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NORM CONVERGENCE OF MOVING AVERAGES FOR τ-INTEGRABLE OPERATORS

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ABSTRACT. It is shown that if α is a positive linear map on $L^1(M,\tau)$ of a von Neumann algebra M with a faithful normal (semi-)finite trace τ which is norm-reducing for both the operator norm and the integral norm associated with τ, then the moving averages converge in L^p-norm, $1 \le p < \infty$. Using this result it has been shown that similar norm convergence results hold for some super-additive processes in $L^p(M,\tau)$ relative to τ-preserving α.

1. Introduction. This article concerns some strong convergence results for moving averages in the von Neumann algebra setting. Beginning with the celebrated theorem of Lance and Yeadon, there has been great interest in extending various results in classical ergodic theory into operator algebras, particularly to von Neumann algebras. For a review, see [**6**], [**8**]. Recently, such activities have been revived in the context of obtaining various weighted ergodic theorems in von Neumann algebras [**7**], [**9**], [**10**]. Study of convergence of moving averages in von Neumann algebra settings is new. Actually we will obtain norm convergence of moving averages for both additive and superadditive processes in a von Neumann algebra.

Let M be a von Neumann algebra with the unit I, and let τ be a faithful normal semi-finite trace on M. For the definition of L^p-spaces, $1 \le p \le \infty$, associated with (M,τ), see [**14**], [**12**], [**15**], [**4**]. $L^p = L^p(M,\tau)$, being noncommutative generalizations of the classical L^p-spaces, inherit most of their important properties. For example, the following form of the Hölder inequality holds [**4**]

$$\|xy\|_r \le \|x\|_p\|y\|_q$$

whenever $p,q,r > 0$ and $p^{-1}+q^{-1} = r^{-1}$. When τ is finite, this implies that if $p > q$ then $L^p \subset L^q$.

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Let $\alpha : L^1 \to L^1$ be a positive linear map with $\alpha(x) \leq I$ and $\tau(\alpha(x)) \leq \tau(x)$ for every $x \in J = L^1 \cap M$ with $0 \leq x \leq I$. Note that if $x = x^* \in J$ then $\|\alpha(x)\|_p \leq \|x\|_p$ for all $1 \leq p \leq \infty$, and α uniquely extends to a linear continuous map in L^p, $1 \leq p < \infty$, which is also denoted by α (see [**16**], [**17**]).

Throughout this article, any sequence of the form $\mathbf{w} = \{(k_n, m_n)\}$ with $k_n \geq 0$, $m_n > 0$ integers and $m_n \to \infty$, will be called a *moving average sequence* (MAS). The sums

$$a_n^{\mathbf{w}}(x) = \frac{1}{m_n} \sum_{i=0}^{m_n-1} \alpha^{k_n+i}(x), \quad x \in L^1,$$

will denote the *moving averages of* x *associated with the* MAS $\mathbf{w}$.

Remarks. The condition $m_n \to \infty$ is a trivial consequence of a more general condition under which $\mathbf{w}$ is called a B-sequence (see, for example, [**1**], [**5**]). Note also that, with $m_n = n$, $k_n = 0$ for every n, $\{a_n(x)\}$ becomes the usual ergodic averages sequence of x.

More generally, if $\{\mu_n\}$ is a sequence of probability measures on Z_+,then for $x \in L^1(M, \tau)$ we define the *weighted averages* along $\{\mu_n\}$ as

$$a_{\mu_n}(x) = \sum_{k \in Z_+} \mu_n(k) \alpha^k(x).$$

(If $\mu_n = (1/m_n)\chi_{[k_n, k_n+m_n)}$, then $a_{\mu_n} = a_n^{\mathbf{w}}$.) When it is clear from the context, we will also denote these averages by $a_n(x)$. Again the inequality $\|\alpha(x)\|_p \leq \|x\|_p$, $x = x^* \in L^p$, implies that $a_n : L^p \to L^p$ and $\|a_n\|_p \leq 2$ for every n, $1 \leq p \leq \infty$.

Let L_s^1 be the self-adjoint part of $L^1(M, \tau)$. A sequence $F = \{x_i\}_{i \geq 0} \subset L_s^1$ is called an α-*superadditive process* if the sequence of partial sums $\{F_n\}_{n \geq 1}$, where $F_n = \sum_{i=0}^{n-1} x_i$ satisfy

$$F_{m+n} \geq F_m + \alpha^m F_n \quad \text{for } m, n > 1.$$

When equality holds, then F is called an α-*additive process*. Clearly α-additive processes are necessarily of the form $\{\sum_{i=0}^{n-1} \alpha^i x_0\}$. If $\sup_{n \geq 1} \|F_n\|_1 / n < \infty$, the process is called *bounded*. It is well known that, if τ is finite, then any bounded α-superadditive process has a dominant, that is, there exists a $\delta \in L^1$ satisfying $\tau(\delta) = \gamma_F := \sup_{n\geq 1} \|F_n\|_1/n < \infty$ such that $F_n \leq \sum_{i=0}^{n-1} \alpha^i \delta$ [**8**, Lemma 2.3.10]. Following the popular terminology, we will refer to this δ as the *exact dominant* in what follows.

Remark. If F is an α-superadditive process, then $G = \{G_n\} = \{\sum_{i=0}^{n-1} \alpha^i x_0\}$ is an α-additive process, and hence $\{F_n - G_n\}$ is a *positive* α-superadditive process.

If F is an α-superadditive process and $\mathbf{w} = \{(k_n, m_n)\}$ is an MAS, we define the averages of F along $\mathbf{w}$ by

$$a_n^{\mathbf{w}} F = \frac{1}{m_n}\, \alpha^{k_n} F_{m_n}.$$

(Hence, if F is α-additive, then the corresponding moving averages along $\mathbf{w}$ will be $a_n^{\mathbf{w}}(x_0) = (1/m_n)\alpha^{k_n} \sum_{i=0}^{m_n-1} \alpha^i x_0$, coinciding with the previous definition). It should be noted here that, in the superadditive setting, it is possible to give alternative definitions of moving averages; however, such averages may fail to converge almost everywhere and in the mean, even in the commutative case as shown in [**3**].

2. Norm convergence of additive processes. The first result of this section is a simple consequence of properties of the reflexive Banach spaces and behavior of MAS's.

Theorem 2.1. *Let M be a* vNA *with unit I, τ a faithful normal semi-finite trace on M and $\alpha : L^1 \to L^1$ a positive linear map with $\alpha(x) \leq I$ and $\tau(\alpha(x)) \leq \tau(x)$ for every $x \in L^1 \cap M$. If $\mathbf{w} = \{(k_n, m_n)\}$ is an* MAS*, then for every $x \in L^p$, $1 < p < \infty$, the sequence $a_n^{\mathbf{w}}(x)$ converges in L^p to some α-invariant $\hat{x} \in L^p$.*

Proof. Let B be the self-adjoint part of $L^p(M, \tau)$. Then B is a reflexive Banach space, while α is a contraction in it. Repeating the steps of the proof of Theorem 9.1 in [**11**] and taking into account the condition $m_n \to \infty$, we see that for every $x \in B$ the moving averages $a_n^{\mathbf{w}}(x)$ converge in B to some α-invariant $\hat{x} \in B$ which ends the proof due to the fact that every element of L^p is a sum of two self-adjoint ones. $\square$

If τ is finite, it is straightforward, via Hölder's inequality, to extend Theorem 2.1 to the case $p = 1$. In this case, however, one can obtain more. Namely, one can develop some tools, along the line of results obtained in [**13**], that enables one to work on the norm convergence of averages along MAS's as well as more general weighted averages in the vNA setting.

In the rest of this section, we will assume that τ is finite and $\tau(I) = 1$. Note that $H_\tau = (L^2, \|\cdot\|_2)$ is a Hilbert space, more precisely, the completion of the pre-Hilbert space M (equipped with the inner product $(x, y) = \tau(y^* x)$, $x, y \in M$). By the Gelfand-Naimark-Segal representation theorem, there exists an embedding $\pi : M \to B(H_\tau)$ such that, if we denote $\bar{x}$ the vector in H_τ generated by $x \in M$, then $\pi(x)\bar{y} = \overline{xy}$, $x, y \in M$.

Besides, $\delta_0 = \bar{I}$ is a bicyclic vector for M in H_τ such that, for every $x \in M$,

$$\tau(x) = \tau(Ix) = (\bar{x}, \xi_0) = (\pi(x)\xi_0, \xi_0),$$

or, abusing notations, $\tau(x) = (x\xi_0, \xi_0)$. We use the same notation $\|\cdot\|_2$ for the norm in $L^2(M, \tau)$ and for the norm in H_τ.

Lemma 2.2. *If $x_n \in M$, $\|x_n\|_\infty \le C < \infty$ and $x_n \to \hat{x}$ in L^2, then $\hat{x} \in M$ and $x_n \to \hat{x}$ in L^p for every $1 \le p < \infty$.*

Proof. Since $x \in M$ implies that $\|x\delta_0\|_2 = \|\bar{x}\|_2 = \tau(x^* x)^{1/2} = \|x\|_2$, for every y' from the commutant M' of M, we have

$$\begin{aligned}\|(x_n - \hat{x})y'\xi_0\|_2 &= \|y'(x_n - \hat{x})\xi_0\|_2 \\ &\le \|y'\|_\infty \|(x_n - \hat{x})\xi_0\|_2 \\ &= \|y'\|_\infty \|x_n - \hat{x}\|_2.\end{aligned}$$

Therefore $\|(x_n - \hat{x})\xi\|_2 \to 0$ for every $\xi \in M'\xi_0$. Because the set $M'\xi_0$ is dense in H_τ and $\{\|x_n\|_\infty\}$ is bounded, we see that the above convergence takes place for all $\xi \in H_\tau$, i.e., $x_n \to \hat{x}$ strongly in M, hence $\hat{x} \in M$.

Now if $1 \le p < 2$, then $x_n \to \hat{x}$ in L^p follows from the Hölder inequality. Let $2 < p < \infty$. Since $x_n \to \hat{x}$ strongly with $\|x_n\| \le C$, we conclude that $\|\hat{x}\|_\infty \le C$ and $\|x_n - \hat{x}\|_\infty \le 2C$. Therefore, we have $|x_n - \hat{x}|^p \le (2C)^{p-2}|x_n - \hat{x}|^2$, and hence

$$\|x_n - \hat{x}\|_p^p \le (2C)^{p-2}\|x_n - \hat{x}\|_2^2 \longrightarrow 0. \qquad \square$$

The following simple fact is an immediate consequence of the Uniform Boundedness principle.

Lemma 2.3. *Let X be a Banach space, and let $a_n : X \to X$, $n \ge 1$, be a sequence of continuous linear maps satisfying the condition $\sup_n\{\|a_n(x)\|\} < \infty$ for all $x \in X$. Then the set $X_0 = \{x \in X : a_n(x)$ converges$\}$ is closed.*

Lemma 2.4 (cf. [**13**]). *The sequence $\{a_n(x)\}$ converges in L^p for all $x \in L^p$, $1 \le p < \infty$, if and only if it is convergent in L^2 for all $x \in L^2$.*

Proof. Fix $1 \le p < \infty$. Since $M = L^\infty$ is dense in L^p and $\|a_n\|_p \le 2$, by Lemma 2.3 it is enough to show that, for $x \in M$, $a_n(x) \to \hat{x}$ in L^2 implies that $a_n(x) \to \hat{x}$ in L^p. But this immediately follows from Lemma 2.2 since, for $x_n = a_n(x)$, we have $\|x_n\|_\infty \le 2\|x\|_\infty$. $\square$

As a corollary of this lemma and Theorem 2.1, we derive

Theorem 2.5. *Let M be a von Neumann algebra with the identity I, and let τ be a faithful normal finite trace on M. Assume that $\alpha : L^1 \to L^1$ is a positive linear map satisfying $\alpha(x) \le I$ and $\tau(\alpha(x)) \le \tau(x)$ if $x \in M$ and $0 \le x \le I$. If $\mathbf{w} = \{(k_n, m_n)\}$ is an* MAS *and $x \in L^p$, $1 \le p < \infty$, then the corresponding moving averages $a_n^{\mathbf{w}}(x)$ converge in L^p to some α-invariant $\hat{x} \in L^p$.*

Remarks. 1. Due to the similarities involved,the results obtained in this section are also valid for *block sequences* (see [**3**]).

2. For $p = 1$, it is essential that τ is finite.

Now, using a method similar to the one utilized in [**13**], we establish the following.

Theorem 2.6. *If α is an automorphism such that $\tau(\alpha(x)) = \tau(x)$ for every $x \geq 0$, then the following conditions are equivalent*

(i) $a_{\mu_n}(x) = \sum_{k\in Z} \mu_n(k)\alpha^k(x)$ *converges in* L^p *for all* $x \in L^p$, $1 \leq p < \infty$;

(ii) *the Fourier transforms* $\hat{\mu}_n(\gamma) = \sum_{k\in Z} \mu_n(k)\bar{\gamma}^k$ *converge for all* $\gamma \in T = \{|z| = 1\}$.

Proof. Assume first that $\hat{\mu}_n$ converges pointwise on T. Let $x \in L^2$ and consider the function $t_x(k) = \tau(x^*\alpha^k(x))$. Then t_x is a positive definite function on Z; so, by the Herglotz theorem, there is a positive regular Borel measure ν_x on T for which $\hat{\nu}_x(k) = t_x(k)$, $k \in Z$, holds. But then

$$\begin{aligned}
\|a_n(x)\|_2^2 &= \tau(a_n(x)^* a_n(x)) \\
&= \sum_{k\in Z}\sum_{l\in Z} \mu_n(k)\overline{\mu_n(l)}\tau(\alpha^l(x^*)\alpha^k(x)) \\
&= \sum_{k\in Z}\sum_{l\in Z} \mu_n(k)\overline{\mu_n(l)}\tau(x^*\alpha^{k-l}(x)) \\
&= \sum_{k\in Z}\sum_{l\in Z} \mu_n(k)\overline{\mu_n(l)}\hat{\nu}_x(k-l) \\
&= \sum_{k\in Z}\sum_{l\in Z} \mu_n(k)\overline{\mu_n(l)} \int_T \bar{\gamma}^{k-l}\, d\nu_x(\gamma) \\
&= \int_T |\hat{\mu}_n(\gamma)|^2\, d\nu_x(\gamma).
\end{aligned}$$

Hence, for $m, n \geq 1$, $\|a_m(x) - a_n(x)\|_2^2 = \int_T |\hat{\mu}_m(\gamma) - \hat{\mu}_n(\gamma)|^2\, d\nu_x(\gamma)$. But $\{\hat{\mu}_n\}$ is a uniformly bounded sequence of continuous functions on T which converges pointwise on T. Therefore, by the bounded convergence theorem, $\{\hat{\mu}_n\}$ converges in $L^2(T, \nu)$. Thus, $\{a_n(x)\}$ is L^2-norm Cauchy and must converge in L^p by Lemma 2.2.

Conversely, let $\gamma \in T$, and let $M = L^\infty(T, \lambda)$ with λ the usual normalized Lebesgue measure on T. If $\tau(f) = \int_T f\, d\lambda$, $0 \leq f \in M$, and $\alpha : L^1(T) \to L^1(T)$ is given by $\alpha(f)(z) = f(\bar{\gamma}z)$, $z \in T$, then for the function $f(z) = z$ we have

$$a_n(f)(z) = \sum_{k \in Z} \mu_n(k) \alpha^k(f)(z) = \sum_{k \in Z} \mu_n(k) \bar{\gamma}^k z = \hat{\mu}_n f(z).$$

Then the convergence of $a_n(f)$ in the L^2-norm implies the converges of $\hat{\mu}_n(\gamma)$. $\square$

Remark. This result suggests that further developments similar to those of [**13**] should be possible to carry out in the noncommutative setting.

3. Convergence of moving averages of superadditive processes. In this section we will extend the main result of the previous section to α-superadditive processes, where α is τ-preserving and τ is finite. First we will obtain some lemmas which are instrumental in obtaining the norm convergence when $p = 1$.

Lemma 3.1. *Let $0 \leq x_1 \leq x_2 \leq \cdots$ be a sequence in $L^1(M, \tau)$ with, for some $x \in L^1$, $x_n \leq x$, $n \geq 1$. Then there exists $\hat{x} \in L^1_+$ such that $\lim_n \|\hat{x} - x_n\|_1 = 0$. Furthermore, $\tau(\hat{x}) = \lim_n \tau(x_n)$.*

Proof. Clearly, $0 \leq \tau(x_1) \leq \tau(x_2) \leq \cdots \tau(x)$. Hence, the sequence $\{\tau(x_n)\}$ is Cauchy. Since

$$\|x_n - x_m\|_1 = \tau(x_n - x_m) = \tau(x_n) - \tau(x_m) \longrightarrow 0$$

(assuming $x_m \leq x_n$), the lemma follows. ($\hat{x} \geq 0$ by [**4**, Theorem 3.2].) $\square$

Remark. The same argument also proves that the assertion of Lemma 3.1 holds if $0 \leq x_1 \leq x_2 \leq \cdots$ is a sequence in $L^1(M, \tau)$ with $\sup_n \|x_n\|_1 < \infty$.

The following is an adaptation of a lemma of Akcoglu and Sucheston (see [**2**]) to the von Neumann algebra setting. Since it is proved similarly, via the techniques in [**8**, Section 2.3], we omit the proof.

Lemma 3.2. *Let $F = \{x_i\} \subset L^1$ be a positive α-superadditive process. If $h_j = (1/j)F_j$, $j \geq 1$, then* (*with the convention that sums over void sets are zero*) $F_n \geq \sum_{i=0}^{n-j-1} \alpha^i h_j$.

For each $j \geq 1$, define the α-additive process H^j by $H_n^j = \sum_{i=0}^{n-1} \alpha^i h_j$.

Lemma 3.3. *Let $F \subset L_s^1$ be a positive bounded α-superadditive process, and* $\mathbf{w}$ *an* MAS. *Then, for each $j \geq 1$, the moving averages $a_n^{\mathbf{w}}(h_j)$ converges in norm to a $g_j \in L^1$. Furthermore, given $\varepsilon > 0$ one can pick j such that $\tau(g_j) > \gamma_F - \varepsilon$.*

Proof. From Theorem 2.5 norm convergence follows immediately. For the second assertion, given $\varepsilon > 0$, find j such that $\|h_j\|_1 = \|(1/j)F_j\|_1 > \gamma_F - \varepsilon/2$. Also, find $N \geq 1$ such that $\|g_j - H_n^j\|_1 < \varepsilon/2$, $n \geq N$. Since α is τ-preserving, we have $\tau(H_n^j) = \tau(h_j)$. Then

$$\begin{aligned} |\tau(g_j) - \gamma_F| &\leq |\tau(g_j) - \tau(H_n^j)| + |\tau(H_n^j) - \gamma_F| \\ &= |\|g_j\|_1 - \|H_n^j\|_1| + |\|H_n^j\|_1 - \gamma_F| \\ &\leq \|g_j - H_n^j\|_1 + |\tau(h_j) - \gamma_F| \\ &\leq \varepsilon/2 + |\|h_j\|_1 - \gamma_F| < \varepsilon \end{aligned}$$

since, by definition, $\tau(|x|) = \|x\|_1$, $x \in L^1$. □

Theorem 3.4. *Let $F \subset L_s^1$ be a bounded α-superadditive process, and* $\mathbf{w}$ *an* MAS. *Then the moving averages $a_n^{\mathbf{w}} F$ converge in norm to some α-invariant $\hat{x} \in L^1$.*

Proof. By Theorem 2.5, we can assume that F is positive. Since F is bounded, an exact dominant δ exists. Given $\varepsilon > 0$, pick j such that $\|h_j\|_1 > \gamma_F - \varepsilon/2$. Then, from Lemma 3.2 and superadditivity, for all $n > j \geq 1$, $H_{n-j}^j \leq F_n \leq G_n$ where $G_n = \sum_{i=0}^{n-1} \alpha^i \delta$. Hence, for n large enough,

$$0 \leq \frac{1}{m_n} (F_{m_n} - H_{m_n - j}^j) \leq \frac{1}{m_n} (G_{m_n} - H_{m_n - j}^j).$$

Both the processes $\{G_n\}$ and $\{H_n^j\}$ are positive α-additive processes. Thus, by Theorem 2.5, the averages $(1/m_n)\alpha^{k_n} G_{m_n}$ and

$(1/m_n)\alpha^{k_n}H^j_{m_n}$ converge in the L^1-norm to some α-invariant δ^* and g_j, respectively. Also, observe that

$$L^1 - \lim_n \frac{1}{m_n}\,\alpha^{k_n}H^j_{m_n} = L^1 - \lim_n \frac{1}{m_n}\,\alpha^{k_n}H^j_{m_n-j}.$$

Hence, $\lim_n(1/m_n)\alpha^{k_n}(G_{m_n} - H^j_{m_n-j}) = \delta^* - g_j$ exists in the L^1-norm, and consequently,

$$0 \le L^1 - \lim \frac{1}{m_n}\,\alpha^{k_n}(F_{m_n} - H^j_{m_n-j}) \le \delta^* - g_j.$$

Next, by superadditivity, for all $j \ge 1$,

$$\begin{aligned} g_{2j} &= L^1 - \lim_n \frac{1}{m_n}\,\alpha^{k_n}H^{2j}_{m_n} \\ &\ge L^1 - \lim_n \frac{1}{2m_n}\,\alpha^{k_n}(H^j_{m_n} + \alpha^j H^j_{m_n}) \\ &= \frac{1}{2}\left[L^1 - \lim_n \frac{1}{m_n}\,(\alpha^{k_n}H^j_{m_n} + \alpha^{k_n+j}H^j_{m_n})\right] \\ &= g_j. \end{aligned}$$

This implies that $\{x_i := g_{2^i j}\}_i$ is an increasing sequence in L^1 which is bounded by $\delta^* \in L^1$. Hence, by Lemma 3.1, there exists a $\hat{x} \in L^1$ such that $x_i \to \hat{x}$ in L^1-norm. So there exists an i such that $\|x_i - \hat{x}\|_1 < (\varepsilon/2)$. Since $\|(1/m_n)\alpha^{k_n}H^j_{m_n}\|_1 \le \|(1/j)F_j\|_1$, it follows that $\|x_i\|_1 \le \gamma_F$ for all i. Also, one can pick i large enough so that $\tau(x_i) > \gamma_F - (e/2)$ by Lemma 3.2. Then

$$\begin{aligned} \|\delta^* - \hat{x}\|_1 &\le \|\delta^* - x_i\|_1 + \|x_i - \hat{x}\|_1 \\ &< [\tau(\delta^*) - \tau(x_i)] + \frac{\varepsilon}{2} < \left(\gamma_F - \gamma_F + \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} \\ &= \varepsilon, \end{aligned}$$

since $\tau(\delta^*) \le \gamma_F$. Arbitrariness of ε implies that $L^1 - \lim_n(1/m_n)\alpha^{k_n}F_{m_n}$ exists (and is equal to $\hat{x}$). The invariance of the limit follows from that of x_i's and δ^*. $\square$

As observed in [**2**], the condition $\sup_n \|(1/n)F_n\|_p < \infty$ does not yield to the norm convergence of (ordinary) averages of superadditive processes. Hence, when $1 < p < \infty$ for the L_p-norm convergence of superadditive processes one needs more than the *boundedness* condition on the process. In [**2**] it has been shown that for a more restrictive class of superadditive processes, namely, *Chacon admissible processes*, one can obtain affirmative results for the convergence in the L_p-norm. That is why, for the rest of this section, we will work with such processes.

Definition. A sequence $\{x_n\} \subset L_s^p(M,\tau)$ is called an α-*admissible sequence* if $\alpha x_i \le x_{i+1}$ for all $i \ge 1$. If $\{x_n\}$ is α-admissible, then the process $F = \{F_n\}$ where $F_n = \sum_{i=0}^{n-1} x_i$ is an α-superadditive process, called an α-*admissible process*. Such a process is called *strongly bounded* if $\sup_n \|x_n\|_p < \infty$.

Theorem 3.5. *Let α be a τ-preserving $*$-automorphism and $\{x_n\} \subset L_s^p(M,\tau)$, $1 < p < \infty$, an α-admissible sequence with $\sup_n \|x_n\|_p < \infty$ where F is the associated strongly bounded α-admissible process. If $\mathbf{w} = \{(k_n, m_n)\}$ is an* MAS*, then the moving averages $a_n^{\mathbf{w}}(\alpha)F$ converge in the L^p-norm, and the limit is α-invariant.*

Proof. Since $\{x_i\} \subset L^p$ is an admissible sequence, $\sum_{j=0}^{n-1} \alpha^j x_0 \le F_n$, and hence we can assume that $x_i \ge 0$, $i \ge 0$. For convenience, define $P_i = x_i - \alpha x_{i-1}$, $i \ge 1$, where we set $P_0 = x_0$. Observe that, by the Clarkson type inequality in $L^p(M,\tau)$ [**4**, Theorem 5.3] and strong boundedness,

$$\|P_r\|_p^p = \tau(x_r - \alpha x_{r-1})^p \le C_p[\tau(x_r^p) - \tau(x_{r-1}^p)] < \infty$$
$$C_p = 2^{q-1}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Now we will use a technique employed in [**3**]: for a fixed positive integer k, define

$$y_n^k = \begin{cases} \alpha^{n-k} x_k & \text{for } n > k \\ x_n & \text{for } 0 \le n \le k. \end{cases}$$

Then it follows that $x_n - y_n^k = \sum_{i=1}^m \alpha^{m-i} P_{k+i}$ for $n > k$ where $m = n - k$, and $x_n - y_n^k = 0$ for $0 \le n \le k$. Defining $D_i = \sum_{n=0}^{m_i - 1} x_n - y_n^k$, we estimate that

$$D_i \le \sum_{n=0}^{m_i-1} \sum_{r=k+1}^{n} \alpha^{n-r} P_r.$$

Next if we let $b_{k,t} = \sum_{r=k+1}^{t} \alpha^r P_r$ and $b_k = \lim_{t\to\infty} b_{k,t}$, then $b_{k,t} \ge 0$ and $b_k \ge 0$. Using the Fack-Kosaki monotone convergence theorem [**4**, Theorem 3.5], we obtain that

$$\tau(b_k^p) = \lim_{t\to\infty} \tau(b_{k,t}^p) \le \sum_{r=k+1}^{\infty} \tau(P_r^p) \le C_p \sup_r \|x_r\|_p^p < \infty.$$

Having $b_{k,t} \uparrow b_k$ and $b_k \in L^P$, we conclude that $\alpha^j b_{k,t} \uparrow \alpha^j b_k$ in L^p for all j, since α is strongly continuous. This in turn implies that

$$\alpha^{k_i} D_i \le \sum_{n=0}^{m_i-1} \alpha^{k_i+n} b_k.$$

On the other hand, by Jensen type inequality in $L^p(M,\tau)$ [**4**, Proposition 4.6] and by the τ-preserving property of α, as $k \to \infty$,

$$\left\| \frac{1}{m_i} \sum_{n=0}^{m_i-1} \alpha^{k_i+n} b_k \right\|_p^p \le \|b_k\|_p^p \le \sum_{i=k+1}^{\infty} \tau(P_i^p) \downarrow 0.$$

By Theorem 2.3 we have $Y_k := L^p - \lim_{i\to\infty} (1/m_i)\alpha^{k_i} \sum_{n=0}^{m_i-1} y_n^k$ exists and is α-invariant. Since, for all $n \ge 1$, $y_n^k \le y_n^{k+1}$, we also have $Y^k \le Y^{k+1}$. Therefore, $\{Y^k\}$ is a monotone increasing sequence in L^p with $\sup_k \tau(Y_k^p) < \infty$ and, consequently, by Lemma 3.1, $Y = \lim_{k\to\infty} Y^k$ exists in L^p and is α-invariant. Now, given $\varepsilon > 0$, find a positive integer K such that for $k \ge K$, $\|b_k\|_p^p < \varepsilon/3$, $\|(1/m_i)\alpha^{k_i} \sum_{n=0}^{m_i-1} y_n^k - Y^k\|_p^p < \varepsilon/3$, and $\|Y - Y^k\|_p^p < \varepsilon/3$. Then

$$\begin{aligned} \left\| \frac{1}{m_i} \alpha^{k_i} \sum_{n=0}^{m_i-1} x_n - Y \right\|_p^p &\le \left\| \frac{1}{m_i} \alpha^{k_i} \sum_{n=0}^{m_i-1} (x_n - y_n^k) \right\|_p^p \\ &\quad + \left\| \frac{1}{m_i} \alpha^{k_i} \sum_{n=0}^{m_i-1} y_n^k - Y^k \right\|_p^p + \|Y - Y^k\|_p^p < \varepsilon, \end{aligned}$$

proving the assertion.

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