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DIFFEOMORPHISMS WITH THE AVERAGE-SHADOWING PROPERTY ON TWO-DIMENSIONAL CLOSED MANIFOLDS

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ABSTRACT. The average pseudo-orbits and the average-shadowing property of diffeomorphisms on two-dimensional closed manifolds are considered, and the C^1 interior of the set of all diffeomorphisms satisfying the average-shadowing property is characterized as the set of all Anosov diffeomorphisms.

The notion of pseudo-orbits very often appears in several branches of the modern theory of dynamical systems, and, especially, the pseudo-orbit shadowing property usually plays an important role in the investigation of the stability theory. In [**1**] Blank introduced the notion of average pseudo-orbits as a certain generalization of the notion of pseudo-orbits (see also [**2**, p. 19]) and it was proved there that, for a certain kind of hyperbolic system f, every average pseudo-orbit of f is shadowed in average by some true orbit of f (the average-shadowing property).

Let M be a C^∞ closed manifold, that is, M is compact connected and $\partial M = \varnothing$, and let d be the distance induced from a Riemannian metric $\|\cdot\|$ on TM. Denote by Diff (M) the set of all diffeomorphisms on M endowed with C^1 topology. For $\delta > 0$, a sequence $\{x_i\}_{i=-\infty}^{\infty}$ of points in M is called a *δ-average pseudo-orbit* of $f \in$ Diff (M) if there is a number $N = N(\delta) > 0$ such that for all $n \geq N$, $k \in \mathbf{Z}$,

$$\frac{1}{n}\sum_{i=1}^{n} d(f(x_{i+k}), x_{i+k+1}) < \delta.$$

We say that f has the *average-shadowing property* if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that every δ-average pseudo-orbit $\{x_i\}_{i=-\infty}^{\infty}$ is

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ε-shadowed in average by some $z \in M$, that is,

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} d(f^i(z), x_i) < \varepsilon.$$

Notice that f has the average-shadowing property if and only if f^n, $n > 0$, has the average-shadowing property. Average pseudo-orbits arise naturally in the realizations of independent Gaussian random perturbations with zero mean and in the investigations of the most probable orbits of the dynamical system with general Markov perturbations, etc. ([**2**, p. 20]). It is proved in [**1**, Theorem 4] that if Λ is a basic set of a diffeomorphism f satisfying Axiom A, then $f_{|\Lambda}$ has the average-shadowing property. We notice that the topological transitivity of $f_{|\Lambda}$ plays an essential role in that proof.

The author has characterized the dynamics of diffeomorphism having the pseudo-orbit shadowing property under some condition (see [**6**]). The purpose of this paper is to analyze the dynamics of diffeomorphisms satisfying the average-shadowing property on a two-dimensional closed manifold. We shall denote by $\mathcal{AS}(M)$ the C^1 interior of the set of all $f \in \mathrm{Diff}\,(M)$ having the average-shadowing property.

In the following theorem and corollary, let M be a two-dimensional closed manifold (recall that M is connected).

Theorem. *$\mathcal{AS}(M)$ is characterized as the set of all Anosov diffeomorphisms.*

Our theorem is obtained by showing the hyperbolicity of the periodic points of $f \in \mathcal{AS}(M)$. To do this we need to control the global behavior of the average-shadowing orbit, and we can do that when M is a two-dimensional closed manifold. The theorem may be true for higher dimensions; however, the author does not know how to control the orbit when $\dim M \geq 3$.

As we stated above, the average-shadowing property is closely related to the topological transitivity. Recall that a diffeomorphism f on M is called *topologically transitive* if there is a dense orbit, and let $\mathcal{TT}(M)$ be the C^1 interior of the set of all topologically transitive diffeomorphisms. Then, since every Anosov diffeomorphism is topologically transitive when $\dim M = 2$, $\mathcal{AS}(M) \subset \mathcal{TT}(M)$ by the theorem. Moreover, we can check the converse.

Corollary. $\mathcal{AS}(M) = \mathcal{TT}(M)$.

As usual, a sequence $\{x_i\}_{i=-\infty}^{\infty}$ of points in M is called a δ-*pseudo-orbit*, $\delta > 0$, of $f \in \mathrm{Diff}\,(M)$ if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbf{Z}$. We say that f has the *shadowing property* if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that for every δ-pseudo-orbit $\{x_i\}_{i=-\infty}^{\infty}$ there exists $z \in M$ satisfying $d(f^i(z), x_i) < \varepsilon$ for all $i \in \mathbf{Z}$. Let $\mathcal{S}(M)$ be the C^1 interior of the set of all diffeomorphisms satisfying the shadowing property. Then $f \in \mathcal{S}(M)$ if and only if f satisfies Axiom A and the strong transversality condition (see [**5**], [**6**]). Clearly $\mathcal{AS}(M) \subset \mathcal{S}(M)$ by the theorem when $\dim M = 2$ but its converse is not true in general. Indeed, there exists a diffeomorphism satisfying Axiom A and the strong transversality condition that is not Anosov (e.g., a Morse-Smale diffeomorphism on the unit sphere $\mathbf{S}^2$).

A diffeomorphism $f \in \mathrm{Diff}\,(M)$ is said to be *expansive* if there is a constant $e > 0$ such that if $d(f^n(x), f^n(y)) \le e$ for all $n \in \mathbf{Z}$, then $x = y$. It is well known that if $f \in \mathcal{S}(M)$ is expansive, then f is Anosov (see [**7**]). Thus, if $f \in \mathcal{S}(M)$ is expansive, then $f \in \mathcal{AS}(M)$ when $\dim M = 2$. We remark that only the two-dimensional closed manifold on which there is an Anosov diffeomorphism is the torus.

Lemmas and proofs of results. Let M, d be as before, and denote the set of all periodic points of $f \in \mathrm{Diff}\,(M)$ by $P(f)$. A hyperbolic set Λ (f-invariant closed set) is called a *basic* set if $f_{|\Lambda}$ has a dense orbit, i.e., topologically transitive, and locally maximal. The *stable manifold*, $W^s(x)$, and the *unstable manifold*, $W^u(x)$ of $x \in \Lambda$ are defined in the usual way, and put $W^\sigma(\Lambda) = \cup_{x\in\Lambda} W^\sigma(x)$, $\sigma = s, u$. When f satisfies Axiom A, the *nonwandering set*, $\Omega(f)$, of f is equal to the closure of $P(f)$ and is decomposed into a union of basic sets $\Omega(f) = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_l$. It is well known that $M = \cup_{i=1}^{l} W^\sigma(\Lambda_i)$, $\sigma = s, u$, and $W^s(\Lambda_i) \cap W^u(\Lambda_i) = \Lambda_i$ for $1 \le i \le l$. If f satisfies Axiom A with no cycles, then there is a sequence of compact sets (which is called a *filtration*) $\varnothing = M_0 \subset M_1 \subset \cdots \subset M_l = M$ such that $f(M_i) \subset \mathrm{int}\, M_i$ and $\cap_{n\in\mathbf{Z}} f^n(M_i \backslash M_{i-1}) = \Lambda_i$ for $1 \le i \le l$ (cf. [**8**]). It can be checked that, for every neighborhood U_i of Λ_i, $1 \le i \le l$, a positive integer m_i exists satisfying $\Lambda_i \subset f^{m_i}(M_i)\backslash f^{-m_i}(M_{i-1}) \subset U_i$. We denote by $\mathcal{F}^1(M)$ the set of all $f \in \mathrm{Diff}\,(M)$ having a C^1 neighborhood $\mathcal{U}(f)$ such that every $p \in P(g)$, $g \in \mathcal{U}(f)$, is hyperbolic. Then it is proved in [**4**] that if $f \in \mathcal{F}^1(M)$, then f satisfies Axiom A with no cycles (its converse is also true, see [**3**]). Our theorem will be obtained by using the following

Proposition. *If M is a two-dimensional closed manifold, then $\mathcal{AS}(M) \subset \mathcal{F}^1(M)$.*

To prove the proposition, we prepare two lemmas.

Lemma 1. *Let $\mathcal{U}(f) \subset \mathrm{Diff}\,(M)$ be a neighborhood of f, and let $f^n(p) = p$, $n > 0$, be a periodic point. Then there are $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that if $\mathcal{O}_p : T_pM \to T_pM$ is a linear isomorphism with $\|\mathcal{O}_p - I\| < \delta_0$, then there is a $g \in \mathcal{U}(f)$ satisfying*

(i) $B_{4\varepsilon_0}(f^i(p)) \cap B_{4\varepsilon_0}(f^j(p)) = \varnothing$ *for* $0 \le i \ne j \le n-1$,

(ii) $g(x) = f(x)$ *if* $x \in \{p, f(p), \dots, f^{n-1}(p)\} \cup \{M \backslash \cup_{i=0}^{n-1} B_{4\varepsilon_0}(f^i(p))\}$,

(iii) $g(x) = \exp_{f^{i+1}(p)} \circ D_{f^i(p)} f \circ \exp^{-1}_{f^i(p)}(x)$ *if* $x \in B_{\varepsilon_0}(f^i(p))$ *for* $0 \le i \le n-2$,

(iv) $g(x) = \exp_p \circ \mathcal{O}_p \circ D_{f^{n-1}(p)} f \circ \exp^{-1}_{f^{n-1}(p)}(x)$ *if* $x \in B_{\varepsilon_0}(f^{n-1}(p))$,

where $I : T_pM \to T_pM$ is the identity map and $B_\varepsilon(x) = \{y \in M : d(x,y) \le \varepsilon\}$ for $\varepsilon > 0$.

Proof. See [**3**, Lemma 1.1.].

Hereafter, let M be a two-dimensional closed manifold.

Lemma 2. *For any $\mathcal{U}(f) \subset \mathrm{Diff}\,(M)$ and $f^n(p) = p$, $n > 0$, let ε_0, $\delta_0 > 0$, be as in Lemma 1. Suppose that p is not hyperbolic. Then we can find a linear isomorphism $\mathcal{O}_p : T_pM \to T_pM$ with $\|\mathcal{O}_p - I\| < \delta_0$ such that for the diffeomorphism $g \in \mathcal{U}(f)$, $g^n(p) = p$, given by Lemma 1 for this $\mathcal{O}_p$, there is a D_pg^{nL}-invariant splitting $T_pM = E \oplus F$, $\dim E = \dim F = 1$, satisfying $D_pg^{nL}(v) = v$, for all*

$v \in E$, for some $L > 0$.

Proof. For any $\mathcal{U}(f)$ and $f^n(p) = p$, $n > 0$, let ε_0, $\delta_0 > 0$ be as in Lemma 1. Let us denote two eigenvalues of $D_p f^n$ by λ and μ, and suppose that $|\lambda| = 1$ (since $f^n(p) = p$ is not hyperbolic). At first, assume that λ is real. If the canonical form of $D_p f^n$ is $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ with respect to some linear isomorphism $G : T_pM \to T_pM$, that is, $D_p f^n = G \circ \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \circ G^{-1}$, then we put $\mathcal{O}_p = I$. Let $g \in \mathcal{U}(f)$ be the diffeomorphism given by Lemma 1 for $\mathcal{O}_p = I$, and let E, F be the corresponding eigenspaces for λ, μ respectively. Then the conclusion will be obtained for $L = 1$ or 2. For the case when the multiplicity of λ is 2 and $\lambda = 1$, the canonical form of Df_p^n is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ with respect to some linear isomorphism G as above. In this case, take $\varepsilon > 0$ such that $\|\mathcal{O}_p - I\| < \delta_0$, where $\mathcal{O}_p = G \circ \begin{pmatrix} 1 & 0 \\ 0 & 1+\varepsilon \end{pmatrix} \circ G^{-1}$. Let $g \in \mathcal{U}(f)$ be the diffeomorphism given by Lemma 1 for the $\mathcal{O}_p$, and let E, F be the eigenspaces for the eigenvalues 1, $1+\varepsilon$ of $D_p g^n$ respectively. Then we have the conclusion for $L = 1$ (the other case when $\lambda = -1$ follows in a similar way).

Next we suppose that λ is a complex number, and let $\alpha + \beta i$, $\alpha - \beta i$ be the eigenvalues of $D_p f^n$, $|\lambda|^2 = \alpha^2 + \beta^2 = 1$. There is a linear isomorphism $G : T_pM \to T_pM$ and $\theta = \theta(\alpha, \beta) \in \mathbf{R}$ such that $D_p f^n = G \circ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \circ G^{-1}$. Take $\theta' \in \mathbf{R}$ near 0 and $L > 0$ such that $\|\mathcal{O}_p - I\| < \delta_0$ and $(\mathcal{O}_p \circ D_p f^n)^L = G \circ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \circ G^{-1}$. Here $\mathcal{O}_p = G \circ \begin{pmatrix} \cos\theta' & -\sin\theta' \\ \sin\theta' & \cos\theta' \end{pmatrix} \circ G^{-1}$. Let $g \in \mathcal{U}(f)$ be the diffeomorphism given by Lemma 1 for the $\mathcal{O}_p$. Then the conclusion is clear.

Proof of Proposition. Let $f \in \mathcal{AS}(M)$. To get the conclusion, it is enough to show that every $p \in P(f)$ is hyperbolic since $\mathcal{AS}(M)$ is an open set. Fix $\mathcal{U}(f) \subset \mathcal{AS}(M)$ and suppose that $f^n(p) = p \in P(f)$, $n > 0$, is not hyperbolic; we shall derive a contradiction. Let ε_0, $g \in \mathcal{U}(f)$, L and $T_pM = E \oplus F$ be given in Lemma 2. Since g has the average-shadowing property, g^{nL} does so. Hereafter in this proof, we shall denote g^{nL} by g for simplicity. Choose $0 < \varepsilon_1 < \varepsilon_0$ with $g(B_{\varepsilon_1}(p)) \subset B_{\varepsilon_0}(p)$ and set $\varepsilon = \varepsilon_1/10 > 0$. Let $0 < \delta = \delta(\varepsilon) < \varepsilon$ be a number as in the definition of the average-shadowing property of g. Fix an integer $n_0 > 0$ such that $\delta(n_0 - 1) < 3\varepsilon_1 \le \delta n_0$. By Lemma 2 there is a $\mu \in \mathbf{R}$ such that $D_p g(v) = v$, for all $v \in E$, and $D_p g(w) = \mu w$, for all $w \in F$. Thus, for every $(v, w) \in E \oplus F$ if $\exp_p(v, w) \in B_{\varepsilon_1}(p)$ then $g(\exp_p(v, w)) = \exp_p(v, \mu w)$.We may suppose further that $\mu > 0$ (for the case when $\mu < 0$, we consider g^2 instead of g and denote it again by g for convenience).

Now we pick two points w_1 and w_2 in $\exp_p(E \cap \exp_p^{-1}(B_{\varepsilon_1}(p)))$ symmetrically with respect to p. More precisely, choose $v \in E$ and put $w_1 = \exp_p(-v)$, $w_2 = \exp_p(v)$ such that $d(w_1, w_2) = \varepsilon_1$. Define a cyclic sequence $\{x_i\}_{i=-\infty}^{\infty}$ of points consisting of w_1 and w_2 by

$$x_{2jn_0+1} = x_{2jn_0+2} = \cdots = x_{(2j+1)n_0} = w_1$$

and

$$x_{(2j+1)n_0+1} = x_{(2j+1)n_0+2} = \cdots = x_{2(j+1)n_0} = w_2$$

for $j \in \mathbf{Z}$. Then it is easy to see that, for all $m > 3n_0$ and $k \in \mathbf{Z}$,

$$\frac{1}{m} \sum_{i=1}^{m} d(g(x_{i+k}), x_{i+k+1}) < \frac{3d(w_1, w_2)}{2n_0} = \frac{3\varepsilon_1}{2n_0} \le \frac{\delta n_0}{2n_0} < \delta$$

and thus $\{x_i\}_{i=-\infty}^{\infty}$ is a cyclic δ-average pseudo-orbit of g. Hence there is a $z \in M$ which ε-shadows $\{x_i\}_{i=-\infty}^{\infty}$ in average. Put $\mathcal{I} = \{x \in B_{\varepsilon_1}(p) : g(x) = x\}$. Then $g^i(x) = x$ for $i \in \mathbf{Z}$ whenever $x \in \mathcal{I}$. If $z \in \mathcal{I}$, then $d(z, w_1) \ge 5\varepsilon$ or $d(z, w_2) \ge 5\varepsilon$, since $d(w_1, w_2) = 10\varepsilon$. In both cases, we have $(1/m) \sum_{i=1}^{m} d(g^i(z), x_i) > \varepsilon$ for any sufficiently large $m > 3n_0$. This is a contradiction and hence $z \notin \mathcal{I}$ is concluded.

If $\mu = 1$, then $\mathcal{I} = B_{\varepsilon_1}(p)$ so that $(1/m) \sum_{i=1}^{m} d(g^i(z), x_i) > \varepsilon$ for all $m > 0$. This is inconsistent with the choice of z. For the case when $\mu > 1$, since we can find $m' \ge 0$ such that $g^{m'+j}(z) \notin B_{\varepsilon_1}(p)$ for all $j \ge 0$, because $\dim M = 2$ and $z \notin \mathcal{I}$, we see $d(g^{m'+j}(z), x_{m'+j}) \ge 2\varepsilon$ for all $j \ge 0$. Thus, if $m > m'$ is large enough, then $(1/m) \sum_{i=1}^{m} d(g^i(z), x_i) > \varepsilon$. This is a contradiction. For the case when $\mu < 1$, we may suppose that there is an $m'' \ge 0$ such that $g^i(z) \notin B_{\varepsilon_1}(p)$ for $0 \le \forall i \le m''$ and $g^{m''+1}(z) \in B_{\varepsilon_1}(p)$. There are $J > 0$ and $i \in \{1, 2\}$ such that $j \ge J$ implies $d(g^{m''+j}(z), w_i) \ge 4\varepsilon$. Thus, for any sufficiently large $m > m'' + J$, we have

$$\frac{1}{m} \sum_{i=1}^{m} d(g^i(z), x_i) \ge \frac{1}{m}(m - J - m'' - 1) \cdot \frac{4}{3}\varepsilon > \varepsilon.$$

This is also a contradiction and thus $\mathcal{AS}(M) \subset \mathcal{F}^1(M)$ is proved. □

Proof of Theorem. Since every Anosov diffeomorphism on a two-dimensional closed manifold has the average-shadowing property, it only remains to show that if $f \in \mathcal{AS}(M)$, then f is Anosov. Let $f \in \mathcal{AS}(M)$. Since f satisfies Axiom A with no-cycles by the proposition, for the spectral decomposition $\Omega(f) = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_l$, there is a filtration $\varnothing = M_0 \subset M_1 \subset \cdots \subset M_l = M$.

Claim. *Under the above notations, we have* $l = 1$.

If this claim is true, then f is Anosov. Actually, $M = W^s(\Lambda_1) \cap W^u(\Lambda_1) = \Lambda_1$. Thus the theorem is proved.

To prove the claim, by assuming that $l \geq 2$ we shall derive a contradiction. For simplicity, we suppose $l = 2$. Take $\varepsilon > 0$ small enough and fix integers $n_1, n_2 \geq 5$ such that

$$(n_1 - 1)\varepsilon < d(M_1, \Lambda_2) \leq n_1\varepsilon, (n_2 - 1)\varepsilon < d(\Lambda_1, \Lambda_2) \leq n_2\varepsilon.$$

Here $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$ for $A, B \subset M$. Since f has the average-shadowing property, there is $0 < \delta = \delta(\varepsilon) < \varepsilon$ such that every δ-average pseudo-orbit $\{x_i\}_{i=-\infty}^{\infty}$ is ε-shadowed in average by some point in M. Finally, let us fix $n_3 \geq 3$ such that $(n_3 - 1)\delta < d(\Lambda_1, \Lambda_2) \leq n_3\delta$. Take $x \in \Lambda_1$, $y \in \Lambda_2$ with $d(x, y) = d(\Lambda_1, \Lambda_2)$. Since $\Omega(f) = \overline{P(f)}$, there are $p \in \Lambda_1 \cap P(f)$ and $q \in \Lambda_2 \cap P(f)$ such that

$$\max\{d(x, p),\ d(y, q),\ d(f(x), f(p)),\ d(f(y), f(q))\} < \delta.$$

Let $l_1, l_2 > 0$ be the periods of p, q, respectively,that is, $f^{l_1}(p) = p$, $f^{l_2}(q) = q$. Fix $l_3 > 0$ such that $l_i l_3 > n_3$ for $i = 1, 2$, and denote a cyclic sequence

$$\{\ldots, y, f(q), f^2(q), \ldots, f^{l_1 l_2 l_3^2 - 1}(q), x, f(p), f^2(p), \ldots, \\ f^{l_1 l_2 l_3^2 - 1}(p), y, f(q), \ldots\}$$

of points, composed of two points $\{x, y\}$ and two periodic orbits, by $\{z_i\}_{i=-\infty}^{\infty}$, $z_0 = y$. Then this is a δ-average pseudo-orbit. Indeed, for every $m > 2l_1l_2l_3^2$ and $k \in \mathbf{Z}$,

$$\frac{1}{m}\sum_{i=1}^{m} d(f(z_{i+k}), z_{i+k+1}) \le \frac{1}{2l_1l_2l_3^2}(4\delta + 3\delta + 3n_3\delta) < \delta.$$

Pick $z \in M$ which ε-shadows $\{z_i\}_{i=-\infty}^{\infty}$ in average. If $z \in \Lambda_2$, then $f^i(z) \in \Lambda_2$ for all $i \ge 0$. Hence, for a sufficiently large $m > 3l_1l_2l_3^2$, $(1/m)\sum_{i=1}^{m} d(f^i(z), z_i) > ((n_2-1)\varepsilon/3) > \varepsilon$. This is a contradiction. If $z \notin \Lambda_2$, then there exists a neighborhood U_2 of Λ_2 with $z \notin U_2$. By using a filtration property we can find $m' > 0$ such that, for all $i > m'$, $f^i(z) \in M_1$. Thus,

$$\frac{1}{m}\sum_{i=1}^{m} d(f^i(z), z_i) = \frac{C'}{m} + \frac{1}{m}\sum_{j=1}^{m-m'} d(f^{m'+j}(z), z_{m'+j}) > \frac{(n_1-1)\varepsilon}{3} > \varepsilon$$

if we take m large enough. Here $C' = \sum_{i=1}^{m'} d(f^i(z), z_i)$. This is also a contradiction and so the proof of the claim is completed. □

Proof of Corollary. To get the conclusion, it is enough to show that if $f \in \mathcal{TT}(M)$, then $f \in \mathcal{F}^1(M)$ when $\dim M = 2$. Suppose that there is a nonhyperbolic periodic point p of $f \in \mathcal{TT}(M)$. Then, by making use of Lemma 1, we can find g (C^1 near f) possessing a sink or a source periodic point p. This is a contradiction since g is topologically transitive.

Remark. A diffeomorphism g exists on the two-dimensional torus $\mathbf{T}^2$, belonging to the boundary of $\mathcal{AS}(\mathbf{T}^2)$, but g does not have the average-shadowing property. Indeed, let $\mathcal{A}(\mathbf{T}^2)$ be the set of all Anosov diffeomorphisms on $\mathbf{T}^2$, and let θ^2 be the set of all C^2 diffeomorphisms g on $\mathbf{T}^2$ such that there are a Dg-invariant continuous splitting $T\mathbf{T}^2 = E^u \oplus E^s$, a Riemannian metric $\|\cdot\|$ and $0 < \lambda < 1$ satisfying

$$\|D_xg|_{E^s}\| < \lambda \quad \text{and} \quad \|D_xg|_{E^u}\| \ge 1.$$

Then every element of θ^2 is C^2 (and so C^1)-approximated by Anosov diffeomorphisms ([**9**, Proposition C]). For $g \in \theta^2 \backslash \mathcal{A}(\mathbf{T}^2)$, let $\Lambda = \{x \in \mathbf{T}^2 : \|D_xg^n|_{E^u}\| = 1 \text{ for } n \in \mathbf{Z}\}$. Then Λ is a nonempty closed invariant set. For the case when each connected component of Λ is not a single point, Λ is a finite union of C^2 arcs and $g(x) = x$ for all $x \in \Lambda$ (see [**9**, Proposition B]). Notice that E^s is uniformly contracting. Since a neighborhood of Λ in M is expressed as a direct product of Λ and a local stable manifold, by the same argument used in a proof of the proposition of the present paper, we see that g does not have the average-shadowing property.

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REFERENCES

1. M.L. Blank, *Metric properties of ε-trajectories of dynamical systems with stochastic behaviour*, Ergodic Theory Dynamical Systems **8** (1988), 365–378.

2. ———, *Small perturbations of chaotic dynamical systems*, Russian Math. Surveys **44** (1989), 1–33.

3. J. Franks, *Necessary conditions for stability of diffeomorphisms*, Trans. Amer. Math. Soc. **158** (1971), 301–308.

4. S. Hayashi, *Diffeomorphisms in $\mathcal{F}^1(M)$ satisfy Axiom* A, Ergodic Theory Dynamical Systems **12** (1992), 233–253.

5. C. Robinson, *Stability theorems and hyperbolicity in dynamical systems*, Rocky Mountain J. Math. **7** (1977), 425–437.

6. K. Sakai, *Pseudo-orbit tracing property and strong transversality of diffeomorphisms on closed manifolds*, Osaka J. Math. **31** (1994), 373–386.

7. ———, *Continuum-wise expansive diffeomorphisms*, Publ. Mat. **41** (1997), 375–382.

8. S. Smale, *The Ω-stability theorem*, in *Global analysis*, Proc. Amer. Math. Soc. **14** 1970, 289–297.

9. N. Sumi, *Diffeomorphisms approximated by Anosov on the* 2*-torus and their SRB measures*, Trans. Amer. Math. Soc. **351** (1991), 3373–3385.

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