ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 30, Number 1, Spring 2000

EQUAL SUMS OF POWERS OF THE TERMS OF ARITHMETIC PROGRESSIONS OF EQUAL LENGTHS

AJAI CHOUDHRY

ABSTRACT. This paper gives a method of obtaining two arithmetic progressions of equal arbitrary length and consisting entirely of positive integers such that the sums of either the squares or the cubes or the fourth powers of the terms of the two arithmetic progressions are equal. It is further shown that an arbitrarily large number of such arithmetic progressions can be obtained such that the sums of the squares of the terms of all these arithmetic progressions are equal.

1. The diophantine equation

$$x_1^k + x_2^k + \cdots + x_m^k = y_1^k + y_2^k + \cdots + y_m^k \tag{1}$$

has been studied by numerous mathematicians [**2, 3**]. In this paper we study equation (1) with the additional stipulation that $x_1, x_2, \dots, x_m$ and $y_1, y_2, \dots, y_m$ are the terms of arithmetic progressions of positive integers. It will be shown that there exist solutions of equation (1) with the x_i, $i = 1, 2, \dots, m$ and y_i, $i = 1, 2, \dots, m$, $m \geq 3$, being the terms of distinct arithmetic progressions of equal length and $k = 2$ or 3 or 4. In fact, when $k = 2$, it will be shown that there is an arbitrarily large number of arithmetic progressions of positive integers, each arithmetic progression having the same given number of terms, such that the sums of the squares of the terms of each of the arithmetic progressions are equal.

We note that equation (1) is homogeneous. It therefore follows that any rational solution of (1) may be multiplied throughout by a suitable integer to yield a solution in integers.

2. Let x_i, $i = 1, 2, \dots, m$ and y_i, $i = 1, 2, \dots, m$ be the terms of two distinct arithmetic progressions of m terms each, $m \geq 3$, with first

Received by the editors on November 3, 1998.

terms a and a_1 and common differences d and d_1, respectively. Then

$$(2) \qquad \begin{aligned} x_i &= a + (i-1)d, \quad i = 1, 2, \dots, m \\ y_i &= a_1 + (i-1)d_1, \quad i = 1, 2, \dots, m. \end{aligned}$$

Substituting these values of x_i, y_i in (1) we get

$$(3) \qquad \sum_{i=1}^{m} \{a + (i-1)d\}^k = \sum_{i=1}^{m} \{a_1 + (i-1)d_1\}^k.$$

When $k = 2$, equation (3) reduces to

$$(4) \quad \begin{aligned} &6a^2 + 6(m-1)ad + (m-1)(2m-1)d^2 \\ &\qquad = 6a_1^2 + 6(m-1)a_1d_1 + (m-1)(2m-1)d_1^2. \end{aligned}$$

The complete solution of equation (4) is obtained simply by writing $a_1 = a + t\alpha_1$, $d_1 = d + t\beta_1$ and substituting in (4) which is then readily solved for t and leads to the following rational solution of (4):

$$(5) \qquad \begin{aligned} a_1 = &-[\{6\alpha_1^2 - (m-1)(2m-1)\beta_1^2\}a \\ &\quad + 2(m-1)\{3\alpha_1 + (2m-1)\beta_1\}\alpha_1 d] \\ &\times \{6\alpha_1^2 + 6(m-1)\alpha_1\beta_1 + (m-1)(2m-1)\beta_1^2\}^{-1} \\ d_1 = &[-6\{2\alpha_1 + (m-1)\beta_1\}\beta_1 a \\ &+ \{6\alpha_1^2 - (m-1)(2m-1)\beta_1^2\}d] \\ &\times \{6\alpha_1^2 + 6(m-1)\alpha_1\beta_1 + (m-1)(2m-1)\beta_1^2\}^{-1} \end{aligned}$$

where a, d, α_1 and β_1 are arbitrary. Thus, given an arithmetic progression with first term a and common difference d, we have obtained another arithmetic progression with first term a_1 and common difference d_1 such that the sums of the squares of m terms of the two arithmetic progressions are equal. As α_1, β_1 are arbitrary integers, we may replace them with different pairs of integers, say α_r, β_r, $r = 2, 3, \dots, n$, to obtain additional arithmetic progressions with first term a_r and common difference d_r, respectively, such that the sums of the squares of m terms of each of these arithmetic progressions is the same. The rth arithmetic progression has first term a_r and common difference d_r given by

$$
\begin{aligned}
a_r &= -[\{6\alpha_r^2 - (m-1)(2m-1)\beta_r^2\}a \\
&\qquad + 2(m-1)\{3\alpha_r + (2m-1)\beta_r\}\alpha_r d] \\
&\qquad \times \{6\alpha_r^2 + 6(m-1)\alpha_r\beta_r + (m-1)(2m-1)\beta_r^2\}^{-1} \\
d_r &= [-6\{2\alpha_r + (m-1)\beta_r\}\beta_r a \\
&\qquad + \{6\alpha_r^2 - (m-1)(2m-1)\beta_r^2\}d] \\
&\qquad \times \{6\alpha_r^2 + 6(m-1)\alpha_r\beta_r + (m-1)(2m-1)\beta_r^2\}^{-1}.
\end{aligned} \tag{6}
$$

For a given m, we choose α_r, β_r to be positive integers such that, for each r,

$$6\alpha_r^2 > (m-1)(2m-1)\beta_r^2 \tag{7}$$

and we choose a and d to be negative integers such that

$$a < d\Big\{6\alpha_r^2 - (m-1)(2m-1)\beta_r^2\Big\}[6\{2\alpha_r + (m-1)\beta_r\}\beta_r]^{-1} \tag{8}$$

for $r = 1, 2, \ldots, n$. When (7) and (8) hold, it follows from (6) that both a_r and d_r are positive. Thus, all the terms of the rth arithmetic progression, $1 \le r \le n$, are positive and the sum of the squares of m terms of this arithmetic progression is equal to the sum of the squares of an arithmetic progression with first term a and common difference d. We will now show that no two of the n arithmetic progressions obtained above have a term in common. The ith term, $1 \le i \le m$, of the rth arithmetic progression, $1 \le r \le n$, and the jth term of the sth arithmetic progression, $1 \le s \le n$, will be equal if

$$a_r + (i-1)d_r = a_s + (j-1)d_s. \tag{9}$$

Substituting the values of a_r, d_r, a_s, d_s given by (6), we obtain a linear equation in a and d which has just one solution for a/d. Moreover, since $1 \le i \le m$, $1 \le j \le m$, $1 \le r \le n$ and $1 \le s \le n$, there are only a finite number of equations of type (9). We can easily choose a and d so as to avoid the finite number of values of a/d satisfying an equation of type (9) and such that the relation (8) holds for $r = 1, 2, \ldots, n$. With such a choice of a and d, we get n arithmetic progressions in positive rational numbers such that no two arithmetic progressions have a term in common and such that the sum of the squares of the m terms of these arithmetic progressions is equal. In the above discussion, both m and n are arbitrary positive integers. Thus we have established a procedure of finding an arbitrarily large number of arithmetic progressions in positive rational numbers, each arithmetic progression having the same given number of terms, such that the sums of the squares of the terms of all these arithmetic progressions are equal. Multiplying by a suitable integer will yield a solution in integers.

When $m = 3$, taking $(\alpha_1, \beta_1) = (2, 1)$, $(\alpha_2, \beta_2) = (3, 1)$, $(\alpha_3, \beta_3) = (4, 1)$, $(\alpha_4, \beta_4) = (5, 1)$, $a = -2$, $d = -1$, we get the following diophantine chain consisting of the terms of four arithmetic progressions such that the sums of the squares of the terms of each of these four arithmetic progressions is the same:

$$\begin{aligned} 3850^2 + 5775^2 + 7700^2 &= 4928^2 + 5929^2 + 6930^2 \\ &= 5550^2 + 5975^2 + 6400^2 \\ &= 5950^2 + 5985^2 + 6020^2. \end{aligned}$$

3. We now consider arithmetic progressions such that the sums of the cubes of the terms of the arithmetic progressions are equal. Substituting $k = 3$ in equation (3), we get the equation

$$\begin{aligned} (10)\quad & 4a^3 + 6(m-1)a^2d + 2(m-1)(2m-1)ad^2 + m(m-1)^2d^3 \\ & = 4a_1^3 + 6(m-1)a_1^2d_1 + 2(m-1)(2m-1)a_1d_1^2 + m(m-1)d_1^3, \end{aligned}$$

or

$$\begin{aligned} (11)\quad & \{2a + (m-1)d\}\{2a^2 + 2(m-1)ad + m(m-1)d^2\} \\ & = \{2a_1 + (m-1)d_1\}\{2a_1^2 + 2(m-1)a_1d_1 + m(m-1)d^2\}. \end{aligned}$$

While it is not difficult to obtain parametric solutions, or even a complete solution of equation (10) in integers, the usual methods of solution lead to arithmetic progressions which contain negative integers. To obtain a solution of our problem in positive integers, we impose the following conditions on a, a_1, d and d_1:

$$(12)\qquad a_1 = at + 3m(dt - d_1)/4$$

and

$$2a + (m-1)d = \{2a_1 + (m-1)\, d_1\}t^2 \tag{13}$$

where t is an arbitrary rational number. Substituting the values of a and a_1 obtained from (12) and (13) in (10), we get the relation

$$(dt - d_1)^2[\{(5t^3-3)m^2 + 4(t^3+1)m + 8\}\, dt \\ - \{(5-3t^3)m^2 + 4(t^3+1)m + 8t^3\}d_1] = 0.$$

We cancel out the factor $(dt - d_1)^2$ and obtain the following solution for d and d_1:

$$\begin{aligned} d &= 2\{(5-3t^3)m^2 + 4(t^3+1)m + 8t^3\} \\ d_1 &= 2\{(5t^3-3)m^2 + 4(t^3+1)m + 8\}t. \end{aligned} \tag{14}$$

This, in turn, gives us

$$\begin{aligned} a &= (7t^3-5)m^3 + (t^3+1)m^2 - 4(2t^3-1)m \\ a_1 &= \{(7-5t^3)m^3 + (t^3+1)m^2 + 4(t^3-2)m\}t. \end{aligned} \tag{15}$$

By choosing t such that

$$(5/7)^{1/3} < t < (7/5)^{1/3},$$

that is,

$$0.8939\ldots < t < 1.1186\ldots\,,$$

we get a solution of (10) in positive rational numbers. Multiplying by a suitable integer yields a solution in positive integers.

As a numerical example, when $m = 3$ and $t = 11/10$, we get the two arithmetic progressions {587970, 1064800, 1541630} and {122463, 880000, 1637537} such that

$$587970^3 + 1064800^3 + 1541630^3 = 122463^3 + 880000^3 + 1637537^3.$$

Similarly, when $m = 4$ and $t = 11/10$, we get the example:

$$29895^3 + 43240^3 + 56585^3 + 69930^3 \\ = 5577^3 + 29359^3 + 53141^3 + 76923^3.$$

4. We next proceed to find arithmetic progressions such that the sums of the fourth powers of the terms of the arithmetic progressions are equal. Substituting $k=4$ in (3), we get the equation

$$\begin{aligned}
&30a^4+60(m-1)a^3d+30(2m^2-3m+1)a^2d^2\\
&\quad+30m(m-1)^2ad^3+(6m^4-15m^3+10m^2-1)d^4\\
&=30a_1^4+60(m-1)a_1^3d_1+30(2m^2-3m+1)a_1^2d_1^2\\
&\quad+30m(m-1)^2a_1d_1^3+(6m^4-15m^3+10m^2-1)d_1^4.
\end{aligned} \tag{16}$$

According to a theorem proved by Choudhry [**1**], a necessary and sufficient condition that a quartic equation of the type

$$\begin{aligned}
&Ax^4+Bx^3y+Cx^2y^2+Dxy^3+Ey^4\\
&\qquad=Au^4+Bu^3v+Cu^2v^2+Duv^3+Ev^4
\end{aligned} \tag{17}$$

has a solution in integers is that there exist rational numbers p,q such that the following two functions $\phi_1(p,q)$ and $\phi_2(p,q)$ given by

$$\begin{aligned}
\phi_1(p,q)&=4Apq^3-3Bpq^2+2Cpq-Dp\\
&\quad-Bq^3+2Cq^2-3Dq+4E,\\
\phi_2(p,q)&=4Ap^3q-Bp^3-3Bp^2q+2Cp^2+2Cpq\\
&\quad-3Dp-Dq+4E
\end{aligned} \tag{18}$$

are either both zero or $-\phi_1(p,q)\phi_2(p,q)$ is a nonzero perfect square. When rational p and q exist such that $-\phi_1(p,q)\phi_2(p,q)$ is a nonzero perfect square, a rational solution of (17) is given by

$$\begin{aligned}
x&=p\phi_1(p,q)-\{-\phi_1(p,q)\phi_2(p,q)\}^{1/2}q,\\
y&=-\phi_1(p,q)+\{-\phi_1(p,q)\phi_2(p,q)\}^{1/2},\\
u&=-p\phi_1(p,q)-\{-\phi_1(p,q)\phi_2(p,q)\}^{1/2}q,\\
v&=\phi_1(p,q)+\{-\phi_1(p,q)\phi_2(p,q)\}^{1/2}.
\end{aligned} \tag{19}$$

We will apply this theorem to the quartic equation (16). Accordingly, we need to consider the functions

$$\begin{aligned}
\phi_1(p,q)&=30p\{4q^3-6(m-1)q^2+2(2m^2-3m+1)q-m(m-1)^2\}\\
&\quad-60(m-1)q^3+60(2m^2-3m+1)q^2-90m(m-1)^2q\\
&\quad+4(6m^4-15m^3+10m^2-1)
\end{aligned}$$

and

$$\begin{aligned}\phi_2(p,q) = 30q\{4p^3 - 6(m-1)^2p^2 + 2(2m^2-3m+1)p - m(m-1)^2\} \\ - 60(m-1)p^3 + 60(2m^2-3m+1)p^2 - 90m(m-1)^2p \\ + 4(6m^4 - 15m^3 + 10m^2 - 1)\end{aligned}$$

and the equation

$$z^2 = -\phi_1(p,q)\phi_2(p,q). \tag{20}$$

We note that, when $p = -1$, $q = 2(m-1)/3$, $z = (4m^4 + 10m^3 - 10m - 4)/3$, equation (20) is satisfied. This, however, leads to a trivial solution of our problem. We must accordingly find another solution of equation (20). We take $p = -1$ in equation (20) when this equation, of type $z^2 =$ quartic in q, with known point at $q = 2(m-1)/3$, represents an elliptic curve, and we can find another solution of (20) by applying the group law. In this manner, we find that when $p = -1$ and

$$\begin{aligned}q = \{5184m^9 + 16848m^8 - 16416m^7 - 76680m^6 + 33408m^5 \\ + 132732m^4 - 72000m^3 - 80904m^2 + 72742m - 14914\} \\ \times \{6480m^8 + 25920m^7 + 1080m^6 - 87480m^5 - 24480m^4 \\ + 127080m^3 + 1920m^2 - 80520m + 30375\}^{-1},\end{aligned}$$

then $-\phi_1(p,q)\phi_2(p,q)$ becomes a nonzero perfect square. Thus, using the solution (19) of equation (17), we get, after multiplying throughout by a suitable constant, cancellation of common factors and by changing the signs of both a_1 and d_1, which is permissible as equation (16) is of even degree, the following solution of equation (16):

$$\begin{aligned}a = 93312m^{12} + 404352m^{11} - 155520m^{10} - 2384640m^9 - 451008m^8 \\ + 6339600m^7 + 1026648m^6 - 9264240m^5 + 492012m^4 \\ + 6810900m^3 - 2017230m^2 - 1526268m + 631457,\end{aligned}$$

$$\begin{aligned}d = 93312m^{12} + 559872m^{11} + 699840m^{10} - 1632960m^9 - 3483648m^8 \\ + 2021760m^7 + 6054048m^6 - 1931040m^5 - 5096448m^4 \\ + 1729440m^3 + 1767540m^2 - 712428m - 64288,\end{aligned}$$

$$a_1 = 93312m^{12} + 715392m^{11} + 1166400m^{10} - 2825280m^9 - 7553088m^8 + 5220720m^7 + 17896248m^6 - 8627040m^5 - 20628108m^4 + 12801180m^3 + 8733810m^2 - 8967888m + 1973717,$$

$$d_1 = 93312m^{12} + 559872m^{11} + 311040m^{10} - 3576960m^9 - 4520448m^8 + 9538560m^7 + 12868848m^6 - 15960240m^5 - 15039648m^4 + 17882640m^3 + 4949040m^2 - 9781728m + 2669462.$$

For any positive integer $m \geq 3$, the above values of a, d, a_1 and d_1 are in positive integers. Thus, given any arbitrary positive integer $m \geq 3$, we can find two arithmetic progressions consisting entirely of positive integers such that the sums of the fourth powers of the terms of the two arithmetic progressions are equal.

As a numerical example, when $m = 3$, we get the two arithmetic progressions {751601, 1184107, 1616613} and {348473, 1016041, 1683609} such that

$$751601^4 + 1184107^4 + 1616613^4 = 348473^4 + 1016041^4 + 1683609^4.$$

For a given m, additional solutions may be found by finding other rational solutions of equation (20) through the further application of the group law to the elliptic curve represented by this equation when p, or, for that matter, q, has a fixed value.

It would be interesting to find two arithmetic progressions such that the sums of the kth powers of the terms of these two arithmetic progressions are equal where $k > 4$.

REFERENCES

1. A. Choudhry, *On the quartic diophantine equation* $f(x, y) = f(u, v)$, J. Number Theory **75** (1999), 34–40.

2. L.E. Dickson, *History of the theory of numbers*, Vol. 2, Chelsea, New York, 1952 (Reprint), Chapters 4–9, 21–24.

3. L.J. Lander, T.R. Parkin and J.L. Selfridge, *A survey of equal sums of like powers*, Math. Comp. **21** (1967), 446–459.

Ambassador of India, Indian Embassy, Kantari Street, Sahmarani Building, P.O. Box 113-5240, Beirut, Lebanon
E-mail address: ambindia@inco.com.lb