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AN EXPLICIT ZERO-FREE REGION FOR THE RIEMANN ZETA-FUNCTION

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ABSTRACT. This paper gives an explicit zero-free region for the Riemann zeta-function derived from the Vinogradov-Korobov method. We prove that the Riemann zeta-function does not vanish in the region $\sigma \geq 1 - .00105 \log^{-2/3} |t| (\log\log |t|)^{-1/3}$ and $|t| \geq 3$.

1. Introduction. It is now well known that the problem involving prime numbers can be related to the study of the Riemann zeta-function. In 1860, Riemann in [**17**] showed that the key to the deeper investigation of the distribution of the primes lies in the study of the function which is now called the Riemann zeta-function. Let $s = \sigma + it$ be a complex variable. For $\sigma > 1$, the Riemann zeta-function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1}$$

The above series converges absolutely and uniformly on the half plane $\sigma \geq \sigma_0$ for any $\sigma_0 > 1$. It can be extended to be a regular function on the whole complex plane $\mathbf{C}$, except at $s = 1$, which is the only pole of the Riemann zeta-function and at which the function has residue 1. The general definition of the Riemann zeta-function may be referred to by its functional equation. That is,

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s). \tag{2}$$

Here Γ is the factorial function of a complex variable and $\Gamma(n) = (n-1)!$ for every positive integer n. The pole of Γ at $s = 0$ corresponds to that of $\zeta(s)$ at $s = 1$. The other poles of Γ at $s = -n$ for positive integers

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n cancel with the so-called trivial zeros of the Riemann zeta-function, which are $s = -2n$ for all positive integers n. For reference, one may see [**7, 9, 12, 16**] or [**21**].

The Riemann hypothesis states that all nontrivial zeros of $\zeta(s)$ are on the line $\sigma = 1/2$, which is not proved or disproved. There exist weaker results concerning the zero-free region for the Riemann zeta-function. It is not difficult to show that $\zeta(s) \neq 0$ for $\sigma > 1$; we shall include it in the next section for completeness. It was proved independently by Hadamard, see [**8**] and de la Vallée Poussin, see [**23**], in 1896 that $\zeta(s) \neq 0$ for $\sigma \geq 1$. In 1899, de la Vallée Poussin established in [**24**] that $\zeta(s)$ does not have zeros in the region

$$\sigma > 1 - \frac{A}{\log t}, \quad t \geq t_0,$$

where A and t_0 are positive constants. For reference, one may see [**7**]. Rosser in [**19**] gave this result with $A = 1/19$ and $t_0 = 1400$ in 1939 for estimating on primes. In fact, the number A can be sharpened if we allow a bigger t_0. Rosser, and later with Schoenfeld, used $A = 1/17.71$ for $t_0 = 1468$ and $A = 1/17.51$ with $t_0 = e^{9.99}$, see [**18**] and [**20**]. For another explicit result, one may also see [**5**], in which Chen and Wang gave a zero-free region in the form

$$\sigma > 1 - \frac{0.057812}{\log\log x} \quad \text{and} \quad |t| \leq \log x,$$

for $x \geq e^{20000}$. In 1922, Littlewood, see [**2**], proved that $\zeta(s) \neq 0$ in the region

$$\sigma > 1 - A\frac{\log\log t}{\log t}, \quad t \geq t_0,$$

where A and t_0 are positive constants. This result was based on an upper bound on exponential sums, see [**2**] or [**27**]. An explicit zero-free region has useful applications in number theory. For reference, one may see the above-mentioned paper of Rosser, that of Rosser and Schoenfeld, or [**5**].

The widest known zero-free region for the Riemann zeta-function is in the form of

$$\sigma > 1 - \frac{A}{\log^{2/3} t(\log\log t)^{1/3}}, \quad \text{and } t \geq t_0, \tag{3}$$

where A and t_0 are positive constants. For reference, one may see [**25, 26, 13**] or the above-mentioned book [**12**]. For more information on this topic, one may see [**14**]. The establishment of this kind of zero-free region is based on an upper bound for the Riemann zeta-function derived from the Vinogradov-Korobov method. In [**6**], the following result is given.

Lemma 1. *For $\sigma \geq (1/2)$ and $t \geq 2$, we have*

$$|\zeta(s)| \leq At^{B(\sigma)} \log^{2/3} t, \tag{4}$$

where A is 175 *and*

(a) $B(\sigma) = 46(1-\sigma)^{3/2}$ *if* $\sigma \leq 1$;

(b) $B(\sigma) = 0$ *if* $\sigma > 1$.

The best known upper bound for the Riemann zeta-function in the region considered in Lemma 1 is due to Heath-Brown [**11**], who claims (4) with constant in $B(\sigma)$ being 18.8, but with the constant A inexplicit. For our application, we will need a completely explicit estimate as afforded by Lemma 1.

The purpose of this paper is to give a zero-free region in the form of (3) explicitly. Like the results in [**19, 18, 20**] and [**5**], this explicit zero-free region can be used to obtain explicit results concerning the distribution of prime numbers, see [**4**]. The main result in this paper is as follows:

Theorem. *The Riemann zeta-function does not vanish in the region*

$$\sigma \geq 1 - 0.00101 \log^{-2/3} |t| (\log\log|t|)^{-1/3} \quad \text{and} \quad |t| \geq 3.$$

2. Some estimates on $\zeta(s)$ and $(\zeta'(s)/\zeta(s))$. Let us begin with an elementary result measuring the size of $\zeta(\sigma)$ for real number σ. We use the decreasing property of $u^{-\sigma}$ as a function of u. We have $(n+1)^{-\sigma} < \int_n^{n+1} u^{-\sigma}\, du < n^{-\sigma}$; if we sum all sides of this inequality over the set of all positive integers n, noting that the resulting series is convergent for $\sigma > 1$, we get $\zeta(\sigma) - 1 < \int_1^\infty u^{-\sigma}\, d\sigma < \zeta(\sigma)$. This gives

$$\frac{1}{\sigma - 1} < \zeta(\sigma) < \frac{\sigma}{\sigma - 1}, \quad \text{for } \sigma > 1. \tag{5}$$

By the property of convergent series, we can differentiate (1) term by term. We get

$$\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\log n}{n^s} \quad \text{for } \sigma > 1. \tag{6}$$

Noting that the function $u^{-\sigma}\log u$ is decreasing for $u \geq 3$, we have that $n^{-\sigma}\log n < \int_{n-1}^{n} u^{-\sigma}\log u\,du$ for $n \geq 4$. Similar to that in the proof of (5), we get $\zeta'(\sigma) + 2^{-\sigma}\log 2 + 3^{-\sigma\log 3} > -\int_3^\infty u^{-\sigma}\log u\,du$. It follows that

$$-\zeta'(\sigma) < \frac{1.001}{(\sigma-1)^2} \quad \text{for } 1 < \sigma < 1.05.$$

In the last step, we have used the fact $\int_3^\infty u^{-\sigma}\log u\,du = (1 = \sigma\log 3 - \log 3)(\sigma-1)^{-2}3^{1-\sigma}$ and $3^{-\sigma+1}(1+\sigma\log 3-\log 3)+(\sigma-1)^2 \times (2^{-\sigma}\log 2 + 3^{-\sigma}\log 3) \leq 1.001$ for $1 < \sigma < 1.05$. Recalling (5), we obtain

$$-\frac{\zeta'(\sigma)}{\zeta(\sigma)} < \frac{1.001}{\sigma-1} \quad \text{for } 1 < \sigma < 1.05. \tag{7}$$

The following identity is called Euler's product identity for the Riemann zeta-function. For $\sigma > 1$, we have

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \tag{8}$$

where p runs through the set of all prime numbers. This identity is an analytic equivalence for the proposition that every natural number can be factorized into prime powers in one and only one way. Using this identity, we can show that $\zeta(s) \neq 0$ for $\sigma > 1$. We have

$$\frac{1}{|\zeta(s)|} = \left|\prod_p \left(1 - \frac{1}{p^s}\right)\right| \leq \prod_p \left(1 + \frac{1}{p^\sigma}\right) \leq \sum_{n=1}^{\infty} \frac{1}{n^\sigma}.$$

The last expression is $\zeta(\sigma)$. Recalling (5), we get

$$|\zeta(s)| > \frac{\sigma-1}{\sigma}. \tag{9}$$

This gives

$$\zeta(s) \neq 0 \quad \text{for } \sigma > 1. \tag{10}$$

Both (9) and (10) shall be needed later in this paper.

We may take logarithms on both sides of (8), in view of convergences for the resulting series. It gives us

$$\log \zeta(s) = -\sum_p \log\left(1 - \frac{1}{p^s}\right) \quad \text{for } \sigma > 1. \tag{11}$$

Hadamard and de la Vallée Poussin used this equation in proving $\zeta(s) \neq 0$ for $\sigma \geq 1$, see [**7**]. A similar technique involved in their proofs shall be used in Section 4. We shall not need $\zeta(s) \neq 0$ for $\sigma = 1$ for the following argument.

Differentiating both sides of (11), we get

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{\log p}{p^s - 1} \quad \text{for } \sigma > 1. \tag{12}$$

The justification of the differentiation term by term is done by noting that, for any arbitrary $\delta > 0$, $|(p^s - 1)^{-1} \log p| < p^{-(\sigma-\delta)}$ for sufficiently large p, $\sum_p p^{-(\sigma-\delta)} < \sum_n n^{-(\sigma-\delta)}$, and the last series is uniformly convergent for $\sigma \geq \sigma_0$ for any $\sigma_0 > 1$. We then rewrite the right side of (11), getting

$$\sum_p \frac{\log p}{p^s - 1} = \sum_p \log p \sum_{m=1}^{\infty} \frac{1}{p^{ms}} = \sum_{p,m} \frac{\log p}{p^{ms}},$$

where the sum is taken over all prime numbers p and all positive integers m. The last double series is absolutely convergent for $\sigma > 1$. We define $\Lambda(n) = \log p$ whenever n is a power of a prime number p or $\Lambda(n) = 0$. The equation (12) becomes

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \quad \text{for } \sigma > 1. \tag{13}$$

The function Λ is the von Mangoldt Lambda-function.

It is this equation (13) which is used in the estimation on the zeros of the Riemann zeta-function. We shall need another tool connecting the upper bound as in (4) for $\zeta(s)$ to that for $-(\zeta'(s)/\zeta(s))$ as well as to the zeros of $\zeta(s)$; see next section.

3. An analytic tool. Traditionally, for the purpose of connecting the upper bound for $\zeta(s)$ to that for $-\zeta'(s)/\zeta(s)$, we have Landau's lemma, see [**2**], though a different approach can be found in [**15**]. The following lemma and its proof can be found in [**10**].

Lemma 2. *Suppose that $f(s)$ is regular in the disc $|s - s_0| \leq r$ and nonvanishing both at $s = s_0$ and on the circle $|s - s_0| = r$. Let $\rho_k = s_0 + r_k e^{i\theta_k}$ be the zeros of $f(s)$ in the disc and n_k the multiplicity of ρ_k. Then*

$$-\Re\left(\frac{f'(s_0)}{f(s_0)}\right) = -\frac{1}{\pi r}\int_{-(\pi/2)}^{(3\pi/2)} \cos(\theta)\log|f(s_0 + re^{i\theta})| + \sum_{\rho_k} n_k\left(\frac{1}{r_k} - \frac{r_k}{r^2}\right)\cos(\theta_k).$$

By the assumption of Lemma 2, we have $(1/r_k) - (r_k/r^2) > 0$. We shall apply Lemma 2 only in the case of $(\pi/2) < \theta_k < 3(\pi/2)$, so that $\cos(\theta_k) < 0$. We see that every term in the last sum is negative. We have the following corollaries.

Corollary 1. *Assume the same as in Lemma 2 and $(\pi/2) < \theta_k < (3\pi/2)$. Then*

$$-\Re\left(\frac{f'(s_0)}{f(s_0)}\right) \leq -\frac{1}{\pi r}\int_0^{2\pi} \cos(\theta)\log|f(s_0) + re^{i\theta})|\, d\theta. \tag{14}$$

If $\rho_0 = s_0 + r_0 e^{i\theta_0}$ is one of the zeros of $f(s)$ with multiplicity n_0 such that $\Im(s_0)$, then $\theta_0 = \pi$ and $\cos(\theta_0) = -1$. In this case we have

$$n_0\left(\frac{1}{r_0} - \frac{r_0}{r^2}\right) \leq -\frac{1}{r_0} + \frac{r_0}{r^2} = -\Re\frac{1}{s_0 - \rho_0} + \frac{r_0}{r^2} \leq -\Re\frac{1}{s_0 - \rho_0} + \frac{1}{r}.$$

Thus we get the following corollary.

Corollary 2. *Assume the same as in Corollary* 1*. If* $\rho_0 = s_0 + r_0 e^{i\theta_0}$ *is one of the zeros of* $f(s)$ *such that* $\Im(\rho_0) = \Im(s_0)$*, then*

$$(15)\quad -\Re\left(\frac{f'(s_0)}{f(s_0)}\right) \le -\frac{1}{\pi r}\int_0^{2\pi} \cos\theta \log|f(s_0+re^{i\theta})|\,d\theta - \Re\frac{1}{s_0-\rho_0} + \frac{1}{r}.$$

To apply Lemma 2, we shall assume that $t_0 \ge e^{e^3}$. For brevity, we denote $L_1 = \log(4t_0 + 1/3)$ and $L_2 = \log(L_1)$. Then we let $\sigma_0 = 1 + \kappa L_1^{-2/3} L_1^{-1/3}$ for some $0 < \kappa < 1$. We have $s_0 = \sigma_0 + it_0$. We note by recalling (10) that the Riemann zeta-function does not vanish at any point s_0. Then we let $r = \lambda L_1^{-2/3} L_2^{-1/3} + \varepsilon$ for some $0 < \lambda < 1$ and ε a positive number tending to zero. We note here that $r < (L_2/L_1)^{2/3} < (3/e^3)^{2/3} < 1/3$.

Suppose s is on the circle $|s - s_0| \le r$. Then $\sigma = \Re(s) = \sigma_0 + r\cos(\theta)$, where $\theta = \arg(s - s_0)$. By the condition that $\sigma_0 > 1$, we have

$$\sigma - 1 > r\cos(\theta).$$

We shall now calculate the integral involved for $-(\pi/2) < \theta < (\pi/2)$ and $(\pi/2) < \theta < (3\pi/2)$ separately.

If $-(\pi/2) < \theta < (\pi/2)$, then $\cos(\theta) > 0$. Recalling (9), we have $\log|\zeta(s)| > \log(\sigma - 1) - \log(1.05)$. Using (16), we get $\log|\zeta(s)| > \log r + \log(\cos\theta) - \log(1.05)$ and

$$-\log|\zeta(s)| < \log\frac{1}{r} + \log\frac{1}{\cos\theta} + 0.05.$$

It follows that

$$(17)\quad -\frac{1}{\pi r}\int_{-\pi/2}^{\pi/2} \cos\theta \log|\zeta(s)|\,d\theta \le \frac{2}{\pi r}\log\frac{1}{r} + \frac{2(1-\log 2)}{\pi r} + \frac{1}{10\pi r}.$$

Here we have used the fact that $-\int_{-\pi/2}^{\pi/2} \cos\theta \log(\cos\theta)\,d\theta = 2(1-\log 2)$.

If $(\pi/2) \le \theta \le (3\pi/2)$, then $\cos\theta \le 0$. Similarly, but recalling (4) and using (16), we get

$$\log|\zeta(s)| \le B(\sigma)\log t + \frac{2}{3}\log\log t + \log A,$$

and $B(\sigma) \le Br^{3/2}(-\cos\theta)^{3/2}$. Note that

$$\int_{\pi/2}^{3\pi/2} (-\cos\theta)^{5/2}\, d\theta = 1.4377... < 1.438.$$

It follows that

$$(18) \quad -\frac{1}{\pi r}\int_{\pi/2}^{3\pi/2} \cos\theta \log|\zeta(s)|\, d\theta \le \frac{1.438B}{\pi} r^{1/2}\log t + \frac{4}{3\pi r}\log\log t + \frac{2\log A}{\pi r}.$$

Combining (17) and (18), we obtain

$$(19) \quad -\frac{1}{\pi r}\int_{0}^{2\pi} \cos\theta \log|\zeta(s)|\, d\theta \le \frac{2}{\pi r}\log\frac{1}{r} + \frac{1.438B}{\pi} r^{1/2}\log t + \left(\frac{4}{3}\log\log t + 2\log A + 0.714\right)\frac{1}{\pi r}.$$

Recalling the definition of r, we deduce

$$(19a) \quad \frac{2}{\pi r}\log\frac{1}{r} < \frac{4-2\log\lambda}{3\pi\lambda} L_1^{2/3} L_2^{1/3},$$

from the fact that $\log(1/\lambda) + (2/3)\log\log(4t_0 + (1/3)) < ((4-2\log\lambda)/3)\log\log(4t_0 + (1/3))$;

$$(19b) \quad \frac{1.438B}{\pi} r^{1/2}\log t \le \frac{1.438B\lambda^{1/2}}{\pi} L_1^{2/3} L_2^{1/3};$$

and

$$(19c) \quad \frac{4\log\log t/3 + 2\log A + 0.714}{\pi r} \le \left(\frac{4}{3\pi\lambda} + \frac{3^{2/3}(2\log A + 0.714)}{\pi e^2 \lambda}\right) L_1^{2/3} L_2^{1/3}.$$

It follows that

$$-\frac{1}{\pi r}\int_0^{2\pi}\cos\theta\log|\zeta(s)|\,d\theta \le CL_1^{2/3}L_2^{1/3}, \tag{20}$$

where

$$\begin{aligned}C &= C(A,B,\lambda)\\ &= \frac{4-2\log\lambda}{3\pi\lambda}+\frac{1.438B\lambda^{1/2}}{\pi}d+\frac{4}{3\pi\lambda}+\frac{3^{2/3}(2\log A+0.714)}{\pi e^2\lambda}.\end{aligned}$$

Applying Corollaries 1 and 2 of Lemma 2, we get

$$-\Re\left(\frac{\zeta'(s_0)}{\zeta(s_0)}\right) \le CL_1^{2/3}L_2^{1/3}; \tag{21}$$

and if $\rho_0 = \mu_0 + it_0$ is a zero of $\zeta(s)$, then

$$-\Re\left(\frac{\zeta'(s_0)}{\zeta(s_0)}\right) \le (C+\lambda^{-1})L_1^{2/3}L_2^{1/3} - \frac{1}{\sigma_0-\mu_0}. \tag{22}$$

4. The final step. We should have recourse to a trigonometric inequality. The inequality serves for the same purpose as $3+4\cos\theta+\cos(2\theta)>0$, see [**7**]. We may choose different ones from those in [**3**] and [**10**] to suboptimize the results. Let us use the following arbitrary form. Let a, b, c and d be positive numbers. Let $\alpha,\beta,\gamma,\delta,\eta$ be determined by

$$\alpha = \frac{5}{8}b^2d^2+\frac{1}{2}a^2d^2+2abcd+\frac{1}{2}b^2c^2+a^2c^2, \tag{23a}$$

$$\beta = \frac{3}{2}abd^2+\frac{3}{2}b^2cd+2a^2cd+2abc^2, \tag{23b}$$

$$\gamma = \frac{1}{2}a^2+2abcd+\frac{1}{2}b^2c^2+\frac{1}{2}b^2d^2, \tag{23c}$$

$$\delta = \frac{1}{2}abd^2+\frac{1}{2}b^2cd, \tag{23d}$$

$$\eta = \frac{1}{8}b^2d^2. \tag{23e}$$

Then

$$(24)\quad \alpha + \beta\cos(\theta) + \gamma\cos(2\theta) + \delta\cos(3\theta) + \eta\cos(4\theta) \\ = (a + b\cos\theta)^2(c + d\cos\theta)^2 \geq 0.$$

Recalling (13), we have

$$-\Re\left(\frac{\zeta'(s)}{\zeta(s)}\right) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma}} \cos(t).$$

It follows from (24) that

$$(25)\quad -\alpha\Re\left(\frac{\zeta'(\sigma)}{\zeta(\sigma)}\right) - \beta\Re\left(\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\right) - \gamma\Re\left(\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}\right) \\ - \delta\Re\left(\frac{\zeta'(\sigma+3it)}{\zeta(\sigma+3it)}\right) - \eta\Re\left(\frac{\zeta'(\sigma+4it)}{\zeta(\sigma+4it)}\right) \geq 0.$$

Now, suppose that $\rho = \mu + i\nu$ is a zero of $\zeta(s)$ with $\mu > (1/2)$ and $\nu \geq e^{e^3}$. We shall take $t_0 = k\nu$ where $k = 1, 2, 3$ or 4. For each $k = 1, 2, 3$ or 4, using the definition of σ_0 in Section 3, we let $s_0 = \sigma_0 + it_0$ be the center for the circle on which we are applying corollaries of Lemma 2. Recall that $\zeta(s_0) \neq 0$. Also, by choosing the value of ε in the definition of r while λ is fixed, we may assume that there is no zero of $\zeta(s)$ on the circle $|s - s_0| = r$, since there exist at most finitely many zeros in any finite region for any regular function. Recalling (10) and $\sigma_0 > 1$, we know that all zeros must have angles in the range $(\pi/2) < \theta < (3\pi/2)$ with respect to the center s_0. Using (7) to $-\Re(\zeta'(\sigma_0)/\zeta(\sigma_0))$, (21) to $-\Re(\zeta'(\sigma_0 + kit_0)/\zeta(\sigma_0 + kit_0))$, for $k = 2$, 3 and 4, (22) to $-\Re(\zeta'(\sigma_0 + it_0)/\zeta(\sigma_0 + it_0))$ and (25), we obtain

$$\frac{1.001\alpha}{\sigma_0 - 1} - \frac{\beta}{\sigma_0 - \mu} + DL_1^{2/3}L_2^{1/3} \geq 0,$$

or

$$\frac{\beta}{\sigma_0 - \mu} \leq DL_1^{2/3}L_2^{1/3} + \frac{1.001\alpha}{\sigma_0 - 1},$$

where

$$D = (\beta + \gamma + \delta + \eta)C + \beta\lambda^{-1},$$

with C being defined in (20). From this, we get

$$1-\mu \geq \frac{\beta}{DL_1^{2/3}L_2^{1/3}+1.001\alpha(\sigma_0-1)^{-1}}-(\sigma_0-1).$$

Substituting σ_0 by its definition, we acquire

$$1-\mu \geq \vartheta L_1^{-2/3}L_2^{-1/3} \tag{26}$$

where

$$\vartheta = [D+1.001\alpha\kappa^{-1}]^{-1}\beta-\kappa.$$

We want to optimize ϑ as a function of κ with fixed a, β and D. We may instead consider the function $f(x) = (\beta/(D+\alpha x)) - (1/x)$ with $x = \kappa^{-1}$. The critical point is $x = D(\sqrt{\alpha}(\sqrt{\beta}-\sqrt{\alpha}))^{-1}$. This leads to

$$\vartheta = \frac{(\sqrt{\beta}-\sqrt{\alpha})(\beta-0.001\alpha-\sqrt{\alpha\beta})}{D(\sqrt{\beta}+0.001\sqrt{\alpha})}. \tag{27}$$

Conclude that we have proved that the Riemann zeta-function does not have zeros in the region

$$\sigma > 1-\vartheta\log^{-2/3}\left(4t+\frac{1}{3}\right)\left(\log\log\left(4t+\frac{1}{3}\right)\right)^{-1/3}, \quad t \geq e^{e^3},$$

with

$$\vartheta = \frac{(\sqrt{\beta}-\sqrt{\alpha})(\beta-0.001\alpha-\sqrt{\alpha\beta})}{(\sqrt{\beta}+0.001\sqrt{\alpha})(\beta+\gamma+\delta+\eta)C\beta\lambda^{-1}}$$

where

$$C = \frac{4-2\log\lambda}{3\pi\lambda}+\frac{1.438B\lambda^{1/2}}{\pi}+\frac{4}{3\pi\lambda}+\frac{3^{2/3}(2\log A+0.714)}{\pi e^2\lambda},$$

with any $0 < \lambda < 1$, A and B from the upper bound in the form of (4), a, b, c and d are any positive numbers, and $\alpha, \beta, \gamma, \delta$ and η are subject to (23a) through (23e).

In the case that $A = 175$ and $B = 46$, we choose $\lambda = 0.43$. It gives $C = 18.5$. We use $a = 5$, $b = 3$, $c = 1$ and $d = 1$ in (24), getting $\vartheta = (1/951)$. In [**22**], it is proved that $\zeta(s) \neq 0$ for $\sigma > 1/2$ and $0 \leq t \leq 545, 439, 823.215$. The last number is greater than e^{e^3}. We let

$t > 3/5$. It follows that $t > (e - 1/3)/4$ or $4t + 1/3 > 0$. Conclude that we have proved that $\zeta(s)$ does not have zeros in the region

$$\sigma > 1 - \frac{1}{951 \log^{2/3}(4t+1/3)(\log\log(4t+1/3))^{1/3}} \quad \text{and } t > 3/5. \tag{28}$$

Recalling Theorem 26 in [**20**] that the Riemann zeta-function does not have zeros in the region

$$\sigma > 1 - 0.0571 \log^{-1} t \quad \text{and } t \geq e^{9.99},$$

we may only use (28) for $t \geq e^{e^{14.7988}}$. Note that

$$\frac{\log^{2/3}(4t+1/3)(\log\log(4t+1/3))^{1/3}}{\log^{2/3} t(\log\log t)^{1/3}} < \frac{952}{951}$$

for $t \geq e^{e^{14.7988}}$. We obtain

$$\sigma > 1 - \frac{1}{952 \log^{2/3} t(\log\log t)^{1/3}} \quad \text{and } t \geq 3. \tag{29}$$

To finish the proof of the main result, we resort to the "symmetric" property of the Riemann zeta-function. That is,

$$\zeta(\bar{s}) = \overline{\zeta(s)}.$$

For $\sigma > 1$, this can be realized easily by using the definition of (1) and is in fact valid everywhere. The latter can be proved by analytical continuation and the principle of reflection, see [**1**]. This tells us that the Riemann zeta-function does not have zero in the region similar to (29) but with t being replaced by $|t|$. We finish the proof of the main result.

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