutes an even number of edges. Thus τ must fix an even number of edges and must therefore have an even number of edges in common with the one factor. But τ can have at most one edge in common with σ, giving a contradiction. Since n is odd each one factor has an odd number of edges. This means it must share exactly one edge with τ. τ has n edges so there are either n or $n-1$ one factors in Γ. Hence $\deg(\Gamma)$ is either n or $n-1$ depending on whether all the edges of τ are edges in Γ or not. $\square$

Parts (a) and (b) in the following are generalizations of [**6**, Theorem 2.1].

Theorem 1.7. *Let $\mathcal{F}$ be j-irreducible. Let $\tau \in \mathrm{Aut}(\mathcal{F})$.*

(a) *If τ has at least $j+1$ fixed points, then $\tau = \hat{1}$.*

(b) *If τ has a fixed point and τ commutes with at least j one factors then* $\tau = \hat{1}$.

(c) $|\mathrm{Aut}(\mathcal{F})|$ *divides* $2n(\gcd[(2n-1)(2n-2)\cdots(2n-j), r(r-1)\cdots(r-j-1)]$ *where* $|V| = 2n$ *and* $\deg(\Gamma) = r$.

Proof. The set of fixed points of τ form a subspace of V. See [**1**, p. 64, Theorem 4.1]. This finishes (a).

Next consider (b). If τ commutes with σ and fixes x, then τ fixes $\sigma(x)$. Hence if τ commutes with j one factors $\sigma_1, \dots, \sigma_j$ then τ fixes the $j+1$ points

$$\{x, \sigma_1(x), \dots, \sigma_j(x)\}.$$

By (a) $\tau = \hat{1}$.

To prove (c) we first show that $|\mathrm{Aut}(\mathcal{F})|$ divides $2n(2n-1)\cdots(2n-j)$. Let

$$S = \{(x_0, \dots, x_j) \mid x_i \text{ are distinct elements of } V\}.$$

We have $|S| = 2n(2n-1)\cdots(2n-j)$. $\mathrm{Aut}(\mathcal{F})$ acts on S by $\tau(x_0, \dots, x_j) = (\tau(x_0), \dots, \tau(x_j))$ for all τ in $\mathrm{Aut}(\mathcal{F})$. All of the orbits of $\mathrm{Aut}(\mathcal{F})$ have the same cardinality since the stability subgroup of $\mathrm{Aut}(\mathcal{F})$ is trivial by (a). $\mathrm{Aut}(\mathcal{F})$ also divides $2n(r)(r-1)\cdots(r-j+1)$ since $\mathrm{Aut}(\mathcal{F})$ acts on

$$S' = \{(x, \sigma_1, \dots, \sigma_j) \mid x \in V \text{ and } \sigma_i \text{ distinct elements of } \mathcal{F}\}$$
$$|S'| = 2nr(r-1)\cdots(r-j+1)$$

and the stability subgroup of $\mathrm{Aut}(\mathcal{F})$ acting on S' is trivial by (b). □

The following result is a generalization of 3.4 in [**3**] which used $\Gamma = K_{2n}$.

Corollary 1.8. *Let $\mathcal{F}$ be a 2-irreducible one factorization. Let* $\tau \in \mathrm{Aut}(\mathcal{F})$.

(a) *If $|F_\tau| = 2$, then there is an integer k such that*

(i) *τ is a product of disjoint k cycles where k divides $2n-2$.*

(ii) $\iota(\tau)$ *is a product of disjoint* k *cycles. Moreover,* k *divides* r *if there is no edge connecting the fixed points of* τ*, and* k *divides* $r-1$ *if there is an edge connecting the fixed points of* τ*.*

(b) *If* $|F_\tau| = 1$*, then there is an integer* k *so that*

(i) τ *is a product of disjoint* k *cycles where* k *divides* $2n-1$*.*

(ii) $\iota(\tau)$ *is a product of disjoint* k *cycles where* k *divides* r*. Hence* $o(\tau)$ *divides* $\gcd(2n-1, r)$*.*

Proof. We start with (a)(i). Let k be the length of the shortest cycle in the disjoint cycle decomposition of τ. We will show that every cycle has length k. Let i be the length of another cycle. Since τ^k has at least $k+2$ fixed points, by Theorem 1.2, $\tau^k = \hat{1}$. Thus i divides k since k was the smallest cycle length $i = k$. Since τ has exactly 2 fixed points, we have $2n-2 = kN$ where N is the number of disjoint cycles of τ. Hence, k divides $2n-2$.

The proof of (a)(ii) is similar. Let m be the smallest cycle length on $\iota(\tau)$. $\iota(\tau)^m$ has at least 2 fixed points, and τ has a fixed point. Therefore $\iota(\tau)^m$ must be the identity, and the length of any other cycle must divide m. Since τ has a fixed point, we have $o(\tau) = m$ by Theorem 1.1(a), so $m = k$. Next observe that $r - |F_{i(\tau)}| = kN$ where N is the number of disjoint cycles of $\iota(\tau)$. If $\sigma \in F_{\iota(\tau)}$, then $\tau\sigma = \sigma\tau$ so σ leaves the fixed points of τ invariant. Hence, no such σ exists if there is no edge connecting the fixed points of τ. Finally, consider the case when there is an edge connecting the fixed points of τ; then there is a σ containing this edge. $\tau\sigma\tau$ also contains this edge so $\tau\sigma\tau = \sigma$. Thus, $\sigma \in F_{\iota(\tau)}$ and $|F_{\iota(\tau)}| = 1$ as desired.

The only difference in the proof of (b) is that $F_{\iota(\tau)} = \varnothing$. If σ existed such that $\sigma\tau = \tau\sigma$ and $\tau(x) = x$, then $\tau(\sigma(x)) = \sigma(\tau(x)) = \sigma(x)$ and τ would have two fixed points. $\square$

2. Modified one factor symmetries.

Definition 2.1. (a) τ is called a modified one factor symmetry of $\mathcal{F}$ if $\tau \in \mathrm{Aut}(\mathcal{F})$, $o(\tau) = 2$ and $|F_\tau| = 2$.

(b) Let τ be a modified one factor symmetry of some $\mathcal{F}$. Define σ_τ

as follows:

$$\sigma_\tau = (i,j)\tau$$

where $F_\tau = \{i,j\}$.

Definition 2.2. A graph Γ is called involution complete with respect to $\mathcal{F}$ if for each $\tau \in \mathrm{Aut}(\mathcal{F})$ with $o(\tau) = 2$ and $|F_\tau| = 2$ and for each $x \in V$ with $\tau(x) \neq x$ we have $\{x, \tau(x)\}$ is an edge of Γ.

Clearly K_{2n} is involution complete with respect to any $\mathcal{F}$. Below we give a class of graphs which also satisfy this property.

Example. Let Γ be any graph (which need not be connected). Suppose Γ contains two disjoint copies of K_n as a subgraph and $r < 3n/2 - 1$. Then Γ is involution complete with respect to any $\mathcal{F}$.

To see this, let $V = V_1 \cup V_2$ where the induced subgraph on V_i is isomorphic to K_n for $i = 1$ and 2. Let $\tau \in \mathrm{Aut}(\mathcal{F})$ with $F_\tau \neq \varnothing$. Any automorphism of a one factorization of Γ must be an automorphism of Γ, and $\tau \in \mathrm{Aut}(\Gamma)$, together with $F_\tau \neq \varnothing$, will be the only properties of τ we will use here. We will show $\tau(V_i) = V_i$. Let $x \in F_\tau$. Assume without loss of generality that $x \in V_1$. We have

$$|\tau(V_1) \cap V_2| + n - 1 \leq \deg(x) < 3n/2 - 1.$$

Hence $|\tau(V_1) \cap V_2| < n/2$ giving $|\tau(V_1) \cap V_1| > n/2$. Assume $\tau(V_1) \cap V_2 \neq \varnothing$. Let $y \in \tau(V_1) \cap V_2$. Now y has every vertex in V_2 as a neighbor and also every vertex in $\tau(V_1)$ as a neighbor since τ is a graph isomorphism. Hence

$$3n/2 - 1 > \deg(y) \geq (|V_2 - \{y\}) \cup (\tau(V_1) \cap V_1)| > n - 1 + n/2$$

which is a contradiction. Hence $\tau(V_1) \cap V_2 = \varnothing$ so $\tau(V_1) = V_1$ which implies $\tau(V_2) = V_2$ as well.

The following result illustrates the usefulness of the property of involution completeness.

Theorem 2.3. *Let Γ be involution complete with respect to $\mathcal{F}$, a one factorization of Γ. Let $\tau \in \mathrm{Aut}(\mathcal{F})$ be a modified one factor symmetry. Then $\sigma_\tau \in \mathcal{F}$.*

Proof. When $n = 1$ there is nothing to prove so let $n \geq 2$ and $x \in V$ so that $\tau(x) \neq x$. There exists $\sigma \in \mathcal{F}$ such that $\sigma(x) = \tau(x)$ since $\{x, \tau(x)\}$ is an edge of Γ. We will show $\sigma = \sigma_\tau$. First consider $y \in V$ such that $\tau(y) \neq y$. Then $\sigma_\tau(y) = \tau(y)$. We will show $\sigma(y) = \tau(y)$. We know $\{y, \tau(y)\}$ is an edge. Hence there exists σ' such that $\sigma'(y) = \tau(y)$. We have

$$\tau\sigma'\tau(y) = \tau\sigma'\sigma'(y) = \tau(y) = \sigma'(y).$$

Therefore $\tau\sigma'\tau = \sigma'$. Thus σ' must leave the fixed point set of τ invariant. Let $F_\tau = \{i, j\}$. Then $\sigma'(i) = j$. Using the same argument with y replaced by x we may also show $\sigma(i) = j$. Hence $\sigma'(i) = \sigma(i)$. Since σ and σ' are one factors $\sigma = \sigma'$. Thus $\sigma(y) = \tau(y) = \sigma_\tau(y)$. If y is a fixed point of τ then y is either i or j. Assume without loss of generality $y = i$ then $\sigma(y) = \sigma(i) = j = (i, j)\tau(i) = (i, j)\tau(y)$. So $\sigma(y) = \sigma_\tau(y)$.

Corollary 2.4. *Let Γ be involution complete with respect to $\mathcal{F}$ and $n > 4$. Let τ and τ' be modified one factor symmetries of $\mathcal{F}$.*

(a) *If $\sigma_\tau = \sigma_{\tau'}$ then $\tau = \tau'$.*

(b) *If $F_\tau = F_{\tau'}$ then $\tau = \tau'$.*

(c) *If $\tau\tau' = \tau'\tau$, then $\tau = \tau'$.*

Proof. Since $\sigma_\tau = \sigma_{\tau'}$ then τ and τ' have at least $n - 2$ edges in common. So $\tau\tau'$ has at least $2n - 4 > n$ fixed points since $n > 4$. So $\tau\tau'$ is the identity because every one factorization is n-irreducible.

(b) follows from (a) since if $F_\tau = F_{\tau'}$ then σ_τ and $\sigma_{\tau'}$ have an edge in common so $\sigma_\tau = \sigma_{\tau'}$.

Next consider (c). Let $F_\tau = \{a, b\}$. Then

$$\begin{aligned}\tau'\sigma_\tau\tau' &= \tau'(a, b)\tau\tau' \\ &= \tau'(a, b)\tau'\tau \\ &= [\tau'(a, b)\tau'](a, b)\sigma_\tau.\end{aligned}$$

Thus $(\tau'\sigma_\tau\tau')\sigma_\tau$ is a product of two 2 cycles. Since there are more than 4 vertices $\tau'\sigma_\tau\tau'\sigma_\tau$ must have a fixed point. Hence $\tau'\sigma_\tau\tau'$ and σ_τ have an edge in common and $\tau'\sigma_\tau\tau' = \sigma_\tau$. Thus σ_τ leaves the fixed points of τ' invariant. Hence σ_τ has an edge in common with $\sigma_{\tau'}$. Therefore $\sigma_\tau = \sigma_{\tau'}$ and (a) gives the result.

Theorem 2.5. *Let Γ be involution complete with respect to an APOF $\mathcal{F}$. Let $n > 4$. Let τ and τ' be two distinct modified one factor symmetries. Then $\sigma_\tau \cup \sigma_{\tau'}$ is Hamiltonian.*

Proof. Let $F_\tau = \{a, b\}$ and $F_{\tau'} = \{a', b'\}$. Let $G = \operatorname{span}\{\sigma_\tau, \sigma_{\tau'}\}$. Then

$$G = \operatorname{span}\{(a,b)\tau, (a',b')\tau'\}.$$

This group has the same orbits in V as

$$G' = \operatorname{span}\{(a,b), (a',b'), \tau, \tau'\}.$$

G' contains two distinct one factors σ_τ and $\sigma_{\tau'}$. We will show it contains a third one factor. In this case G' and hence G will act transitively on V, and $\sigma_\tau \cup \sigma_{\tau'}$ will be Hamiltonian. To find the third one factor note $\langle \tau, \tau' \rangle \simeq \mathbf{D}_k$ where $k = o(\tau\tau')$, and $\mathbf{D}_k$ contains k elements of order two which are either conjugate to τ or τ'.

First suppose $k \geq 3$; then $\langle \tau, \tau' \rangle$, and hence G' contains at least three modified one factor symmetries. There exists $\tilde{\tau}$ such that $F_\tau \neq F_{\tau'} \neq F_{\tilde{\tau}}$. Suppose $F_{\tilde{\tau}} = \{i, j\}$. Then there exists $g \in \langle \tau, \tau' \rangle$ such that

$$g(i,j)g^{-1} = (a,b)$$

or

$$g(i,j)g^{-1} = (a',b'),$$

which implies $(i,j) \in G'$; therefore $(i,j)\tilde{\tau} = \sigma_{\tilde{\tau}} \in G'$. Hence G' contains three one factors and the case where $k \geq 3$ is finished.

When $k = 2$ we have $\tau\tau'\tau\tau' = \hat{1}$ so $\tau\tau' = \tau'\tau$ and the result follows from Corollary 2.4(c).

Corollary 2.6. *Let $\mathcal{F}$ be an APOF of K_{2n}. Let τ and τ' be two distinct modified one factor symmetries. Then $\sigma_\tau \cup \sigma_{\tau'}$ is Hamiltonian.*

Corollary 2.7. *Let Γ be involution complete with respect to an APOF $\mathcal{F}$. Let $n > 4$. Let* $\operatorname{Aut}(\mathcal{F})$ *act transitively on the one factors. If* $\operatorname{Aut}(\mathcal{F})$ *has a modified one factor symmetry then $\mathcal{F}$ is a POF.*

Proof. Since $\mathrm{Aut}(\mathcal{F})$ acts transitively on the one factors, and since there is a modified one factor symmetry, every one factor arises from a modified one factor symmetry. The result follows from Corollary 2.6.

Corollary 2.8. *Let Γ be involution complete with respect to an* APOF $\mathcal{F}$. *Let $n > 4$. Let $\mathrm{Aut}(\mathcal{F})$ have two distinct modified one factor symmetries. Then one of the following is true*:

(a) *$\mathcal{F}$ is the perfect one factorization GK_{2n} of K_{2n} (and $2n-1$ is prime).*

(b) *$\mathcal{F}$ contains the n-modified one factor symmetries of GA_{2n} (and n is prime).*

Proof. Let τ and τ' be the two modified one factorizations. Then, as before, $\langle \tau, \tau' \rangle \simeq \mathbf{D}_k$ where $k = o(\tau\tau'')$. By [**3**] we have two cases:

(a) $k = 2n-1$. Thus we have $2n-1$ one factors and $\Gamma = K_{2n}$, $\mathcal{F}$ is perfect and by [**3**] $\mathcal{F}$ is GK_{2n} and $2n-1$ is prime.

(b) $k = n$. $\mathbf{D}_n$ contains n elements of order two corresponding to the n modified one factors of $\mathcal{F}$. Suppose τ_i and τ_j are distinct modified one factor symmetries. Then $\sigma_{\tau_i} \cup \sigma_{\tau_j}$ is Hamiltonian. We see n is odd since $\sigma_{\tau_i}\sigma_{\tau_j}$ is a product of two distinct n cycles and if n is even $o((\sigma_{\tau_i}\sigma_{\tau_j})^2) = n/2$ implying $\sigma_{\tau_i}\sigma_{\tau_j}$ is a product of 4 disjoint $n/2$ cycles. Similarly we see n must be prime.

It is possible to explicitly construct the one factors arising from two modified one factors as in case (b) above. Without loss of generality we let

$$\tau_0 = (3,4)\dots(2n-1,2n)$$
$$\tau_1 = (1,4)(3,6)\dots(\widehat{n,n+3})\dots(2n-1,2)$$

where $(\widehat{n,n+3})$ indicates this two cycle is deleted. Then

$$\begin{aligned}\tau_0\tau_1 = {} & (1,4,6,\dots,n+1,n,n-2,\dots,3) \\ & \times (2,2n-1,2n-3,\dots,n+2,n+3,\dots,2n).\end{aligned}$$

Relabel the vertices with 0 through $n-1$ corresponding to the vertices in the first n cycle, and 0^* through $(n-1)^*$ the vertices in the second cycle. Then $\tau_0 = x_1 \dots x_{k/2}, x_1^*, \dots, x_{k/2}^*$ where $x_i = (a,b)$ and $x_i^* = (a_i^*, b_i^*)$ with $a + b \equiv 0 \bmod n$. $\tau_1 = x_1 \cdots x_{k/2}, x_1^*, \dots x_{k/2}^*$

where $a + b \equiv 1 \bmod n$. These are the n modified one factors of GA_{2n} (see [**3**]). We call the subgraph of K_{2n} obtained from the one factors associated with the above modified one factors L_n and the associated one factorization GD_{2n}. Another way to describe this one factorization is in terms of the bipartite graph $K_{n,n}$. L_n is the complement of $K_{n,n} - \sigma$ where σ is any one factor of $K_{n,n}$. GD_{2n} can then be defined as follows. Identify the vertex set with $\mathbf{Z}_n \times \{-1, 1\}$. For each $g \in \mathbf{Z}_n$ define a one factor which connects $(g, 1)$ to $(g, -1)$ and connects (x, ε) to $(2g - x, \varepsilon)$ for all $x \neq g$ and all ε.

Theorem 2.9. *Let* Γ *be involution complete with respect to an* APOF $\mathcal{F}$. *Let* $n > 4$. *Let* $\mathrm{Aut}(\mathcal{F})$ *act transitively on the one factors. If* $\mathrm{Aut}(\mathcal{F})$ *has a modified one factor symmetry, then* $\mathcal{F}$ *is one of the two following POFs*:

(a) $\Gamma = K_{2n}$ *and* $\mathcal{F} = GK_{2n}$ *where* $2n - 1$ *is prime.*

(b) $\Gamma = L_n$ *and* $\mathcal{F} = GD_{2n}$ *where* n *is prime.*

Proof. By Corollary 2.7, $\mathcal{F}$ is perfect. Since every one factor arises from a modified one factor symmetry, we have $\tau, \tilde{\tau} \in \mathrm{Aut}(\mathcal{F})$ such that $\tau \neq \tilde{\tau}$. Then, for $\langle \tau, \tilde{\tau} \rangle \simeq \mathbf{D}_{2n-1}$ we have (a).

Suppose we have $\langle \tau, \tilde{\tau} \rangle \simeq \mathbf{D}_n$. If there exists $\sigma_{\hat{\tau}}$ such that $\hat{\tau} \notin \langle \tau, \tilde{\tau} \rangle$ then $\langle \tau, \tilde{\tau} \rangle$ and $\langle \tau, \hat{\tau} \rangle$ would yield $2n - 1$ modified one factors. Then $\Gamma = K_{2n}$ and $\mathcal{F} = GK_{2n}$ or GA_{2n} both of which are not possible. In GK_{2n} we have $\langle \tau, \tilde{\tau} \rangle \simeq \mathbf{D}_{2n-1}$ and GA_{2n} has only n modified one factors. Hence the only possibility is $\Gamma = L_n$ and $\mathcal{F} = GD_{2n}$.

The following example illustrates the necessity of the assumption of the existence of a modified one factor symmetry. It was given by Cameron in a slightly different form [**1**, p. 73].

Example. There is a one factor transitive APOF on K_{12} which is not a POF. We call this one factorization PC_{12}. The vertex set is $\mathbf{Z}_{11} \cup \{\infty\}$. The one factorization is starter induced by using a Mullin-Nemeth starter. The starter is

$$\sigma = (0, \infty)(1, 2)(3, 6)(4, 8)(5, 10)(7, 9).$$

This one factorization has the property that the union of any two distinct one factors consists of two six cycles. The symmetry group, $PSL(2,11)$, acts two transitively on the one factors.

Corollary 2.10. *If $\mathcal{F}$ is an* APOF *on K_{2n}, and if* Aut($\mathcal{F}$) *acts 2-transitively on $\mathcal{F}$, one of the following holds*:

(a) *$\mathcal{F}$ is GK_{2n} for $2n-1$ prime.*

(b) *$\mathcal{F} = PC_{12}$ on K_{12}.*

Proof. Since Aut($\mathcal{F}$) is 2 transitive we know that $2n-2$ divides $|\mathrm{Aut}(\mathcal{F})|$. This means Aut$\mathcal{F}$ has an element of order 2. By [**1**, p. 118, Theorem 6.6] we have that Aut($\mathcal{F}$) has a fixed point or $\mathcal{F} = PC_{12}$. Therefore if $\mathcal{F} \neq PC_{12}$ then Aut($\mathcal{F}$) contains a modified one factor symmetry. The result follows from Corollary 2.10.

Corollary 2.11. *Let Γ be involution complete with respect to an* APOF *$\mathcal{F}$. Let* Aut($\mathcal{F}$) *act transitively on the one factors. If n is odd and $\mathcal{F}$ is not the unique perfect one factorization on K_6, then* Aut($\mathcal{F}$) *is solvable.*

Proof. We start by considering the case in which $n > 4$. Let S denote a Sylow 2-subgroup of Aut($\mathcal{F}$). Consider the action of S on V, the vertex set. If S_i is not trivial, it contains an element of order two which must be a modified one factor symmetry. Then use Theorem 2.9 to give the one factorizations explicitly. The symmetry group of GK_{p+1} is $[\mathbf{Z}_p]\mathbf{Z}_{p-1}$, and the symmetry group of GD_{2p} is $\mathbf{Z}_2 \oplus [\mathbf{Z}_p]\mathbf{Z}_{p-1}$. Both groups are solvable.

Otherwise S_i must be trivial for each i. Hence, each orbit of S has the same number of elements as S. This means $|S|$ divides $2n$. Since n is odd, we have $|S|$ divides 2. Thus $S = \{1\}$ or $\mathbf{Z}_2$. In either case, using the Feit Thompson theorem (together with a theorem of Burnside in the $\mathbf{Z}_2$ case) we have Aut($\mathcal{F}$) is solvable.

Now consider the case in which $n \leq 4$. Since n is odd, and $n = 1$ is trivial, we must only consider $n = 3$. On K_6 there is a unique to within isomorphism one factorization, GK_6, which is isomorphic to GA_6. The symmetry group is S_5 which is not solvable. For any other one factorization with $n = 3$ we have $|\mathcal{F}| \leq 4$. Thus $\iota(\mathrm{Aut}(\mathcal{F})) \subset S_4$

which is solvable. Also $\ker(\iota)$ is a subgroup of $\mathbf{D}_6$ which is solvable. This is true since the union of any two edge disjoint one factors will be a Hamiltonian cycle and so $\ker(\iota)$ will be a subgroup of the symmetry group of a 6 cycle. □

Theorem 2.12 *Let* Γ *be involution complete with respect to* $\mathcal{F}$*. Let* n *be even.*

(a) *If* $\mathrm{Aut}(\mathcal{F})$ *contains a unique modified one factor symmetry, then there is a subgroup* $G \subset \mathrm{Aut}(\mathcal{F})$ *so that* $|G|$ *is odd and*

$$\mathrm{Aut}(\mathcal{F}) = \mathbf{Z}_2 \oplus G$$

(b) *Assume* $\mathcal{F}$ *is an* APOF *and* $n > 4$*. If* $\mathrm{Aut}(\mathcal{F})$ *has a modified one factor symmetry,* $\mathrm{Aut}(\mathcal{F})$ *is solvable.*

Proof. Assume $\mathrm{Aut}(\mathcal{F})$ has a unique modified one factor symmetry τ with fixed point set $\{i, j\}$. Since τ is central in $\mathrm{Aut}(\mathcal{F})$ then $\mathrm{Aut}(\mathcal{F})$ leaves $\{i, j\}$ invariant. Hence $\mathrm{Aut}(\mathcal{F})$ leaves $X = V - \{i, j\}$ invariant. Let S be a Sylow 2-subgroup of $\mathrm{Aut}(\mathcal{F})$. Let $k \in X$. We claim S_k is trivial. If not, it contains an element of order two which has k as a fixed point. Hence, it is a modified one factor symmetry. But τ does not fix k, so it cannot be τ. This contradicts τ being unique. Therefore, every orbit of S on X has cardinality $|S|$. Thus, $|S|$ divides $|X - \{i, j\}| = 2(n-1)$. Hence, $|S|$ divides 2. Thus $S \simeq \mathbf{Z}_2$, which is central, and hence normal. Also, since $S \simeq \mathbf{Z}_2$, $\mathrm{Aut}(\mathcal{F})$ is solvable. Let G be a Hall subgroup corresponding to the odd primes. Since G has index 2, G is also normal. Thus $\mathrm{Aut}(\mathcal{F}) \simeq S \oplus G$ as desired.

For (b) if $\mathrm{Aut}(\mathcal{F})$ has two distinct modified one factors then by Corollary 2.8 $\mathcal{F}$ is GK_{2n} or contains the modified one factor symmetries of GA_{2n}. If $\mathcal{F}$ is GK_{2n}, then $\mathrm{Aut}\mathcal{F} = [\mathbf{Z}_{2n-1}]\mathbf{Z}_{2n-2}$ which is solvable. Next consider the case when $\mathcal{F}$ contains the modified one factor symmetries of GA_{2n}. Since $\mathrm{Aut}(\mathcal{F})$ will permute the modified one factor symmetries of $\mathcal{F}$ among themselves, we have that $\mathrm{Aut}(\mathcal{F})$ will be a subgroup of the group of automorphisms of these 1-factors. This group is $[\mathbf{Z}_{2n}]\mathbf{Z}_{n-1}$ which is solvable. Hence $\mathrm{Aut}\mathcal{F}$ is solvable. Finally, if $\mathrm{Aut}\mathcal{F}$ has only one modified one factor symmetry, then we may use part (a).

3. One factor symmetries. Throughout this section we will assume that $n \geq 3$.

Definition 3.1. σ is called a one factor symmetry if $\sigma \in \mathcal{F} \cap \mathrm{Aut}(\mathcal{F})$.

Example. The following is an APOF on K_8 with a one factor symmetry. Let

$$\sigma_0 = (1,5)(2,6)(3,7)(4,8)$$
$$\sigma_1 = (1,3)(2,4)(5,8)(6,7), \quad \sigma_2 = (1,4)(2,3)(5,7)(6,8)$$
$$\sigma_3 = (1,8)(2,7)(3,4)(5,6), \quad \sigma_4 = (1,2)(3,6)(4,5)(7,8)$$
$$\sigma_5 = (1,6)(2,8)(3,5)(4,7), \quad \sigma_6 = (1,7)(2,5)(3,8)(4,6).$$

Note that σ_0 is a one factor symmetry and the pairs $\{\sigma_1, \sigma_2\}, \{\sigma_3, \sigma_4\}$, $\{\sigma_5, \sigma_6\}$ are not Hamiltonian. By the following

Lemma 3.2. *Let τ be a one factor symmetry of an* APOF *$\mathcal{F}$. Let $\sigma \in \mathcal{F}$. Then either σ commutes with τ or $\sigma \cup \tau$ is a Hamiltonian circuit.*

Proof. Assume that $\tau\sigma\tau \neq \sigma$. We note that $|\{\sigma, \tau, \tau\sigma\tau\}| = 3$ since we also know $\sigma \neq \tau$. Hence $G = \langle \sigma, \tau, \tau\sigma\tau \rangle$ acts transitively on V since $\mathcal{F}$ is an APOF. But we know $G = \langle \sigma, \tau \rangle$. Since $\langle \sigma, \tau \rangle$ acts transitively on V, we know $\sigma \cup \tau$ is connected, and hence a Hamiltonian circuit.

Theorem 3.3. *Let n be even and $n \neq 4$. If $\mathcal{F}$ is an* APOF*, then any two one factor symmetries commute. Also $\mathcal{F}$ has at most two one factor symmetries.*

Proof. Let σ and $\tilde{\sigma}$ be one factor symmetries which do not commute. Then $\sigma \cup \tilde{\sigma}$ is a Hamiltonian cycle. Thus [**3**] $\sigma\tilde{\sigma}$ is a product of two disjoint n-cycles. Hence $\sigma\tilde{\sigma}\sigma\tilde{\sigma}$ is a product of four disjoint $n/2$ cycles since n is even. Thus $\sigma \cup \tilde{\sigma}\sigma\tilde{\sigma}$ is not a Hamiltonian circuit. But $\tilde{\sigma}\sigma\tilde{\sigma}$ is a one factor symmetry because it is the conjugate of a one factor symmetry by a symmetry. Thus σ and $\tilde{\sigma}\sigma\tilde{\sigma}$ must commute by the lemma. Thus

$$\sigma\tilde{\sigma}\sigma\tilde{\sigma} = \tilde{\sigma}\sigma\tilde{\sigma}\sigma.$$

This gives $(\sigma\tilde{\sigma})^4 = \tilde{1}$. We know $\langle\sigma, \tilde{\sigma}\rangle \simeq \mathbf{D}_k$ where $k = o(\sigma\tilde{\sigma})$; therefore, $k\backslash 4$. σ and $\tilde{\sigma}$ do not commute so $o(\sigma\tilde{\sigma}) \neq 2$ giving $k = 4$. But, since $\sigma \cup \tilde{\sigma}$ is Hamiltonian, we have $\langle\sigma, \tilde{\sigma}\rangle \simeq \mathbf{D}_n$. Thus, $n = 4$. Thus, we have shown that any two one factor symmetries commute when $n \neq 4$.

Now suppose we have three one factor symmetries $\sigma, \tilde{\sigma}$ and $\tilde{\tilde{\sigma}}$. Let $G = \langle\sigma, \tilde{\sigma}, \tilde{\tilde{\sigma}}\rangle$. Then G is abelian, and every nontrivial element of G has order two. Hence $G \simeq (\mathbf{Z}_2)^k$ where $k = 2$ or 3. Since $\mathcal{F}$ is APOF, G acts transitively on V. Thus $|V|$ divides $|G|$. These numbers are actually equal since G is abelian. Hence $2n = 8$ or 4. Thus $n = 4$ or 2 as desired. $\square$

Lemma 3.4. *Let n be odd. Let $\mathcal{F}$ be a 2-irreducible* APOF. *Then any two one factor symmetries commute.*

Proof. Suppose σ and $\tilde{\sigma}$ are both one factor symmetries which do not commute. Let $G = \langle\sigma, \tilde{\sigma}\rangle$. We know $G \simeq \mathbf{D}_n$ by Lemma 3.2 and [**3**, Lemma 2.6]. Let $\tau = \sigma\tilde{\sigma}$. We have $o(\tau) = n$ and $\sigma_i = \tau^i\sigma$ are all elements of order two which are conjugate to either σ or $\tilde{\sigma}$. Since each σ_i is conjugate by a symmetry element to a one factor symmetry, we have that each of the σ_i are one factor symmetries. Now let

$$W = \{\tau^k(v) \mid k = 1, \dots, n\}$$

where v is some element of V. We claim W is a subspace. Let $w \in W$ and $\sigma \in \mathcal{F}$ such that $\sigma(w) \in W$. We need to show $\sigma(x) \in W$. If not, then

$$\sigma(x) \in V\backslash W = \{\tau^k\sigma(v) \mid k = 1, \dots, n\}$$

since G acts transitively on V. Hence, $\sigma(x) = \sigma_k(x)$ for some k. Both σ and σ_k are one factors in $\mathcal{F}$ so $\sigma = \sigma_k$. This means $\sigma(w) = \sigma_k(w) \notin W$ which is a contradiction. Now, since W is a subspace, we have $|W| \leq 2$ by assumption since $|W| = n < 2n$ and $\mathcal{F}$ is 2-irreducible. This means $n = 2$ which is not possible. Hence any two one factor symmetries must commute. $\square$

Corollary 3.5. *Let n be odd. Let $\mathcal{F}$ be a 2-irreducible* APOF. *Then $\mathcal{F}$ has at most* 1 *one factor symmetry.*

Proof. Let σ and $\tilde{\sigma}$ be two distinct one factor symmetries. By Lemma 3.4 they commute. Since n is odd, both σ and $\tilde{\sigma}$ are odd permutations. Hence, $\tau = \sigma\tilde{\sigma}$ is an even permutation and is a symmetry of order two. If τ had no fixed points, then τ would be odd (see [**3**, 3.5]). Since σ and $\tilde{\sigma}$ commute with τ they must leave the fixed point set of τ invariant. Thus both σ and $\tilde{\sigma}$ have an edge connecting these two fixed points of τ. This contradicts the assumption that σ and $\tilde{\sigma}$ are distinct one factors. Hence $\mathcal{F}$ cannot have two one factor symmetries. $\square$

4. Steiner triple systems. In this section we review the structure of automorphism groups of one factorizations that arise from Steiner triple systems and provide for easy access the details of the result of Cameron that every finite group occurs as the automorphism group of some one-factorization.

Definition 4.1. Let X be a set and $\mathcal{B}$ a collection of subsets of X. $(X, \mathcal{B})$ is called a Steiner triple system if

(a) $S \in \mathcal{B}$ implies $|S| = 3$.

(b) If T is a subset of X with $|T| = 2$, there is one and only one $S \in \mathcal{B}$ so that $T \subset S$.

Given any Steiner triple system, there is a well-known technique for constructing a one factorization of $K_{|X|+1}$. This is given below.

Definition 4.2. Let $(X, \mathcal{B})$ be a Steiner triple system. Define $\mathcal{F}_{\mathcal{B}}$, a one factorization on $K_{|X|+1}$ as follows: Let $V = X \cup \{\infty\}$. For each $x \in X$ we define σ_x. In the case when $x \neq y$, then

$$\sigma_x(y) = z \iff \{x, y, z\} \in \mathcal{B}.$$

In addition we specify that

$$\sigma_x(x) = \infty \text{ and } \sigma_x(\infty) = x.$$

Then let $\mathcal{F}_{\mathcal{B}} = \{\sigma_x : x \in X\}$.

Remark 4.3. We define

$$\mathrm{Aut}(\mathcal{B}) = \{\tau \in S_X : \tau(S) \in \mathcal{B} \text{ whenever } S \in \mathcal{B}\}.$$

Define $j : \mathrm{Aut}(\mathcal{B}) \to \mathrm{Aut}(\mathcal{F}_{\mathcal{B}})$ by $j(\tau)(\infty) = \infty$ and $j(\tau)(x) = \tau(x)$ for each $x \in X$. A simple calculation shows that

$$j(\tau)\sigma_x j(\tau)^{-1} = \sigma_{\tau(x)},$$

and hence $j(\tau) \in \mathrm{Aut}(\mathcal{F}_{\mathcal{B}})$ for each $\tau \in \mathcal{B}$.

Lemma 4.4.

$$j(\mathrm{Aut}(\mathcal{B})) = \mathrm{Aut}(\mathcal{F}_{\mathcal{B}})_\infty.$$

Proof. Let $\tau \in \mathrm{Aut}(\mathcal{F}_{\mathcal{B}})_\infty$. Since $\tau(\infty) = \infty$ we have $\tau(X) = X$. Define $\rho = \tau|_X$. Let S be in $\mathcal{B}$. We must show $\rho(S) \in \mathcal{B}$. Let

$$S = \{x, y, z\}.$$

We know that there is a $t \in X$ so that

$$\tau\sigma_x\tau^{-1} = \sigma_t$$

since $\tau \in \mathrm{Aut}(\mathcal{F}_{\mathcal{B}})$. Apply this equation to ∞ and use that $\tau(\infty) = \infty$ to find $t = \tau(x)$. Next apply this equation to $\tau(y)$ to find $\tau(z) = \sigma_t(\tau(y))$. Thus

$$\{\tau(x), \tau(y), \tau(z)\} = \{t, \tau(y), \tau(z)\} \in \mathcal{B}$$

as desired. $\square$

Definition 4.5. A point $x \in X$ is called a Steiner point, the union of any two distinct one factors in $\mathcal{F}$ has a four cycle through x.

Theorem 4.6.

$$\mathrm{Aut}(\mathcal{F}_{\mathcal{B}}) \simeq [(\mathbf{Z}_2)^m]\mathrm{Aut}(\mathcal{B})$$

where $(\mathbf{Z}_2)^m$ *is isomorphic to* $\ker\iota$ *defined in Definition* 1.5.

Proof. We know that $\ker\iota \simeq (\mathbf{Z})^m$ for some m by [**1**, p. 11, Theorem 1.4]. We also know $\ker\iota$ is a normal subgroup of $\mathrm{Aut}(\mathcal{F}_{\mathcal{B}})$ and

$$\ker\iota \cap \mathrm{Aut}(\mathcal{F}_{\mathcal{B}})_\infty = \hat{1}.$$

To complete the result we must only show that

$$\mathrm{Aut}(\mathcal{F}_{\mathcal{B}}) = (\ker \iota)\mathrm{Aut}(\mathcal{F}_{\mathcal{B}})_{\infty}.$$

Let $\tau \in \mathrm{Aut}(\mathcal{F}_{\mathcal{B}})$. We have that $\tau(\infty)$ is a Steiner point. Hence, there is a $\rho \in \ker \iota$ so that $\rho(\infty) = \tau(\infty)$ by [**1**, p. 51, Theorem 3.5]. Hence $\tau = \rho(\rho^{-1}\tau)$ shows $\tau \in \ker \iota \mathrm{Aut}(\mathcal{F}_{\mathcal{B}})_{\infty}$ as desired. □

The following is an interesting corollary of an unpublished result of Cameron's relating the results presented above and a result by Mendelsohn [**7**].

Theorem 4.7. *Given any finite group G, there is an n and a one factorization $\mathcal{F}$ of K_{2n} such that*

$$\mathrm{Aut}(\mathcal{F}) \simeq G.$$

Proof. In [**7**], Mendelsohn shows that for any finite group G, there is a Steiner triple system $\mathcal{B}$ which has G as its automorphism group. Unfortunately, the potential presence of a nontrivial $\ker \iota$ means that Theorem 4.2 does not immediately follow from this result. In fact it is necessary to examine Mendelsohn's construction in detail to verify that the $\ker \iota$ is trivial. Fortunately, we need only single out one facet of the Mendelsohn construction. Let X be the vertex set of $\mathcal{B}$. We have $|X| = 2^n - 1$. Also there is a subset $Y \subset X$ so that $|Y| = 15$ which has the following properties:

(a) If $|S \cap Y| \geq 2$ and $S \in \mathcal{B}$ then $S \subset Y$.

(b) If $|S \cap Y| \leq 1$ and $S \in \mathcal{B}$ then S is a block in the projective plane Steiner triple system on $(\mathbf{Z}_2)^n - \{0\}$. These blocks are of the form $H - \{0\}$ where H is any subgroup of $\mathbf{Z}_2^n$ with order 4.

(c) The blocks $S \in \mathcal{B}$ with $S \subset Y$ form a special predetermined triple system $(Y, \mathcal{B}')$ given in [**7**, Table 1].

It should be noted that there are, in fact, many subsets Y as described above. Now let x and y be in Y. (a) above shows that σ_x and σ_y both leave $Y \cup \{\infty\}$ invariant. We consider the cycle structure of $\sigma_x \cup \sigma_y$.

(b) says that outside of $Y \cup \{\infty\}$, both σ_x and σ_y act in exactly the same way as one factors in $(\mathbf{Z})^n$ with

$$\mathcal{F}' = \{\sigma : \sigma \in \mathbf{Z}_2^n \text{ and } \sigma \neq 0\}.$$

The union of any two of these one factors consists of disjoint four cycles so $\sigma_x \cup \sigma_y$ consists only of disjoint four cycles outside of $Y \cup \{\infty\}$. Now let $\tau \in \ker \iota$ leave each one factor invariant. Thus τ leaves $\sigma_x \cup \sigma_y$ invariant. Hence any cycle of $\sigma_x \cup \sigma_y$ of length larger than four inside $Y \cup \{\infty\}$ must remain inside $Y \cup \{\infty\}$ after τ is applied to it.

Now using (c) we have that σ_x and σ_y when restricted to $Y \cup \{\infty\}$ are the same as the one factors coming from the original block design $(Y, \mathcal{B}')$. An explicit calculation shows that for each $z \in Y$ there are x and $y \in Y$ so that z is part of a cycle longer than 4 in $\sigma_x \cup \sigma_y$. Hence all of $Y \cup \{\infty\}$ is invariant under τ. This says that τ is an automorphism of $(Y, \mathcal{B}')$. But $(Y, \mathcal{B}')$ is automorphism free, which means τ must fix all the vertices in $Y \cup \{\infty\}$. However, the only transformation in $\ker \iota$ that has a fixed point is the identity. Hence $\ker \iota = \hat{1}$ as desired.

REFERENCES

1. P. Cameron, *Parallelisms of complete designs*, London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, 1976.

2. A. Hartman and A. Rosa, *Cyclic one factorizations of the complete graph*, European J. Combin. **6** (1985), 45–48.

3. E. Ihrig, *Symmetry groups related to the construction of perfect one factorizations*, J. Combin. Theory Ser. B **40** (1986), 121–151.

4. ———, *The structure of the symmetry groups of perfect* 1*-factorizations of* K_{2n}, J. Combin. Theory Ser. B **47** (1989), 307–329.

5. ———, *Cyclic perfect one factorizations of* K_{2n}, Ann. Discrete Math. **34** (1987), 259–272.

6. ———, *Symmetry groups of perfect one factorizations of* K_{2n}, Congr. Numer. **60** (1987), 105–130.

7. E. Mendelsohn, *On the groups of automorphisms of Steiner triple and quadruple systems*, J. Combin. Theory Ser. A **25** (1978), 97–104.

8. E. Seah, *Perfect one factorizations of the complete graph—A survey*, Bulletin Inst. Combin. Appl. **1** (1991), 59–70.

Department of Mathematics, California State University, Fresno, CA 93711
E-mail address: della@csufresno.edu

P.O. Box 871804, Department of Mathematics, Arizona State University, Tempe, AZ 85287-1804
E-mail address: ihrig@math.la.asu.edu