ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 30, Number 3, Fall 2000

EXISTENCE OF POSITIVE SOLUTIONS OF HIGHER ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS

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ABSTRACT. The neutral differential equation

$$\frac{d^n}{dt^n}[x(t)+h(t)x(t-\tau)]+\sigma f(t,x(g(t)))=0 \tag{1.1}$$

is considered under the following conditions: $n \geq 2$; $\sigma = \pm 1$; $\tau > 0$; $h \in C[t_0-\tau,\infty)$; $g \in C[t_0,\infty)$, $\lim_{t\to\infty} g(t)=\infty$; $f \in C([t_0,\infty)\times(0,\infty))$, $f(t,u)\geq 0$ for $(t,u)\in[t_0,\infty)\times(0,\infty)$, and $f(t,u)$ is nondecreasing in $u \in (0,\infty)$ for each fixed $t \in [t_0,\infty)$. It is shown that, for the case where $h(t) > -1$ and $h(t)=h(t-\tau)$ on $[t_0,\infty)$, equation (1.1) has a positive solution $x(t)$ satisfying

$$x(t)=\left[\frac{c}{1+h(t)}+o(1)\right]t^k \quad \text{as } t\to\infty$$

for some $c>0$ if and only if

$$\int^{\infty} t^{n-k-1} f(t,a[g(t)]^k)\,dt<\infty \quad \text{for some } a>0.$$

Here k is an integer with $0 \leq k \leq n-1$.

1. Introduction. In this paper we consider the higher order neutral differential equation

$$\frac{d^n}{dt^n}[x(t)+h(t)x(t-\tau)]+\sigma f(t,x(g(t)))=0, \tag{1.1}$$

where $n \geq 2$, $\sigma=\pm 1$ and $\tau>0$, and the following conditions (i)–(iii) are assumed:

(i) $h:[t_0-\tau,\infty)\to\mathbf{R}$ is continuous;

Received by the editors on February 3, 1999, and in revised form on June 1, 1999.

(ii) $g : [t_0, \infty) \to \mathbf{R}$ is continuous and $\lim_{t\to\infty} g(t) = \infty$;

(iii) $f : [t_0, \infty) \times (0, \infty) \to \mathbf{R}$ is continuous, $f(t, u) \ge 0$ for $(t, u) \in [t_0, \infty) \times (0, \infty)$, and $f(t, u)$ is nondecreasing in $u \in (0, \infty)$ for each fixed $t \in [t_0, \infty)$.

By a solution of (1.1), we mean a function $x(t)$ which is continuous and satisfies (1.1) on $[t_x, \infty)$ for some $t_x \ge t_0$.

There has been much current interest in studying the problem of the existence of positive solutions of higher order neutral differential equations. We refer the reader to [**1**], [**3**], [**6**], [**8**–**15**], [**17**], [**19**–**21**] for the existence of positive solutions and to [**2**–**6**], [**8**], [**12**], [**18**], [**19**] for the nonexistence of positive solutions. We note that neutral differential equations find numerous applications in natural science and technology. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines. See, for example, Hale [**7**].

Consider neutral differential equations of the form

$$\frac{d^n}{dt^n}[x(t) + \lambda x(t-\tau)] + \sigma f(t, x(g(t))) = 0, \tag{1.2}$$

where $-1 < \lambda < \infty$. It is known ([**12**] for the case $-1 < \lambda < 1$, [**11**] for the case $\lambda = 1$ and [**17**] for the case $1 < \lambda < \infty$) that (1.2) has a positive solution $x(t)$ satisfying

$$\lim_{t\to\infty} \frac{x(t)}{t^k}$$

exists and is a positive finite value, if and only if

$$\int^{\infty} t^{n-k-1} f(t, a[g(t)]^k)\, dt < \infty \quad \text{for some } a > 0. \tag{1.3}$$

In the above, k is a nonnegative integer with $0 \le k \le n-1$. The purpose of this paper is to extend this result to equation (1.1) for the case:

$$h(t) > -1 \quad \text{and} \quad h(t) = h(t-\tau), \quad t \ge t_0. \tag{1.4}$$

Of course, (1.4) means that $h(t)$ is a τ-periodic function satisfying $h(t) > -1$, $t \ge t_0$, and hence there are constants μ and λ such that $-1 < \mu \le h(t) \le \lambda < \infty$ for $t \ge t_0$.

We inductively define the functions $\omega_k(t)$, $k = 0, 1, 2, \dots$, by

$$\omega_k(t) = \begin{cases} \dfrac{1}{1+h(t)}, & k = 0, \\ \dfrac{t^k}{1+h(t)} - \dfrac{h(t)}{1+h(t)} \displaystyle\sum_{i=0}^{k-1} \binom{k}{i} (-\tau)^{k-i} \omega_i(t), & k = 1, 2, \dots . \end{cases}$$

Then we easily see that $\omega_k(t)$ satisfies

$$\omega_k(t) + h(t)\omega_k(t-\tau) = t^k$$

and

$$\omega_k(t) = \left[\frac{1}{1+h(t)} + o(1)\right] t^k \quad \text{as } t \to \infty \tag{1.5}$$

for all $k = 0, 1, 2, \dots$. If $k \in \{0, 1, 2, \dots, n-1\}$, a positive constant multiple of $\omega_k(t)$ is a positive solution of the unperturbed equation

$$\frac{d^n}{dt^n}[x(t) + h(t)x(t-\tau)] = 0,$$

and so it is natural to expect that, if f is small enough in some sense, (1.1) possesses a positive solution $x(t)$ which behaves like the function $c\,\omega_k(t)$ as $t \to \infty$, $c > 0$. Indeed, we have the following theorem.

Theorem. *Let $k \in \{0, 1, 2, \dots, n-1\}$. Suppose that* (1.4) *holds. Then* (1.1) *has a positive solution $x(t)$ such that*

$$x(t) = \left[\frac{c}{1+h(t)} + o(1)\right] t^k \quad \text{as } t \to \infty \tag{1.6}$$

for some $c > 0$ if and only if (1.3) *holds.*

2. Proof of the Theorem. In this section we give the proof of the Theorem. First we prove the "only if" part of the theorem.

Proof of the "only if" part of the Theorem. Let $x(t)$ be a positive solution of (1.1) satisfying (1.6). Put $y(t) = x(t) + h(t)x(t-\tau)$.

From (1.1) it follows that $\sigma y^{(n)}(t) \le 0$ for all large t. We conclude that $y^{(i)}(t)$, $i = 0,1,2,\dots,n-1$, are eventually monotonic, and hence $\lim_{t\to\infty} y^{(i)}(t)$, $i = 0,1,2,\dots,n-1$, exist in $\mathbf{R} \cup \{-\infty,\infty\}$. In view of (1.6), we easily see that $\lim_{t\to\infty} y(t)/t^k = c$, so that $\lim_{t\to\infty} y^{(k)}(t) = ck!$ and $\lim_{t\to\infty} y^{(i)}(t) = 0$ for $i = k+1,\dots,n-1$. Repeated integration of (1.1) yields

$$y^{(k)}(t) = ck! + (-1)^{n-k-1}\sigma \int_t^\infty \frac{(s-t)^{n-k-1}}{(n-k-1)!} f(s, x(g(s)))\, ds$$

for all large t. Consequently we obtain

$$\int^\infty s^{n-k-1} f(s, x(g(s)))\, ds < \infty.$$

By virtue of (1.6) and the monotonicity of f, we conclude that (1.3) holds.

Now let us show the "if" part of the theorem.

Let $k \in \{0,1,2,\dots,n-1\}$. Suppose that (1.3) and (1.4) hold. We can take a sufficiently large number $T \ge t_0 + k\tau$ and positive constants c_1 and c_2 such that

$$h(T) = \max\{h(t) : t \in [t_0,\infty)\},$$

$$T_* \equiv \min\{T-\tau, \inf\{g(t) : t \ge T - k\tau\}\} > \max\{t_0, 0\},$$

and

$$0 < c_1 \le \frac{\omega_k(t)}{t^k} \le c_2, \quad t \ge T_*. \tag{2.1}$$

Because of (1.5) it is possible to take $c_1, c_2 > 0$ satisfying (2.1). Let $C[T_*,\infty)$ denote the Fréchet space of all continuous functions on $[T_*,\infty)$ with the topology of uniform convergence on every compact subinterval of $[T_*,\infty)$. Put

$$\eta(t) = \tau^k \int_{t-k\tau}^\infty s^{n-k-1} f(s, a[g(s)]^k)\, ds, \quad t \ge T.$$

Then $\eta \in C[T,\infty)$, $\eta(t) \geq 0$ for $t \geq T$ and $\lim_{t\to\infty} \eta(t) = 0$. We consider the set Y_0 of all functions $y \in C[T_*,\infty)$, which is nonincreasing on $[T,\infty)$ and satisfies

$$y(t) = y(T) \quad \text{for } t \in [T_*,T], \quad 0 \leq y(t) \leq \eta(t) \quad \text{for } t \geq T.$$

It is easy to check that Y_0 is a closed convex subset of $C[T_*,\infty)$. From the proposition in [**16**], a mapping $\Phi_0 : Y_0 \to C[T_*,\infty)$ exists which has the following properties:

(i) For each $y \in Y_0$, $\Phi_0[y]$ satisfies $\lim_{t\to\infty} \Phi_0[y](t) = 0$ and

$$\Phi_0[y](t) + h(t)\Phi_0[y](t-\tau) = y(t), \quad t \geq T;$$

(ii) Φ_0 is continuous on Y_0 in the $C[T_*,\infty)$-topology, i.e., if $\{y_j\}_{j=1}^{\infty}$ is a sequence in Y_0 converging to $y \in Y_0$ uniformly on every compact subinterval of $[T_*,\infty)$, then $\Phi_0[y_j]$ converges to $\Phi_0[y]$ uniformly on every compact subinterval of $[T_*,\infty)$.

We use the following notation:

$$\Delta[u](t) = u(t) - u(t-\tau);$$
$$\Delta^0[u] = u; \qquad \Delta^j[u] = \Delta^{j-1}[\Delta[u]], \quad j = 1,2,\dots,.$$

Define the sets Y_i, $i = 1,2,\dots$, inductively as follows:

$$Y_i = \{y \in C[T_* - i\tau,\infty) : \Delta[y] \in Y_{i-1}\}, \quad i = 1,2,\dots.$$

We see that

$$Y_i = \{y \in C[T_* - i\tau,\infty) : \Delta^i[y] \in Y_0\}, \quad i = 1,2,\dots, \tag{2.2}$$

and that Y_i, $i = 1,2,\dots$, are closed convex subsets of $C[T_* - i\tau,\infty)$. For each $y \in Y_i$, we define the functions $\Phi_i[y]$, $i = 1,2,\dots$, on $[T_*,\infty)$ by

$$\Phi_i[y](t) = \begin{cases} \dfrac{y(t)}{1+h(t)} + \dfrac{h(t)}{1+h(t)}\Phi_{i-1}[\Delta[y]](t) & t \geq T-\tau, \\ \Phi_i[y](T-\tau) & t \in [T_*, T-\tau], \end{cases}$$
$$i = 1,2,\dots.$$

The method of induction shows that, for every $i \in \{1, 2, \dots\}$, Φ_i is well defined on Y_i and is continuous on Y_i in the $C[T_* - i\tau, \infty)$-topology and satisfies

$$\Phi_i[y](t) + h(t)\Phi_i[y](t-\tau) = y(t), \quad t \geq T, \quad y \in Y_i.$$

We need the following lemmas.

Lemma 1. *Let $i \in \{0, 1, 2, \dots\}$. Assume that $y \in Y_i$ and $y(t)/t^i \to 0$ as $t \to \infty$. Then $\Phi_i[y](t)/t^i \to 0$ as $t \to \infty$.*

Proof. Note that $\lim_{t\to\infty} \Phi_0[y](t)/t^0 = 0$ for each $y \in Y_0$. The conclusion follows from induction on i.

Lemma 2. *Let $i \in \{1, 2, \dots\}$. Suppose that $u \in C^i[t_1 - i\tau, \infty)$. Then, for every $t \in [t_1, \infty)$, there is a number $\alpha \in (t - i\tau, t)$ such that $\Delta^i[u](t) = \tau^i u^{(i)}(\alpha)$.*

Proof. Let $t \geq t_1$ be arbitrary. Note that $(\Delta^l[u])'(t) = (\Delta^l[u'])(t)$ for $l = 0, 1, 2, \dots$. By the mean value theorem, there is a number $\alpha_1 \in (t - \tau, t)$ such that

$$\begin{aligned}
\Delta^i[u](t) &= \Delta^{i-1}[u](t) - \Delta^{i-1}[u](t-\tau) \\
&= \tau(\Delta^{i-1}[u])'(\alpha_1) \\
&= \tau\Delta^{i-1}[u'](\alpha_1).
\end{aligned}$$

In exactly the same way, we obtain

$$\Delta^{i-1}[u'](\alpha_1) = \tau\Delta^{i-2}[u''](\alpha_2)$$

for some $\alpha_2 \in (\alpha_1 - \tau, \alpha_1)$ and there are numbers $\alpha_3, \alpha_4, \dots, \alpha_i$ such that $\alpha_l \in (\alpha_{l-1} - \tau, \alpha_{l-1})$ and

$$\Delta^{i-(l-1)}[u^{(l-1)}](\alpha_{l-1}) = \tau\Delta^{i-l}[u^{(l)}](\alpha_l), \quad l = 3, 4, \dots, i.$$

Consequently, we have

$$\begin{aligned}
\Delta^i[u](t) &= \tau\Delta^{i-1}[u'](\alpha_1) = \tau^2\Delta^{i-2}[u''](\alpha_2) = \cdots \\
&= \tau^i\Delta^{i-i}[u^{(i)}](\alpha_i) = \tau^i u^{(i)}(\alpha_i).
\end{aligned}$$

Since

$$(\alpha_{i-1}-\tau,\alpha_{i-1})\subset(\alpha_{i-2}-2\tau,\alpha_{i-2})\subset\cdots \\ \subset(\alpha_1-(i-1)\tau,\alpha_1)\subset(t-i\tau,t),$$

we see that $\alpha_i\in(t-i\tau,t)$. This completes the proof.

Lemma 3. *Let $i\in\{1,2,\dots\}$ and $\overline{T}\ge\overline{T}_*$. Suppose that $u\in C^i[\overline{T}_*-i\tau,\infty)$.*

(i) *If $u^{(i)}(t)\ge 0$ for $t\ge\overline{T}-i\tau$, then $\Delta^i[u](t)\ge 0$ for $t\ge\overline{T}$.*

(ii) *Assume that $v\in C[\overline{T},\infty)$. If $u^{(i)}(t)\le\tau^{-i}v(t+i\tau)$ for $t\ge\overline{T}-i\tau$ and $v(t)$ is nonincreasing on $[\overline{T},\infty)$, then $\Delta^i[u](t)\le v(t)$ for $t\ge\overline{T}$.*

(iii) *If $u^{(i)}(t)$ is nonincreasing on $[\overline{T}-i\tau,\infty)$, then $\Delta^i[u](t)$ is nonincreasing on $[\overline{T},\infty)$.*

(iv) *If $u^{(i)}(t)=u^{(i)}(\overline{T})$ for $t\in[\overline{T}_*-i\tau,\overline{T}]$, then $\Delta^i[u](t)=\Delta^i[u](\overline{T})$ for $t\in[\overline{T}_*,\overline{T}]$.*

Proof. (i) The conclusion follows from Lemma 2.

(ii) Put

$$V(t)=\tau^{-i}\int_{\overline{T}}^{t+i\tau}\frac{(t+i\tau-s)^{i-1}}{(i-1)!}v(s)\,ds$$

for $t\ge\overline{T}-i\tau$ and $w(t)=V(t)-u(t)$ for $t\ge\overline{T}-i\tau$. Then $w^{(i)}(t)=\tau^{-i}v(t+i\tau)-u^{(i)}(t)\ge 0$ for $t\ge\overline{T}-i\tau$. In view of (i), we have $\Delta^i[w](t)\ge 0$ for $t\ge\overline{T}$. We note that $\Delta^i[w](t)=\Delta^i[V](t)-\Delta^i[u](t)$. Lemma 2 implies that

$$\Delta^i[u](t)\le\Delta^i[V](t)=\tau^iV^{(i)}(\alpha)=v(\alpha+i\tau),\quad t\ge\overline{T}$$

for some $\alpha\in(t-i\tau,t)$. Since v is nonincreasing on $[\overline{T},\infty)$, we get

$$\Delta^i[u](t)\le v(t),\quad t\ge\overline{T}.$$

(iii) Let $\varepsilon>0$ be arbitrary. Set $z(t)=u(t)-u(t+\varepsilon)$. Since $u^{(i)}(t)$ is nonincreasing on $[\overline{T}-i\tau,\infty)$, we have $z^{(i)}(t)=u^{(i)}(t)-u^{(i)}(t+\varepsilon)\ge 0$ for $t\ge\overline{T}-i\tau$. From (i), we obtain

$$\Delta^i[u](t)-\Delta^i[u](t+\varepsilon)=\Delta^i[z](t)\ge 0,\quad t\ge\overline{T}.$$

Consequently, $\Delta^i[u](t)$ is nonincreasing on $[\overline{T}, \infty)$.

(iv) Let $t \in [\overline{T}_*, \overline{T}]$. By virtue of Lemma 2 there is a number $\alpha \in (t - i\tau, t)$ such that $\Delta^i[u](t) = \tau^i u^{(i)}(\alpha)$. Since $u^{(i)}(s) = u^{(i)}(\overline{T})$ for $s \in [\overline{T}_* - i\tau, \overline{T}]$ and $\alpha \in [\overline{T}_* - i\tau, \overline{T}]$, we have $\Delta^i[u](t) = \tau^i u^{(i)}(\overline{T})$. In particular, $\Delta^i[u](\overline{T}) = \tau^i u^{(i)}(\overline{T})$. Hence $\Delta^i[u](t) = \Delta^i[u](\overline{T})$ for $t \in [\overline{T}_*, \overline{T}]$.

Lemma 4. *Let $i \in \{1, 2, \dots\}$. Assume that $u \in C^i[T_* - i\tau, \infty)$ and $u^{(i)}(t)$ is nonincreasing on $[T - i\tau, \infty)$ and satisfies*

$$0 \le u^{(i)}(t) \le \tau^{-i}\eta(t + i\tau), \quad t \ge T - i\tau,$$

and

$$u^{(i)}(t) = u^{(i)}(T), \quad t \in [T_* - i\tau, T].$$

Then $u \in Y_i$.

Proof. Applying Lemma 3, we easily see that $\Delta^i[u] \in Y_0$. From (2.2) it follows that $u \in Y_i$.

Proof of the "if" part of the Theorem. From (2.1) there are constants $c > 0$, $\delta > 0$ and $\varepsilon > 0$ such that

$$0 < (\delta + \varepsilon)t^k \le c\,\omega_k(t) \le (a - \varepsilon)t^k, \quad t \ge T_*. \tag{2.3}$$

Here a is a number in the integral condition (1.3). For each $y \in Y_k$ we denote the function $\Psi[y](t)$ by

$$\Psi[y](t) = c\,\omega_k(t) + (-1)^{n-k-1}\sigma\Phi_k[y](t), \quad t \ge T_*. \tag{2.4}$$

Notice that Ψ is continuous on Y_k in the $C[T_* - k\tau, \infty)$-topology and that, for each $y \in Y_k$,

$$\Psi[y](t) + h(t)\Psi[y](t - \tau) = c\,t^k + (-1)^{n-k-1}\sigma y(t), \quad t \ge T. \tag{2.5}$$

Define the mapping $I : Y_k \longrightarrow C[T_* - k\tau, \infty)$ as follows:

$$I[y](t) = \begin{cases} \displaystyle\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\varphi(s, \Psi[y](g(s)))\,ds, & t \ge T, \\ I[y](T), & t \in [T_* - k\tau, T], \end{cases}$$

for the case $k = 0$ and

$$I[y](t) = \begin{cases} \displaystyle\int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} \varphi(r, \Psi[y](g(r)))\, dr\, ds, \\ \hfill t \geq T, \\ \displaystyle\frac{(t-T)^k}{k!} \int_T^\infty \frac{(r-T)^{n-k-1}}{(n-k-1)!} \varphi(r, \Psi[y](g(r)))\, dr, \\ \hfill t \in [T_* - k\tau, T], \end{cases}$$

for the case $k \neq 0$, where

$$\varphi(t,u) = \begin{cases} f(t, a[g(t)]^k), & u \geq a[g(t)]^k, \\ f(t,u), & \delta[g(t)]^k \leq u \leq a[g(t)]^k, \\ f(t, \delta[g(t)]^k), & u \leq \delta[g(t)]^k. \end{cases}$$

It is easy to check that I is well defined and, if $k = 0$, then I maps Y_0 into itself. We observe that, for each $y \in Y_k$, if $k \neq 0$,

$$(I[y])^{(k)}(t) = \begin{cases} \displaystyle\int_t^\infty \frac{(r-t)^{n-k-1}}{(n-k-1)!} \Phi(r, \Psi[y](g(r)))\, dr, & t \geq T, \\ \displaystyle\int_T^\infty \frac{(r-T)^{n-k-1}}{(n-k-1)!} \varphi(r, \Psi[y](g(r)))\, dr, & t \in [T_* - k\tau, T], \end{cases}$$

so that $(I[y])^{(k)}(t)$ is nonincreasing on $[T - k\tau, \infty)$ and satisfies

$$0 \leq (I[y])^{(k)}(t) \leq \tau^{-k} \eta(t + k\tau), \quad t \geq T - k\tau, \tag{2.6}$$

and

$$(I[y])^{(k)}(t) = (I[y])^{(k)}(T), \quad t \in [T_* - k\tau, T].$$

From Lemma 4 it follows that if $k \neq 0$, then $I[y] \in Y_k$ for every $y \in Y_k$, and hence I maps Y_k into itself.

The Lebesgue dominated convergence theorem shows that I is continuous on Y_k.

Since $\{(I[y])'(t) : y \in Y_k\}$ is uniformly bounded on every compact subinterval of $[T_* - k\tau, \infty)$, the mean value theorem implies that $\{(I[y])(t) : y \in Y_k\}$ is equicontinuous on every compact subinterval of $[T_* - k\tau, \infty)$. Obviously, $\{(I[y])(t) : y \in Y_k\}$ is uniformly bounded on every compact subinterval of $[T_* - k\tau, \infty)$. Hence, by the Ascoli-Arzelà theorem, $\{(I[y])(t) : y \in Y_k\}$ is relatively compact.

Consequently we are able to apply the Schauder-Tychonoff fixed point theorem to the operator I and so an element $\tilde{y} \in Y_k$ exists such that $\tilde{y} = I[\tilde{y}]$. Set $x(t) = \Psi[\tilde{y}](t)$. The inequality (2.6) and the fact that $\lim_{t\to\infty} \eta(t) = 0$ lead us to $\lim_{t\to\infty} \tilde{y}(t)/t^k = 0$. In view of Lemma 1 together with (1.5), (2.4) and (2.3), we find that $x(t)$ satisfies (1.6) and $\delta[g(t)]^k \le x(g(t)) \le a[g(t)]^k$ for all large t, say $t \ge \widetilde{T}$, so that $\varphi(t, x(g(t))) = f(t, x(g(t)))$ for $t \ge \widetilde{T}$. From (2.5) it follows that

$$\begin{aligned} &x(t) + h(t)x(t-\tau) \\ &\quad = c\,t^k + (-1)^{n-k-1}\sigma\, I[\tilde{y}](t) \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!}\, \varphi(r, x(g(r)))\, dr\, ds \end{aligned}$$

for $t \ge \widetilde{T}$. Differentiation of the above equality yields

$$\frac{d^n}{dt^n}[x(t) + h(t)x(t-\tau)] = -\sigma\varphi(t, x(g(t))) = -\sigma f(t, x(g(t)))$$

for $t \ge \widetilde{T}$. This means that $x(t)$ is a solution of (1.1). The proof of the "if" part is complete.

Acknowledgment. The author thanks Professor Manabu Naito for many helpful suggestions.

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