ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 30, Number 1, Spring 2000

LOCAL COHOMOLOGY FOR COMMUTATIVE BANACH ALGEBRAS

M.S. MOSLEHIAN AND A. NIKNAM

ABSTRACT. The purpose of this paper is to introduce a local cohomology theory in the unital commutative Banach algebras context and to describe a connection between the local cohomology functor and direct limit of hom functors.

1. Introduction. The local cohomology theory in the context of commutative ring theory was introduced by Grothendieck [**3**] and developed by Brodmann, Mcdonald and Sharp and some other mathematicians [**2**]. In this paper we introduce a version of local cohomology for unital commutative Banach algebras and Banach modules. After introduction, we specialize in Section 2 to local cohomology functor, torsion and torsion-free modules with respect to a given ideal and also some related examples. Section 3 provides the connected right sequence of local cohomology functors. In the last section the local cohomology functor is described as a direct limit of some hom functors.

Throughout the paper, A is a fixed unital commutative Banach algebra with unit e, $\|e\| = 1$, and I is a fixed closed ideal of A. We follow the notation and terminology of [**5**] or [**6**], but with some exceptions as the following:

Definition 1.1. A Banach A-module is a Banach space with an algebraic unital symmetric A-bimodule structure satisfying

$$(\diamondsuit) \qquad \begin{gathered} \|ax\| \leq \|a\| \cdot \|x\|; \quad a \in A, x \in X \\ ex = x; x \in X \end{gathered}$$

Received by the editors in revised form in February, 1999.
1991 AMS *Mathematics Subject Classification.* Primary 13D45, 46H25, 46J05, Secondary 18G10, 46J20.
Key words and phrases. Unital commutative Banach algebra, injective resolution, local cohomology functors, connected right sequence of covariant functors, direct limit of Banach modules.

A module morphism between Banach A-modules X and Y is a linear mapping $f : X \to Y$ such that

$$f(ax) = af(x); \quad x \in X, a \in A.$$

Notation 1.2. We denote the category of all Banach A-modules and module morphisms between them by $\mathcal{C}$, the corresponding positive (cochain) complexes by $\overline{\mathcal{C}}$, and the category of algebraic A-modules with underlying complete semi-normed spaces satisfying ($\diamondsuit$) and their module morphisms by $\langle\mathcal{C}\rangle$.

Let S be a subset of A and $X \in \mathcal{C}$; then the set of all $x \in X$ such that $Sx = 0$ is a Banach A-submodule of X which is denoted by $(0 :_X S)$.

2. Local cohomology functor with respect to an ideal.

Definition 2.1. Let $X \in \mathcal{C}$; then the local cohomology of X with respect to I, denoted by $\Gamma_I(X)$, is the closure of $\{x \in X; I^n x = 0 \text{ for some } n \in \mathbf{N}\}$, where I^n is the closed ideal generated by $a_1 a_2 \cdots a_n$, $1 \le i \le n$, $a_i \in I$. $\Gamma_I(X)$ is a Banach A-submodule of X and so it belongs to $\mathcal{C}$. It follows therefore that

$$\Gamma_I(X) = \overline{\cup_{n=1}^{\infty}(0 :_X I^n)}.$$

An A-module X is said to be I-torsion, respectively, I-torsion-free, whenever $\Gamma_I(X) = X$, respectively $\Gamma_I(X) = 0$.

Examples 2.2. (i) If I has a bounded approximate identity, then clearly $I^n = I$. Hence $\Gamma_I(X) = (0 :_X I)$ for each Banach A-module X; in particular, $\Gamma_I(A) = \mathrm{Ann}\,(I)$, where $\mathrm{Ann}\,(I)$ is the annihilator of I. Moreover, if I has a bounded approximate identity for X, i.e., there is a net $\{e_\lambda\}_{\lambda\in\Lambda}$ with $e_\lambda \in I$ such that $\lim_\lambda e_\lambda x = x$ for all $x \in X$ and $\sup\{\|e_\lambda\|; \lambda \in \Lambda\} < \infty$, e.g., $X = I$, then $\Gamma_I(X) = 0$. Thus, in this case, X is I-torsion-free.

(ii) If X is a Banach A-module and $\tilde{\Gamma}_I(X) = \cup_{n=1}^{\infty}(0 :_X I^n)$, then

$$\Gamma_I(X) = \overline{\tilde{\Gamma}_I(X)} = \overline{\tilde{\Gamma}_I(\tilde{\Gamma}_I(X))} \subseteq \overline{\tilde{\Gamma}_I(\Gamma_I(X))} = \Gamma_I(\Gamma_I(X)) \subseteq \Gamma_I(X).$$

Hence, $\Gamma_I(\Gamma_I(X)) = \Gamma_I(X)$. Thus $\Gamma_I(X)$ is I-torsion.

(iii) Let A be an abelian von Neumann algebra and I a weak-operator closed ideal of A. Then $A \simeq C(\Omega)$ for some extremely disconnected compact Hausdorff space Ω [**7**, Theorem 5.2.1] and I is of the form Ac for a projection $c \in A$ [**8**, Theorem 6.8.8]. Since I has the identity c, Example 2.2 (i) shows that $\Gamma_I(X) = \{x \in X; cx = 0\}$. If $X = A$ and I is nontrivial, A is clearly neither I-torsion nor I-torsion-free.

If $f : X \to Y$ is a module morphism in $\mathcal{C}$ and x is annihilated by I^n for some n, then for each $a \in I^n$, $af(x) = f(ax) = 0$, so I^n annihilates $f(x)$. Hence, $f(\Gamma_I(X)) \subseteq \Gamma_I(Y)$, by the continuity of f. Let $\Gamma_I(f)$ be the restriction and corestriction of f to $\Gamma_I(X)$ and $\Gamma_I(Y)$, respectively. Then it can be checked that $\Gamma_I(\cdot)$ is a functor from $\mathcal{C}$ to $\mathcal{C}$. We call this the local cohomology functor with respect to I. It is additive, $\mathbf{C}$-linear and A-linear, i.e., for objects X and Y in $\mathcal{C}$, module morphisms $f : X \to Y$ and $g : X \to Y$ $\lambda \in \mathbf{C}$ and $a \in A$:

$$\Gamma_I(f+g) = \Gamma_I(f) + \Gamma_I(g), \Gamma_I(\lambda f) = \lambda\Gamma_I(f) \quad \text{and} \quad \Gamma_I(af) = a\Gamma_I(f).$$

3. The ith local cohomology functors with respect to an ideal, $i \geq 0$. We recall that if E is a Banach space, then $\mathcal{B}(A, E)$ is a Banach A-module together with the following action:

$$(a\phi)(b) = \phi(ba); \phi \in \mathcal{B}(A, E), \quad a, b \in A.$$

It is an injective Banach A-module [**6**, Chapter III, Section 1.4].

Consider the normalized injective resolution for $X \in \mathcal{C}$ [**6**, Chapter III, Section 2]:

$$0 \longrightarrow J^0(X) \xrightarrow{d^0} J^1(X) \longrightarrow \cdots \longrightarrow J^i(X) \xrightarrow{d^i} J^i(X) \longrightarrow \cdots (\mathcal{J}(X)).$$

This complex has the property that the following complex is admissible:

$$0 \longrightarrow X \xrightarrow{\tilde{\pi}} J^0(X) \xrightarrow{d^0} J^1(X) \longrightarrow \cdots \longrightarrow J^i(X) \xrightarrow{d^i} J^{i+1}(X) \longrightarrow \cdots .$$

In the latter complex, $\tilde{\pi}_X : X \to \mathcal{B}(A, X)$ is given by $(\tilde{\pi}(x))(a) = ax$; also if $C(X) = \mathcal{B}(A, X)/\operatorname{Im}\tilde{\pi}_X, C^{-1}(X) = X$ and $C^i(X) =$

$C(C^{i-1}(X))$, $i \geq 0$, then for each $i \geq 0$, $J^i(X) = \mathcal{B}(A, C^{i-1}(X))$ and d^i is the composition of $J^i(X) \xrightarrow{\text{nat.}} C^i(X) \xrightarrow{\tilde{\pi}_{C^i(X)}} J^{i+1}(X)$. In fact $\mathcal{J}$ is a functor from $\mathcal{C}$ to $\overline{\mathcal{C}}$.

Definition 3.1. The nth injective derived functor of $\Gamma_I(\cdot)$, i.e., $H^i \circ \overline{\Gamma}_I \circ \mathcal{J} = H^i_I$ is called the ith local cohomology functor with respect to I, [**6**, Chapter III, Section 3]. $H^i_I(X)$ is called the ith local cohomology module of X with respect to I. The functors H^i_I, $i \geq 0$, are additive, **C**-linear, A-linear and covariant functors from $\mathcal{C}$ to $\langle \mathcal{C} \rangle$. $H^i_I(X)$ is independent of the choice of injective resolution for X up to an isomorphism in $\langle \mathcal{C} \rangle$ [**6**, Theorem 3.3.10].

Remark 3.2. If Q is an injective Banach A-module, then $H^i_I(Q) = 0$ for all $i > 0$. In fact, the exact complex $0 \to Q \overset{1_Q}{\to} Q \to 0 \to \cdots$ shows that $0 \to Q \to 0 \to \cdots (\mathcal{Q})$ is an injective resolution for Q so, for all $i > 0$, $H^i_I(Q) = H^i(\Gamma_I(\mathcal{Q})) = 0$.

Definition 3.3. A sequence $(T^i)_{i \geq 0}$ of covariant functors from $\mathcal{C}$ to $\langle \mathcal{C} \rangle$ is called a connected right sequence of covariant functors if the following conditions are satisfied:

(i) If $0 \to X \to Y \to Z \to 0$ is an admissible short complex in $\mathcal{C}$, there are defined continuous connecting morphisms $T^n(Z) \to T^{n+1}(X)$, $n \geq 0$, in $\langle \mathcal{C} \rangle$ such that $0 \to T^0(X) \to T^0(Y) \to T^0(Z) \to T^1(X) \to \cdots$ is a complex.

(ii) Whenever

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & 0 \end{array}$$

is a commutative diagram in $\mathcal{C}$ with admissible rows, there is defined a morphism between the corresponding complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T^0(X) & \longrightarrow & T^0(Y) & \longrightarrow & T^0(Z) & \longrightarrow & T^1(X) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T^0(X') & \longrightarrow & T^0(Y') & \longrightarrow & T^0(Z') & \longrightarrow & T^1(X') & \longrightarrow & \cdots \end{array}$$

[**9**, Section 6.5].

Theorem 3.4. *Let $0 \to X \to Y \to Z \to 0$ $(\mathcal{S})$ be an admissible short complex in $\mathcal{C}$. Then there exists a long exact complex in $\langle \mathcal{C} \rangle$ as*

$$0 \longrightarrow H_I^0(X) \longrightarrow H_I^0(Y) \longrightarrow H_I^0(Z) \xrightarrow{\zeta_0} H_I^1(X) \longrightarrow \cdots$$

with continuous connecting morphisms $\zeta_n : H_I^n(Z) \to H_I^{n+1}(X)$. Moreover, $(H_I^i)_{i \geq 0}$ is a connected right sequence of covariant functors.

Proof. Suppose that $0 \to X \xrightarrow{\phi} Y \xrightarrow{\psi} Z \to 0$ $(\mathcal{S})$ is an admissible short complex in $\mathcal{C}$ so that there exist continuous operators $\rho : Y \to X$ and $\sigma : Z \to Y$ such that $\rho \circ \phi = 1_X$, $\psi \circ \sigma = 1_Z$, $\phi \circ \rho + \sigma \circ \psi = 1_Y$ [**6**, Proposition 3.1.8]. Then the following short sequence of complexes is admissible:

$$0 \longrightarrow C(X) \longrightarrow C(Y) \longrightarrow C(Z) \longrightarrow 0.$$

Applying Proposition III.1.5 and Theorem III.1.9 of [**6**] to the functor $\mathcal{B}(A, ?)$, we conclude that, for any $i \geq 0$, $J^i(\mathcal{S})$ splits. Thus $0 \to \Gamma_I(\mathcal{J}(X)) \to \Gamma_I(\mathcal{J}(Y)) \to \Gamma_I(\mathcal{J}(Z)) \to 0$ is exact. Now we may use the fundamental lemma of homological algebra [**6**, Theorem 0.5.7] in order to get

$$o \longrightarrow H_I^0(X) \longrightarrow H_I^0(Y) \longrightarrow H_I^0(Z) \xrightarrow{\zeta_0} H_I^1(X) \longrightarrow \cdots$$

with continuous connecting morphisms $\zeta_n : H_I^n(Z) \to H_I^{n+1}(X)$. The rest is a well-known technique in homological algebra, [**9**, Section 6.3] and [**6**, Chapter 0, Section 5]. $\square$

4. Direct limit and local cohomology functors. Let $\{X_\alpha\}_{\alpha \in K}$ be a family of Banach A-modules. Recall that the topological direct sum of $\{X_\alpha\}_{\alpha\in K}$ is defined as the completion of algebraic direct sum $\oplus_{\alpha\in K}X_\alpha$ with respect to the norm $\|(x_\alpha)_{\alpha\in K}\| = \sum_{\alpha\in K}\|x_\alpha\|$ which will also be denoted by $\oplus_{\alpha\in K}X_\alpha$. This is in $\mathcal{C}$ and consists of all elements $(x_\alpha)_{\alpha\in K}$ of the algebraic direct product $\prod_{\alpha\in K}X_\alpha$ such that $\sum_{\alpha\in K}\|x_\alpha\| < \infty$, see [**10**, Section 2.1] and [**1**, Section 9].

Definition 4.1. (i) Let $(D,\leq)$ be a directed set. A direct system over D in $\mathcal{C}$ consists of families $\{X_\alpha\}_{\alpha\in D}$ of Banach A-modules and $\{\pi_{\beta\alpha}\}_{(\alpha,\beta)\in D\times D,\alpha\leq\beta}$ of module morphisms such that for all $\alpha,\beta,\gamma\in D$, $\pi_{\alpha\alpha} = 1_{X_\alpha}$ and $\pi_{\gamma\beta}\circ\pi_{\beta\alpha} = \pi_{\gamma\alpha}$ whenever $\alpha\leq\beta\leq\gamma$ and $\sup_{\alpha\leq\beta}\|\pi_{\beta\alpha}\| < \infty$. It is denoted by (X,π,D).

(ii) Let (X,π,D) and (Y,η,D) be direct systems. A map ϕ from (X,π,D) to (Y,η,D) is a family $\{\phi_\alpha\}_{\alpha\in D}$ of module morphisms $\phi_\alpha : X_\alpha \to Y_\alpha$ such that it is uniformly bounded, i.e., $\sup_{\alpha\in D}\|\phi_\alpha\| < \infty$, and for $\alpha\leq\beta$ the following diagram commutes:

$$\begin{tikzcd} X_\alpha \arrow[r, "\phi_\alpha"] \arrow[d, "\pi_{\beta\alpha}"'] & Y_\alpha \arrow[d, "\eta_{\beta\alpha}"] \\ X_\beta \arrow[r, "\phi_\beta"'] & Y_\beta \end{tikzcd}$$

(iii) A direct limit of a direct system (X,π,D) is a Banach A-module X_∞ together with a family $\{\pi_\alpha\}_{\alpha\in D}$ of module morphisms $\pi_\alpha : X_\alpha \to X_\infty$ such that $\sup_{\alpha\in D}\|\pi_\alpha\| < \infty$ and for all $\alpha,\beta\in D$ if $\alpha\leq\beta$ then the diagram

$$\begin{tikzcd} X_\alpha \arrow[r, "\pi_{\beta\alpha}"] \arrow[d, "\pi_\alpha"'] & X_\beta \arrow[dl, "\pi_\beta"] \\ X_\infty & \end{tikzcd}$$

commutes. In addition, it has the following "universal property":

For each Banach A-module X and each family $\{\rho_\alpha\}_{\alpha\in D}$ of module morphisms $\rho_\alpha : X_\alpha \to X$ such that $\sup_{\alpha\in D}\|\rho_\alpha\| < \infty$ and for $\alpha\leq\beta$, the diagram

$$\begin{tikzcd} X_\alpha \arrow[r, "\pi_{\beta\alpha}"] \arrow[dr, "\rho_\alpha"'] & X_\beta \arrow[d, "\rho_\beta"] \\ & X \end{tikzcd}$$

commutes, we have a unique module morphism $\omega : X_\infty \to X$ such that, for all $\alpha \in D$ we have the following commutative diagram:

$$\begin{tikzcd} X_\alpha \arrow[d, "\pi_\alpha"'] \arrow[dr, "\rho_\alpha"] & \\ X_\infty \arrow[r, "\omega"'] & X \end{tikzcd}$$

It is obvious that X_∞ and $\{\pi_\alpha\}_{\alpha \in D}$ are unique up to a module isomorphism in $\mathcal{C}$. We denote the direct limit of (X, π, D) by $(X_\infty, \{\pi_\alpha\}_{\alpha\in D})$ or $\varinjlim_\alpha X_\alpha$.

Proposition 4.2. *Any direct system in $\mathcal{C}$ has a direct limit.*

Proof. Suppose that (X, π, D) is a direct system in $\mathcal{C}$. Let $i_\alpha : X_\alpha \to \oplus_{\alpha\in D} X_\alpha$ be the natural injection and R the Banach A-submodule of $\oplus_{\alpha\in D} X_\alpha$ generated by $i_\beta(\pi_{\beta\alpha}(x_\alpha)) - i_\alpha(x_\alpha)$, $(\alpha, \beta) \in D$, $\alpha \le \beta$, $x_\alpha \in X_\alpha$. Set $X_\infty = (\oplus_{\alpha\in D} X_\alpha)/R$ and suppose that, for each $\alpha \in D$, π_α is the composition $X_\alpha \overset{i_\alpha}{\to} \oplus_{\alpha\in D} X_\alpha \overset{\text{nat}}{\to} X_\infty$. Obviously $\sup_{\alpha\in D} \|\pi_\alpha\| \le 1$. We shall show that $(X_\infty, \{\pi_\alpha\}_{\alpha\in D})$ is the direct limit of (X, π, D).

Let $X \in \mathcal{C}$, $\rho_\alpha : X_\alpha \to X$, $\alpha \in D$ be module morphisms such that for each $\alpha \le \beta$ the diagram

$$\begin{tikzcd} X_\alpha \arrow[r, "\pi_{\beta\alpha}"] \arrow[dr, "\rho_\alpha"'] & X_\beta \arrow[d, "\rho_\beta"] \\ & X \end{tikzcd}$$

commutes and $M = \sup_{\alpha\in D} \|\rho_\alpha\| < \infty$. Suppose $(x_\alpha)_{\alpha\in D}$ is an element of the algebraic direct sum of $\{X_\alpha\}_{\alpha\in D}$. Then $x_\alpha = 0$ for all except finitely many α. If $\Theta((x_\alpha)_{\alpha\in D}) = \sum_{\alpha\in D}\rho_\alpha(x_\alpha)$, then $\|\Theta((x_\alpha)_{\alpha\in D})\| \le \sum_{\alpha\in D}\|\rho_\alpha(x_\alpha)\| \le M\sum_{\alpha\in D}\|x_\alpha\| = M\|(x_\alpha)_{\alpha\in D}\|$. So we can extend Θ by the continuity to $\oplus_{\alpha\in D}X_\alpha$, denoted by the same Θ. For $(\alpha,\beta)\in D$, $\alpha\le\beta$ and $x_\alpha\in X_\alpha$, $\Theta(i_\beta(\pi_{\beta\alpha}(x_\alpha)) - i_\alpha(x_\alpha)) = \rho_\beta(\pi_{\beta\alpha}(x_\alpha)) - \rho_\alpha(x_\alpha) = \rho_\alpha(x_\alpha)-\rho_\alpha(x_\alpha) = 0$. Hence $R\subseteq\ker\Theta$. Thus we have module morphism $\omega : X_\infty\to X$ defined by $\omega(u+R)=\Theta(u)$, $u\in\oplus_{\alpha\in D}X_\alpha$. We next have $(\omega\circ\pi_\alpha)(x_\alpha) = \omega(i_\alpha(x_\alpha)+R) = \Theta(i_\alpha(x_\alpha)) = \rho_\alpha(x_\alpha)$, $x_\alpha\in X_\alpha$; hence, $\omega\circ\pi_\alpha=\rho_\alpha$. $\square$

If $\{\phi_\alpha\}_{\alpha\in D}$ is a mapping from (X,π,D) to (Y,η,D), then it is easy to verify that there exists a unique, in a certain meaning, module morphism $\phi_\infty : X_\infty\to Y_\infty$. Moreover, it is possible to consider "direct limit" as a functor [**4**].

Example 4.3. Suppose that D is a directed set, $\{X_\alpha\}_{a\in D}$ a family of Banach A-submodules of a given $X\in\mathcal{C}$ and whenever $\alpha\le\beta$, $X_\alpha\subseteq X_\beta$ and $\pi_{\beta\alpha}: X_\alpha\to X_\beta$ is the inclusion map, then it is clear that $\varinjlim_\alpha X_\alpha = \overline{\cup_{\alpha\in D}X_\alpha}$ and $\pi_\alpha : X_\alpha\to\varinjlim_\alpha X_\alpha$ is also the inclusion map.

Now let, for $X\in\mathcal{C}$ and $n\in\mathbf{N}$, $X_n = {}_Ah((A/I^n),X)$; for $m\le n$, $\pi_{nm}: X_m\to X_n$ given by $\pi_{nm}(\alpha)=\alpha\circ\delta_{nm}$ where $\delta_{nm}: A/I^n\to A/I^m$ is defined naturally by $\delta_{nm}(a+I^n) = a+I^m$ (note that $I^n\subseteq I^m$ whenever $m\le n$). It follows that $\{X_n\}_{n\in\mathbf{N}}$ together with $\{\pi_{nm}\}_{(m,n)\in\mathbf{N}\times\mathbf{N},m\le n}$ is a direct system. Also if $f: X\to Y$ is a module morphism and $\phi_n: X_n\to Y_n$ is given by $\phi_n(\alpha)=f\circ\alpha$, then $\sup_n\|\phi_n\|\le\|f\|$ and the commutativity of

$$\begin{array}{ccc} X_m & \xrightarrow{\phi_m} & Y_m \\ {\scriptstyle {}_Ah(\delta_{nm},X)}\downarrow & & \downarrow{\scriptstyle {}_Ah(\delta_{nm},Y)} \\ X_n & \xrightarrow[\phi_n]{} & Y_n \end{array}$$

shows that $\{\phi_n\}_{n\in\mathbf{N}}$ is a mapping between corresponding direct systems. So there exists a module morphism $\phi_\infty : \varinjlim_n {}_Ah((A/I^n),X)\to\varinjlim_n {}_Ah((A/I^n),Y)$. Moreover we can consider $\varinjlim_n {}_Ah((A/I^n),\cdot)$ as a functor from $\mathcal{C}$ to $\mathcal{C}$.

The next theorem is the Banach theory version of an important purely algebraic theorem, [**2**, Theorem 1.2.11].

Theorem 4.4. *The functors* $\Gamma_I(\cdot)$ *and* $\varinjlim_n {}_Ah((A/I^n),\cdot)$ *are naturally equivalent.*

Proof. For each n, $\psi_n : {}_Ah((A/I^n),X) \to (0 :_X I^n)$ given by $\psi_n(f) = f(e+I^n)$ is obviously a module isomorphism and $\sup_n \|\psi_n\| \leq 1$. In addition the following diagram commutes:

$$\begin{array}{ccc}
{}_Ah((A/I^m),X) \simeq & \xrightarrow{\phi_m} & (0 :_X I^m) \\
\Big\downarrow {\scriptstyle {}_Ah(\delta_{nm},X)} & & \Big\downarrow {\scriptstyle \mathrm{inc}} \\
{}_Ah((A/I^n),X) \simeq & \xrightarrow[\phi_n]{} & (0;_X I^n)
\end{array}$$

For the direct limit is a functor, we have a module isomorphism $\psi(X) : \varinjlim_n {}_Ah((A/I^n),X) \simeq\to \varinjlim_n (0 :_X I^n)$.

By Example 4.3 and $(0 :_X I^n) \subseteq (0 :_X I^{n+1})$, $\varinjlim_n (0 :_X I^n) = \Gamma_I(X)$ and so $\varinjlim_n {}_Ah((A/I^n),X) \simeq \Gamma_I(X)$. Also it is easy to check that if $f : X \to Y$ is a morphism in $\mathcal{C}$, the following diagram is well-defined and commutative.

$$\begin{array}{ccc}
\varinjlim_n {}_Ah((A/I^n),X) \simeq & \longrightarrow & \Gamma_I(X) \\
\Big\downarrow {\scriptstyle \varinjlim_n {}_Ah(1_{A/I^n},f)} & & \Big\downarrow {\scriptstyle \Gamma_I(f)} \\
\varinjlim_n {}_Ah((A/I^n),Y) \simeq & \longrightarrow & \Gamma_I(Y)
\end{array}$$

Thus $\varinjlim_n {}_Ah((A/I^n),\cdot)$ is naturally equivalent to $\Gamma_I(\cdot)$. $\square$

REFERENCES

1. F.F. Bonsal and J. Duncan, *Complete normed algebras*, Springer-Verlag, 1973.

2. M.P. Brodmann and R.Y. Sharp, *Local cohomology: An algebraic introduction with geometric applications*, Cambridge Univ. Press, Cambridge, 1998.

3. A. Grothendieck, *Local cohomology*, Lecture Notes in Math. **41**, Springer-Verlag, New York, 1967.

4. A. Guichardet, *Special topics in topological algebras*, Gordon and Breach, New York, 1968.

5. A.Ya. Helemskii, *Banach and locally convex algebras*, Oxford University Press, New York, 1993.

6. ———, *The homology of Banach and topological algebras*, Kluwer Academic Publishers, New York, 1989.

7. R.V. Kadison and J.R. Ringrose, *Fundamentals of the theory of operator algebras*, Vol. I, Academic Press, New York, 1983.

8. ———, *Fundamentals of the theory of operator algebras*, Vol. II, Academic Press, New York, 1986.

9. D.G. Northcott, *An introduction to homological algebra*, Cambridge Univer. Press, Cambridge, 1972.

10. G.K. Pedersen, *Analysis now*, Springer-Verlag, New York, 1989.

Department of Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, P.O. Box 1159-91775, Mashhad-Iran
E-mail address: `msalm@science2.um.ac.ir`

Department of Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, P.O. Box 1159-91775, Mashhad-Iran
E-mail address: `niknam@science2.um.ac.ir`