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SHARP REGULARITY THEORY FOR ELASTIC AND THERMOELASTIC KIRCHOFF EQUATIONS WITH FREE BOUNDARY CONDITIONS

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ABSTRACT. We consider mixed problems for, initially, a two-dimensional model of an elastic Kirchoff equation with free boundary conditions (BC) and provide sharp trace and interior regularity results. The problem does not satisfy Lopatinski's conditions.

Pseudo-differential operator/micro-local analysis techniques are used. These results, in turn, yield a sharp regularity theory for the corresponding thermoelastic plate equation. The described sharp regularity theory, besides being of interest in itself, is critically needed in establishing a structural decomposition result of the corresponding thermoelastic semigroup with free BC [**12**], as well as in exact controllability problems.

1. Introduction and statement of main results.

Dynamical model. Let $\Omega \subset R^2$ be a bounded domain with smooth boundary Γ, say of class C^2. On Ω we consider the following two mixed (dual) problems for the so-called Kirchoff plate equation with free boundary conditions (BC) in the vertical displacement $w(t,\xi)$ or $u(t,\xi)$, $\xi = [\xi_1, \xi_2]$, respectively

(1.1a)
$$\mathcal{P}w \equiv w_{tt} - \gamma\Delta w_{tt} + \Delta^2 w = q, \qquad u_{tt} - \gamma\Delta u_{tt} + \Delta^2 u = 0 \quad \text{in } Q,$$

(1.1b)
$$w(0,\cdot) = w_0,\ w_t(0,\cdot) = w_1, \quad u(T,\cdot) = 0,\ u_t(T,\cdot) = 0 \quad \text{in } \Omega,$$

(1.1c)
$$\mathcal{B}_1 w \equiv \Delta w + B_1 w = 0, \qquad \Delta u + B_1 u = g_1 \quad \text{in } \Sigma,$$

(1.1d)
$$\mathcal{B}_2 w \equiv \frac{\partial \Delta w}{\partial \nu} + B_2 w - \gamma\frac{\partial w_{tt}}{\partial \nu} \equiv 0; \qquad \frac{\partial \Delta u}{\partial \nu} + B_2 u - \gamma\frac{\partial u_{tt}}{\partial \nu} = g_2 \quad \text{in } \Sigma;$$

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$Q = (0,T] \times \Omega$; $\Sigma = (0,T] \times \Gamma$. In (1.1) and throughout this paper, the constant γ is positive: $\gamma > 0$ (physically, γ is proportional to the square of the thickness of the plate). The second- and third-order boundary operators B_1 and B_2 in (1.1c)–(1.1d) are usually given in the literature in terms of the two scalar spatial variables ξ_1, ξ_2 and take the form

(1.2)
$$B_1 w \equiv (1-\mu)[2\nu_1\nu_2 w_{\xi_1\xi_2} - \nu_1^2 w_{\xi_2\xi_2} - \nu_2^2 w_{\xi_1\xi_2}],$$

(1.3)
$$B_2 w \equiv (1-\mu)\,\frac{\partial}{\partial\tau}[(\nu_1^2-\nu_2^2)w_{\xi_1\xi_2} + \nu_1\nu_2(w_{\xi_2\xi_2} - w_{\xi_1\xi_1})],$$

see [**2**], [**3**] and references quoted therein, where $0 < \mu < 1$ is the Poisson's modulus, $\nu = [\nu_1, \nu_2]$ is the unit outward normal and $\tau = [-\nu_2, \nu_1]$ is a tangent unit vector along the boundary curve, oriented counterclockwise. However, for purposes of mathematical analysis such as that in the present paper, it is far more convenient—and indeed, essential—to rewrite the boundary operators B_1 and B_2 in terms of the normal and tangential vectors ν and τ. The following main expressions are proved in [**13**, Proposition 3C.6, Appendix C of Chap. 3, p. 305].

First BC (1.1c). Here we may write [**13**],

$$\text{on } \Gamma : \Delta w|_\Gamma \equiv \frac{\partial^2 w}{\partial\nu^2} + \frac{\partial^2 w}{\partial\tau^2} + k(\xi)\frac{\partial w}{\partial\nu}; \qquad (1.4)$$
$$B_1 w = -(1-\mu)\left[\frac{\partial^2 w}{\partial\tau^2} + k(\xi)\frac{\partial w}{\partial\nu}\right],$$

where $k(\xi) \equiv \operatorname{div}\nu(\xi)$ is the mean curvature. Thus, by (1.4), the first boundary operator $\mathcal{B}_1 w$ in (1.1c) is more conveniently rewritten as

$$\mathcal{B}_1 w \equiv \Delta w + B_1 w \equiv \frac{\partial^2 w}{\partial\nu^2} + \mu\frac{\partial^2 w}{\partial\tau^2} + \mu k(\xi)\frac{\partial w}{\partial\nu} \quad \text{on } \Gamma. \qquad (1.5)$$

Second BC (1.1d). Here we may write [**13**, Proposition 3C.8, Appendix C of Chap. 3, p. 306.]

$$\text{on } \Gamma : B_2 w \equiv (1-\mu)\frac{\partial}{\partial\tau}\frac{\partial}{\partial\nu}\frac{\partial w}{\partial\tau}, \qquad (1.6)$$

so that the second boundary operator $\mathcal{B}_2 w$ in (1.1d) is more conveniently rewritten as

$$\begin{aligned}\mathcal{B}_2 w \equiv \frac{\partial \Delta w}{\partial \nu} + B_2 w = \frac{\partial^3 w}{\partial \nu^3} + (2-\mu)\frac{\partial}{\partial \nu}\frac{\partial^2 w}{\partial \tau^2} \\ + k(\xi)\frac{\partial^2 w}{\partial \nu^2} + \text{l.o.t.} \quad \text{on } \Gamma,\end{aligned} \tag{1.7}$$

where l.o.t. denotes lower order terms.

Preliminary interior regularity of the w-problem in (1.1). If one sets $q = 0$ in (1.1a), the corresponding homogeneous w-problem (1.1) generates a strongly continuous (sc) contraction semigroup $\{w_0, w_1\} \to \{w(t), w_t(t)\}$ on a space norm-equivalent to $H^2(\Omega) \times H^1(\Omega)$. This can be readily proved [**1**], [**2**], [**13**, Chap. 3, Sect. 5] by invoking the Lumer-Phillips theorem [**18**]. As a consequence, the following known optimal interior regularity result [**2**], [**12**] may then be given as a preliminary starting point.

Proposition 1.0. *With reference to the w-problem in* (1.1)*, we have that the map*

$$\begin{aligned}\{w_0, w_1, q\} \in H^2(\Omega) \times H^1(\Omega) \times L_1(0, T; [H^1(\Omega)]') \\ \to \{w, w_t\} \in C([0, T]; H^2(\Omega) \times H^1(\Omega))\end{aligned} \tag{1.8}$$

is continuous.

Main regularity results. A first main goal of the present paper is to provide the following new trace and interior regularity results of the w-problem and u-problem in (1.1), respectively, which are dual to each other.

Theorem 1.1 (Trace regularity of the w-problem). *With reference to the w-problem in* (1.1a)–(1.1d) (*left*)*, the following trace regularity result holds true*: *the map*

$$\begin{aligned}\{w_0, w_1, q\} \in H^2(\Omega) \times H^1(\Omega) \times L_1(0, T; [H^1(\Omega)]') \\ \to \frac{\partial w_t}{\partial \nu} \in L_2(0, T; H^{-1/2}(\Gamma))\end{aligned} \tag{1.9}$$

is continuous.

By duality, [**12**], [**13**], Theorem 1.1 yields, see also Appendix C:

Theorem 1.2 (Interior regularity of the u-problem). *With reference to the u-problem in* (1.1a)–(1.1d) (*right*)*, the following interior regularity result holds true*: *the map*

(1.10)

$$\begin{cases} g_1 \in L_2(0,T;H^{1/2}(\Gamma)) \\ \\ g_2 \in L_2(0,T;H^{-1/2}(\Gamma)) \end{cases} \Rightarrow \{u,u_t\} \in C([0,T];H^2(\Omega)\times H^1(\Omega))$$

is continuous.

The critical regularity is that which involves g_1. We shall give below the proof of Theorem 1.1 which critically uses the a-priori interior regularity provided by Proposition 1.0. Comparing (1.8) with (1.10), we see that this a-priori interior regularity in (1.8) for the w-problem is precisely the same as that guaranteed by the *boundary* datum g for problem u in (1.1) (right) via (1.10). As a consequence, the same proof of Theorem 1.1, this time applied to the u-problem (1.1) (right) yields

Theorem 1.3 (Trace regularity of the u-problem). *With reference to the u-problem* (1.1a)–(1.1d) (*right*)*, the following trace regularity holds true*: *the map*

$$\begin{cases} g_1 \in L_2(0,T;H^{1/2}(\Gamma)) \\ \\ g_2 \in L_2(0,T;H^{-1/2}(\Gamma)) \end{cases} \Rightarrow \frac{\partial u_t}{\partial \nu} \in L_2(0,T;H^{-1/2}(\Gamma)) \tag{1.11}$$

is continuous.

Remark 1.1. The trace regularity (1.9), respectively (1.11), does not follow from the interior regularity (1.8), respectively (1.10), by trace theory. The application of trace theory on w_t from (1.8) to (1.9), respectively on u_t from (1.10) to (1.11), is only *formal*.

Consequence on thermoelastic plate equations with free BC. Theorems 1.1–1.3, besides being new and of interest in themselves, have the following implication on thermoelastic plate equations with free BC. Supplement the elastic Kirchoff equation in the displacement u by considering also thermal effects due to the relative temperature θ about the stress-free state $\theta = 0$, as to obtain the following thermoelastic plate equation with free BC [**2**], [**5**],

$$z_{tt} - \gamma\Delta z_{tt} + \Delta^2 z + \Delta\theta = q \quad \text{in } (0,T] \times \Omega = Q, \tag{1.12a}$$

$$\theta_t - \Delta\theta - \Delta z_t = 0 \quad \text{in } Q, \tag{1.12b}$$

$$z(0,\cdot) = z_0,\ z_t(0,\cdot) = z_1;\ \theta(0,\cdot) = \theta_0 \quad \text{in } \Omega, \tag{1.12c}$$

$$\Delta z + B_1 z + \theta = 0 \quad \text{in } (0,T] \times \Gamma \equiv \Sigma, \tag{1.12d}$$

$$\frac{\partial \Delta z}{\partial \nu} + B_2 z - \gamma \frac{\partial z_{tt}}{\partial \nu} + \frac{\partial \theta}{\partial \nu} = 0 \quad \text{in } \Sigma, \tag{1.12e}$$

$$\frac{\partial \theta}{\partial \nu} + b\theta = 0 \quad b \geq 0 \quad \text{in } \Sigma. \tag{1.12f}$$

Notice that both the equations (1.12a)–(1.12b) as well as the BC (1.12d)–(1.12e) *couple* the mechanical and the thermal variables, z and θ, respectively. The following result is known [**2**], [**3**], [**4**], [**1**], [**12**], [**13**].

Proposition 1.4. (a) *Problem* (1.12) *with* $q = 0$ *generates a* sc *contraction semigroup*

$$\{z_0, z_1, \theta_0\} \to \{z(t), z_t(t), \theta(t)\}$$

on a space norm-equivalent to $H^2(\Omega) \times H^1(\Omega) \times L_2(\Omega)$*. Thus,*

$$\begin{aligned} &\{z_0, z_1, \theta_0\} \in H^2(\Omega) \times H^1(\Omega) \times L_2(\Omega) \Rightarrow \\ &\qquad \{z, z_t, \theta\} \in C([0,T]; H^2(\Omega) \times H^1(\Omega) \times L_2(\Omega)); \end{aligned} \tag{1.13}$$

(b) *moreover, continuously in* $\{z_0, z_1, \theta_0\} \in H^2(\Omega) \times H^1(\Omega) \times L_2(\Omega)$*, we have*

$$\theta \in L_2(0,T; H^1(\Omega)), \quad \textit{hence} \quad \theta|_\Gamma \in L_2(0,T; H^{1/2}(\Gamma)); \tag{1.14}$$

$$\Delta\theta = -\mathcal{A}_R \theta \in L_2(0,T; [H^1(\Omega)]'), \tag{1.15}$$

where $\mathcal{A}_R : L_2(\Omega) \supset \mathcal{D}(\mathcal{A}_R) \to L_2(\Omega)$ *is the positive, self-adjoint operator*

$$\begin{aligned} \mathcal{A}_R f &= -\Delta f, \\ \mathcal{D}(\mathcal{A}_R) &= \left\{ f \in H^2(\Omega) : \frac{\partial f}{\partial \nu} + bf = 0 \text{ on } \Gamma \right\}, \\ \mathcal{D}(\mathcal{A}_R^{1/2}) &= H^1(\Omega). \end{aligned} \tag{1.16}$$

If $b = 0$, *we shall write* $\mathcal{A}_N$ (*Neumann*) *instead of* $\mathcal{A}_R$ (*Robin*).

(c) *The same regularity for* $\{z, z_t, \theta\}$ *continues to hold if, in addition,* $q \in L_1(0, T; [H^1(\Omega)]')$.

The first statement (a) is readily proved by the Lumer-Phillips theorem [**1**], [**13**, Chap. 3, Sect. 13]; the second statement (b) by a dissipation energy argument [**1**], [**11**], [**12**].

Rewriting the thermoelastic problem (1.12) with forcing term q via the a-priori regularity asserted by Proposition 1.4(b) as

(1.17a)
$$\mathcal{P}z \equiv z_{tt} - \gamma \Delta z_{tt} + \Delta^2 z = \mathcal{A}_R \theta + q \in L_2(0, T; [H^1(\Omega)]'),$$
(1.17b)
$$z(0, \cdot) = z_0, \; z_t(0, \cdot) = z_1 \quad \text{in } H^2(\Omega) \times H^1(\Omega),$$
(1.17c)
$$\mathcal{B}_1 z \equiv \Delta z + B_1 z = -\theta|_\Gamma \in L_2(0, T; H^{1/2}(\Gamma)),$$
(1.17d)
$$\mathcal{B}_2 z \equiv \frac{\partial \Delta z}{\partial \nu} + B_2 z - \gamma \frac{\partial z_{tt}}{\partial \nu} = b\, \theta|_\Gamma \in L_2(0, T; H^{1/2}(\Gamma)),$$

we can then apply Theorem 1.1 and Theorem 1.3 and obtain

Theorem 1.5. (Trace regularity of the thermoelastic problem (1.12) or (1.17)). *With reference to the thermoelastic problem* (1.12) *or* (1.17), *the following trace regularity holds true*: *the map*

$$\begin{cases} q \in L_1(0, T; [H^1(\Omega)]') \\ \{z_0, z_1, \theta_0\} \in H^2(\Omega) \times H^1(\Omega) \times L_2(\Omega) \end{cases} \tag{1.18}$$
$$\Rightarrow \frac{\partial z_t}{\partial \nu} \in L_2(0, T; H^{-1/2}(\Gamma))$$

is continuous.

Remark 1.2. The trace regularity (1.18) does not follow from the interior regularity (1.13) on z_t by trace theory.

In order to state a corresponding dual result, we introduce the following (boundary nonhomogeneous) mixed thermoelastic problem

$$\begin{aligned}
&\text{(1.19a)} && y_{tt} - \gamma \Delta y_{tt} + \Delta^2 y + \Delta \alpha = 0 && \text{in } (0,T] \times \Omega \equiv Q;\\
&\text{(1.19b)} && \alpha_t - \Delta \alpha - \Delta y_t = 0 && \text{in } Q;\\
&\text{(1.19c)} && y(0,\cdot) = 0;\ y_t(0,\cdot) = 0;\ \alpha(0,\cdot) = 0 && \text{in } \Omega;\\
&\text{(1.19d)} && \Delta y + B_1 y + \alpha = g_1 && \text{in } (0,T] \times \Gamma \equiv \Sigma;\\
&\text{(1.19e)} && \frac{\partial \Delta y}{\partial \nu} + B_2 y - \gamma \frac{\partial y_{tt}}{\partial \nu} + \frac{\partial \alpha}{\partial \nu} = g_2 && \text{in } \Sigma;\\
&\text{(1.19f)} && \frac{\partial \alpha}{\partial \nu} + b\alpha \equiv 0, \quad b \geq 0 && \text{in } \Sigma.
\end{aligned}$$

Then, by duality [**12**], [**13**], see Appendix C, Theorem 1.5 yields

Theorem 1.6. (Interior regularity of the thermoelastic $\{y, \alpha\}$-problem (1.19)). *With reference to problem* (1.19), *the following interior regularity holds true: the map*

$$\text{(1.20)} \quad \begin{cases} g_1 \in L_2(0,T; H^{1/2}(\Gamma)) \\ g_2 \in L_2(0,T; H^{-1/2}(\Gamma)) \end{cases}$$
$$\Rightarrow \{y, y_t, \alpha\} \in C([0,T]; H^2(\Omega) \times H^1(\Omega) \times L_2(\Omega))$$

is continuous. □

Further regularity results for problem (1.19) will be stated and proved in Section 9.

Literature. There is a conspicuous body of literature concerning the Kirchoff problem(s) (1.1) with $\gamma > 0$ and free BC. Most of the works are focused on problems such as exact controllability (continuous observability estimates), uniform stabilization, etc., see, e.g., [**2**], [**3**], [**4**], the latter one in the thermoelastic case. However, a sharp regularity theory for these Kirchoff elastic problems—and, a fortiori, for their more complicated thermoelastic versions such as (1.12)—is altogether missing at present. (A notable exception is the trace results in [**9**]). This lamentable fact has been noted often in interested circles. (By contrast, this is *not* the case with the other, simpler BC's associated with the Kirchoff equation (1.1a), where in fact an optimal regularity theory is available [**5**], [**7**]). The reason may be that a regularity analysis of the Kirchoff problem (1.1) with free BC is somewhat akin to that of the wave equation, (or, more generally, of second-order hyperbolic equations) with Neumann BC: more technically, both problems share the property—which is in fact a known source of difficulty—that they do not satisfy the so-called Lopatinski conditions. In the latter case of second-order hyperbolic equations with Neumann BC—which is far more difficult to analyze in regularity properties than the corresponding Dirichlet case—sharp regularity results have emerged only recently [**6**]. They require sophisticated pseudo-differential/micro-local analysis techniques to get the sought-after "trace regularity estimates." By contrast, the reverse "continuous observability inequalities" for the canonical *wave equation* with Neumann BC, at least in the energy space, are more amenable to obtain, purely within energy methods in differential (not pseudo-differential) form. (But pseudo-differential methods provide vast refinements and generalizations.)

A counterpart situation may be said to exist in the case of the elastic Kirchoff problem (1.1) with free BC. Accordingly, Theorems 1.1, 1.2 and 1.3 for the elastic problem (1.1), as well as Theorem 1.5 for the thermoelastic version (1.12), are new. Moreover, they have important implications. Indeed, Theorem 1.2 is critically needed in the study of a structural decomposition property of the s.c. semigroup guaranteed by Proposition 1.4. For $\gamma > 0$, this semigroup is akin to an exponentially stable group (Kirchoff equation with damping) [**12**]. (By contrast, for $\gamma = 0$, such a semigroup is analytic [**11**]). It was precisely in the course of the structural decomposition study [**12**] that the need arose to establish an interior sharp result such as Theorem 1.2 for problem (1.1). As in the case of second-order hyperbolic equations [**6**], the proofs below use pseudo-differential operator techniques. By contrast, a sharp (optimal) trace regularity result for Kirchoff elastic or thermoelastic equations, with (coupled) thermal Neumann/mechanical hinged BC was recently obtained in the companion paper [**14**] by use of differential (rather than pseudo-differential) energy methods. Paper [**6**] used the general form of a second-order hyperbolic problem on a half-space, in the style of the Japanese school, e.g., [**17**]. By contrast, we employ here the important canonical form of the Laplacian in local coordinates near the boundary due to [**16**].

Remark 1.3. The analysis below, culminating with Theorem 6.1, equation (6.1), on the regularity of the solution $w_1 = \mathcal{X} w_c$ of the localized problem (4.12), is sharp. Instead, the analysis in Section 7 of the regularity of the localized solution $w_2 = (1 - \mathcal{X})w_c$ may surely be improved. However, such a task will require the use of a very technical apparatus, of the type used in [**6**] for another case of mixed problem, which does not satisfy the Lopatinski conditions (second-order hyperbolic equations with Neumann BC). For the purpose of achieving the sharp regularity result of Theorem 1.1, and its critical implication in [**12**], Section 7 is adequate. $\square$

2. An auxiliary problem. In this section we consider the following auxiliary problem, which will be invoked in the sequel:

$$\begin{aligned}
&(2.1a) && \mathcal{P}v \equiv v_{tt} - \gamma \Delta v_{tt} + \Delta^2 v = F && \text{in } Q = (0,T] \times \Omega,\\
&(2.1b) && v(0,\cdot) = v_0,\ v_t(0,\cdot) = v_1 && \text{in } \Omega,\\
&(2.1c) && \mathcal{B}_1 v \equiv \Delta v + B_1 v = \beta_1 && \text{in } \Sigma = (0,T] \times \Gamma,\\
&(2.1d) && \mathcal{B}_2 v \equiv \frac{\partial \Delta v}{\partial \nu} + B_2 v - \gamma \frac{\partial v_{tt}}{\partial \nu} = \beta_2 && \text{in } \Sigma,
\end{aligned}$$

under the following assumptions:

(i) that the solution v satisfies:

$$\{v, v_t\} \in C([0,T]; H^2(\Omega) \times H^1(\Omega)); \tag{2.2}$$

(ii) that the nonhomogeneous terms are such that the integrals

$$\int_Q F h \cdot \nabla v \, dQ, \ \int_\Sigma \beta_1^2 \, d\Sigma; \quad \int_\Sigma \beta_2 \frac{\partial v}{\partial \nu} \, d\Sigma, \tag{2.3}$$

are well-defined (finite).

In (2.3) $h(\xi) \in C^2(\overline{\Omega})$ is any vector field. The relevance of problem (2.1), in particular of the following regularity result, to our original problem (1.1), will become apparent in Section 4, where the v-problem (2.1) will in fact be the localized w_1-problem (4.12).

Theorem 2.1. *With reference to problem* (2.1) *satisfying assumptions* (2.2) *and* (2.3), *the following estimate holds true*:

$$\begin{aligned}\frac{\gamma}{2}\int_\Sigma |\nabla v_t|^2\,d\Sigma - C_\mu \int_\Sigma \left[\frac{\partial^2 v}{\partial\nu\partial\tau} + \frac{\partial^2 v}{\partial\tau^2}\right] d\Sigma& \\ = \mathcal{O}\bigg(\int_Q Fh\cdot\nabla v\,dQ, \int_\Sigma \beta_1^2\,d\Sigma,& \\ \int_\Sigma \beta_2 \frac{\partial v}{\partial\nu}\,d\Sigma, \|\{v,v_t\}\|^2_{C([0,T];H^2(\Omega)\times H^1(\Omega))}\bigg),&\end{aligned} \tag{2.4}$$

where $C_\mu > 0$ is a suitable positive constant depending on μ, see (1.2) *and* (1.3), *and where $h(\xi) \in C^2(\overline{\Omega})$ is any vector field such that $h|_\Gamma = \nu$.*

Proof of Theorem 2.1. We shall use energy methods. Assumptions (2.2) and (2.3) are in force.

Step 1.

Proposition 2.2. *The solution of the mixed problem* (2.1) *satisfies the following estimate*

$$\begin{aligned}\frac{\gamma}{2}\int_\Sigma |\nabla v_t|^2\,d\Sigma + \int_\Sigma \Delta v \frac{\partial^2 v}{\partial\nu^2}\,d\Sigma - \frac{1}{2}\int_\Sigma |\Delta v|^2\,d\Sigma& \\ - \int_\Sigma \left(\frac{\partial\Delta v}{\partial\nu} - \gamma\frac{\partial v_{tt}}{\partial\nu}\right)\frac{\partial v}{\partial\nu}\,d\Sigma& \\ = -\int_Q Fh\cdot\nabla v\,dQ + \mathcal{O}(\|\{v,v_t\}\|^2_{C([0,T];H^2(\Omega)\times H^1(\Omega))}),&\end{aligned} \tag{2.5}$$

where $h(\xi) \in C^2(\overline{\Omega})$ is a vector field such that $h|_\Gamma = \nu$.

Proof. Because of the regularity assumption (2.2), we may invoke Proposition A.1, equation (A.1) of Appendix A, with F in (2.1a) now replacing q in (1.1a), (A.1). We then readily see that the righthand side (RHS) of identity (A.1) becomes the righthand side of (2.5), where, moreover, in the $\mathcal{O}(\quad)$-term, we may also include the boundary term: $\int_\Sigma v_t^2 \, d\Sigma$ by trace theory. This way (2.5) is obtained. $\square$

Step 2. We now concentrate on the boundary terms on the lefthand side (LHS) of identity (2.5). Use will be made of the second BC (2.1d), and of the boundary relation in (1.4).

Proposition 2.3. *The following relations hold true for the boundary terms on the LHS of identity* (2.5):

(i)

$$\int_\Sigma \left(\frac{\partial \Delta v}{\partial \nu} - \gamma \frac{\partial v_{tt}}{\partial \nu} \right) \frac{\partial v}{\partial \nu} \, d\Sigma = (1-\mu) \int_\Sigma \left(\frac{\partial}{\partial \nu} \frac{\partial v}{\partial \tau} \right)^2 d\Sigma + \int_\Sigma \beta_2 \frac{\partial v}{\partial \nu} \, d\Sigma + \mathcal{O}(\|v\|^2_{C([0,T];H^2(\Omega))}); \tag{2.6}$$

(ii) *for any* $\varepsilon > 0$,

$$\left| \int_\Sigma \Delta v \frac{\partial^2 v}{\partial \nu^2} \, d\Sigma - \frac{1}{2} \int_\Sigma |\Delta v|^2 \, d\Sigma \right| \leq \varepsilon \int_\Sigma \left(\frac{\partial}{\partial \tau} \frac{\partial v}{\partial \tau} \right)^2 d\Sigma + C_\varepsilon \int_\Sigma |\Delta v|^2 \, d\Sigma + \mathcal{O}(\|v\|^2_{C([0,T];H^2(\Omega))}); \tag{2.7}$$

(iii) *for any* $\varepsilon > 0$,

$$\frac{\gamma}{2} \int_\Sigma |\nabla v_t|^2 \, d\Sigma - (1-\mu) \int_\Sigma \left(\frac{\partial}{\partial \nu} \left(\frac{\partial v}{\partial \tau} \right) \right)^2 d\Sigma - \varepsilon \int_\Sigma \left(\frac{\partial}{\partial \tau} \frac{\partial v}{\partial \tau} \right)^2 d\Sigma \leq C_\varepsilon \int_\Sigma |\Delta v|^2 \, d\Sigma + \mathcal{O}(\|\{v, v_t\}\|^2_{C([0,T];H^2(\Omega) \times H^1(\Omega))}) + \int_\Sigma \beta_2 \frac{\partial v}{\partial \nu} \, d\Sigma - \int_Q F h \cdot \nabla v \, dQ. \tag{2.8}$$

Proof. (i) Using the second BC (2.1d) with $B_2 v = (1-\mu)(\partial/\partial\tau)(\partial/\partial\nu)(\partial v/\partial\tau)$ by (1.6), we obtain

$$\int_\Sigma \left(\frac{\partial \Delta v}{\partial \nu} - \gamma \frac{\partial v_{tt}}{\partial \nu}\right)\frac{\partial v}{\partial \nu}\,d\Sigma - \int_\Sigma \beta_2 \frac{\partial v}{\partial \nu}\,d\Sigma$$

$$= -\int_\Sigma (B_2 v)\frac{\partial v}{\partial \nu}\,d\Sigma = -(1-\mu)\int_\Sigma \left(\frac{\partial}{\partial \tau}\left(\frac{\partial}{\partial \nu}\frac{\partial v}{\partial \tau}\right)\right)\frac{\partial v}{\partial \nu}\,d\Sigma \tag{2.9}$$

$$= (1-\mu)\int_\Sigma \left(\frac{\partial}{\partial \nu}\frac{\partial v}{\partial \tau}\right)\left(\frac{\partial}{\partial \tau}\frac{\partial v}{\partial \nu}\right)d\Sigma \tag{2.10}$$

$$= (1-\mu)\int_\Sigma \left(\frac{\partial}{\partial \nu}\frac{\partial v}{\partial \tau}\right)^2 d\Sigma + \text{ l.o.t.}, \tag{2.11}$$

where in going from (2.9) to (2.10) we have integrated by parts, while the lower order term l.o.t. in (2.11) is

$$\begin{aligned} \text{l.o.t.} &= (1-\mu)\int_\Sigma \left(\frac{\partial}{\partial \nu}\frac{\partial v}{\partial \tau}\right)\left(\left[\frac{\partial}{\partial \tau}, \frac{\partial}{\partial \nu}\right]v\right)d\Sigma \\ &= \mathcal{O}(\|v\|^2_{C([0,T];H^2(\Omega))}). \end{aligned} \tag{2.12}$$

Then (2.11) and (2.12) yield (2.6).

(ii) Using identity (1.4) to eliminate $\partial^2 v/\partial \nu^2$, we obtain

$$\left|\int_\Sigma \Delta v \frac{\partial^2 v}{\partial \nu^2}\,d\Sigma - \frac{1}{2}\int_\Sigma |\Delta v|^2\,d\Sigma\right|$$

$$= \left|\int_\Sigma \Delta v\left[\Delta v - \frac{\partial^2 v}{\partial \tau^2} - k\frac{\partial v}{\partial \nu}\right]d\Sigma - \frac{1}{2}\int_\Sigma |\Delta v|^2\,d\Sigma\right| \tag{2.13}$$

$$= \left|\frac{1}{2}\int_\Sigma |\Delta v|^2\,d\Sigma - \int_\Sigma \Delta v\left[\frac{\partial^2 v}{\partial \tau^2} + k\frac{\partial v}{\partial \nu}\right]d\Sigma\right| \tag{2.14}$$

$$\le \varepsilon \int_\Sigma \left(\frac{\partial^2 v}{\partial \tau^2}\right)^2 d\Sigma + C_\varepsilon \int_\Sigma |\Delta v|^2\,d\Sigma + 2\int_\Sigma k^2\left(\frac{\partial v}{\partial \nu}\right)^2 d\Sigma, \tag{2.15}$$

for any $\varepsilon > 0$. Then (2.15) readily yields (2.7) by trace theory.

(iii) We return to identity (2.5), where we use (2.6) and (2.7) to obtain (2.8), as desired. □

Step 3. Comparing estimate (2.8) with the desired estimate (2.4) of Theorem 2.1, we see that we need to estimate the integral term on Δv in the RHS of (2.8). To this end, we invoke the first BC (2.1c).

Proposition 2.4. *For any $\varepsilon > 0$, we have*

$$\int_\Sigma |\Delta v|^2 \, d\Sigma \leq [(1-\mu)^2 + \varepsilon] \int_\Sigma \left(\frac{\partial^2 v}{\partial \tau^2}\right)^2 d\Sigma + \mathcal{O}\left(\int_\Sigma \beta_1^2 \, d\Sigma, \|v\|^2_{C([0,T];H^2(\Omega))}\right). \tag{2.16}$$

Proof. We recall the BC (2.1c) where $B_1 v$ is given by (1.4). This way, we compute

$$\int_\Sigma |\Delta v|^2 \, d\Sigma = \int_\Sigma |\beta_1 - B_1 v|^2 \, d\Sigma$$

$$= \int_\Sigma \left|\beta_1 + (1-\mu)\left[\frac{\partial^2 v}{\partial \tau^2} + k\frac{\partial v}{\partial \nu}\right]\right|^2 d\Sigma \tag{2.17}$$

$$\leq [(1-\mu)^2 + \varepsilon] \int_\Sigma \left(\frac{\partial^2 v}{\partial \tau^2}\right) d\Sigma + \mathcal{O}\left(\int_\Sigma \beta_1^2 \, d\Sigma + \int_\Sigma k^2 \left(\frac{\partial v}{\partial \nu}\right)^2 d\Sigma\right). \tag{2.18}$$

Using trace theory on the last integral term of (2.18), we then obtain (2.16). □

Step 4. Using estimate (2.16) into the RHS of estimate (2.8) yields the desired estimate (2.4). Theorem 2.1 is proved. □

Remark 2.1. *Claim.* Assumptions (2.3) hold true, for instance, when (a)

$$F \equiv f_t, \quad \text{where } f \in C([0,T]; L_2(\Omega)), \tag{2.19}$$

(b)

$$\beta_2 \in L_2(0,T;H^{-1/2}(\Gamma)), \tag{2.20}$$

and of course $\beta_1 \in L_2(0,T;L_2(\Gamma))$.

Indeed, (2.20) combined with $(\partial v/\partial \nu) \in L_2(0,T;H^{1/2}(\Gamma))$ (by trace theory on the a-priori interior regularity of v in (2.2)) makes the third integral term in (2.3) well-defined.

Moreover, with F as in (2.19), integrating by parts in t yields, as desired:

$$\begin{aligned}\int_Q Fh\cdot\nabla v dQ &= \int_\Omega \int_0^T f_t h\cdot\nabla v\, dt\, d\Omega \\ &= \left[\int_\Omega fh\cdot\nabla v\, d\Omega\right]_0^T - \int_Q fh\cdot\nabla v_t dQ \\ &= \text{well-defined}\end{aligned}$$

in view also of the a-priori regularity of $\{v, v_t\}$ in (2.2). The above Claim will be critically invoked in Section 9 in appealing to Theorem 2.1.

3. Reduction to Melrose-Sjöstrand coordinates over a collar domain. As $\Delta = (\partial^2/\partial\xi_1^2) + (\partial^2/\partial\xi_2^2)$ in problem (1.1) over the original domain Ω is a second-order differential operator on Ω with real (principal) symbol $-(\zeta_1^2+\zeta_2^2)$ and with noncharacteristic boundary, then near any point $\xi \in \Gamma = \partial\Omega$ we may choose [**16**, pp. 597–598] local coordinates (x, y), centered at ξ, such that Ω is locally given by $x \geq 0$ and the Laplacian Δ is replaced by

$$\tilde{\Delta} = D_x^2 + R(x,y,D_y), \qquad D_x = \frac{\partial}{\partial x}; \quad D_y = \frac{\partial}{\partial y}, \tag{3.1}$$

where R is a *second-order differential operator in the y variable only*, with smooth coefficients of real principal type for each fixed x. Hence, in our two-dimensional case, $R(x,y,D_y)$ is given explicitly by

$$R(x,y,D_y) = \rho(x,y)D_y^2 + \text{l.o.t. in } D_y, \tag{3.2}$$

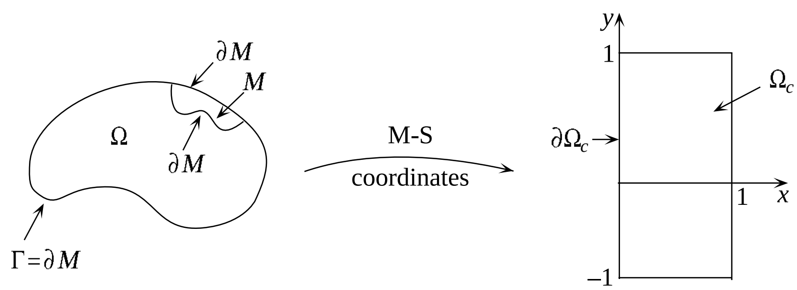

FIGURE 1.

with $\rho(x, y)$ real and smooth. Thus, henceforth, we may consider the original problem (1.1) as defined on the collar domain

$$\Omega_c = \{0 \leq x < 1; |y| < 1\}, \tag{3.3}$$

where Δ is replaced by $\tilde{\Delta}$ as given in Ω_c by (3.1), (3.2) and $\rho(x, y)$ is real and smooth on $\overline{\Omega}_c$.

Such a new problem over Ω_c may be viewed as corresponding to the original problem (1.1), defined however only over a boundary (collar) subdomain M of Ω and acting on the solution w having compact support on $\partial M \cap \Omega$ after the change of coordinates $\xi = (\xi_1, \xi_2) \in M \to (x, y) \in \Omega_c$. Consequently, the new problem over Ω_c with $\tilde{\Delta}$ given here by (3.1) may be considered for a solution w vanishing as follows

$$w \text{ has compact support} \quad \text{for } x = 1 \quad \text{and for } |y| = 1. \tag{3.4}$$

As finitely many subdomains such as M will cover the full collar of $\Gamma = \partial\Omega$, boundary estimates at $x = 0$ obtained for the new problem over Ω_c provide corresponding boundary estimates of the original problem over Γ.

Henceforth, we shall work with the w-problem (1.1) on the domain Ω_c in (3.3) with $\tilde{\Delta}$ given by (3.1) and (3.2) and with solutions w vanishing as in (3.4).

4. Beginning of proof of Theorem 1.1 in the new variables over Ω_c: Time and dual space localization.

Time localization. Let $\phi(t) \in C_0^\infty(-\infty, \infty)$ be a cut-off function such that

$$\begin{aligned} &\phi(t) \equiv 1, \quad t \in [0, T], \\ \text{and} \quad &\phi(t) \equiv 0 \quad \text{for } t \leq -\frac{T}{2} \\ \text{and for} \quad &t \geq \frac{3}{2}T; \quad \operatorname{supp}\phi \in \left(-\frac{T}{2}, \frac{3}{2}T\right), \end{aligned} \tag{4.1}$$

and set a new variable w_c (the subscript "c" reminds us that w_c is a cut-off of w), defined on Ω_c in (3.3) by

$$\begin{aligned} w_c(t, x, y) &= \phi(t) w(t, x, y); \quad (x, y) \in \Omega_c; \\ w(t, 1, y) &\equiv w(t, x, \pm 1) \equiv 0. \end{aligned} \tag{4.2}$$

Lemma 4.1. *In terms of the new variable w_c, the original problem*: $\mathcal{P}w = q$ *in* Q; $\mathcal{B}_1 w = 0$ *and* $\mathcal{B}_2 w = 0$ *in* Σ, *in* (1.1), *becomes over* $Q_{c,\infty} = (-\infty, \infty) \times \Omega_c$, *and* $\Sigma_{c,\infty} = (-\infty, \infty) \times (\Omega_c|_{x=0})$ *and with* Δ *given by* (3.1), *as follows,*

(4.3a)
$$\tilde{\mathcal{P}} w_c \equiv w_{c,tt} - \gamma \tilde{\Delta} w_{c,tt} + \tilde{\Delta}^2 w_c = [\tilde{\mathcal{P}}, \phi] w + (\phi q), \quad \text{in } Q_{c,\infty},$$
(4.3b)
$$\tilde{\mathcal{B}}_1 w_c \equiv \tilde{\Delta} w_c + \tilde{B}_1 w_c \equiv 0, \quad \text{in } \Sigma_{c,\infty},$$
(4.3c)
$$\tilde{\mathcal{B}}_2 w_c = \frac{\partial \tilde{\Delta} w_c}{\partial \nu} + \tilde{B}_2 w_c - \gamma \frac{\partial w_{c,tt}}{\partial \nu} \equiv [\tilde{\mathcal{B}}_2, \phi] w, \quad \text{in } \Sigma_{c,\infty},$$

where $(\partial/\partial\nu) = (\partial/\partial x)$ *and the commutators are*

(4.4)
$$[\tilde{\mathcal{P}}, \phi] w = -2\gamma \phi_t \tilde{\Delta} w_t - \gamma \phi_{tt} \tilde{\Delta} w + \phi_{tt} w + 2\phi_t w_t, \quad \text{in } Q_{c,\infty};$$
(4.5)
$$[\tilde{\mathcal{B}}_2, \phi] w = -\gamma \phi_{tt} \frac{\partial w}{\partial \nu} - 2\gamma \phi_t \frac{\partial w_t}{\partial \nu} \quad \text{in } \Sigma_{c,\infty}.$$

Proof. Direct verification. □

Dual space localization. Given the original variables t (time) and y (tangential direction at the boundary), let σ and η be the corresponding dual Fourier variables: $t \to \sigma$; $y \to \eta$. We shall need to micro-localize problem (4.3). To this end, by symmetry, we may restrict our attention to the quarter space $\mathbf{R}^3_{+,\sigma\eta} = \{\sigma > 0, \eta_1 > 0, \eta_2 > 0\}$ of the $\{\sigma, \eta\}$-space $\mathbf{R}^3_{\sigma\eta}$. As in [**8**], [**9**], define the following cones:

(4.6)
$$\mathcal{R}_1 = \{[\sigma,\eta] \in \mathbf{R}^3_{+,\sigma\eta} : \sigma \geq c_1|\eta|\},$$

(4.7)
$$\mathcal{R}_{\mathrm{tr}} = \{[\sigma,\eta] \in \mathbf{R}^3_{+,\sigma\eta} : c_2|\eta| < \sigma < c_1|\eta|\}, \quad c_2 = c_1 - \delta,\ \delta > 0,$$

(4.8)
$$\mathcal{R}_2 = \{[\sigma,\eta] \in \mathbf{R}^3_{+,\sigma\eta} : 0 < \sigma \leq c_2|\eta|\},$$

for constant $c_1 > 0$ to be determined sufficiently large in Section 5 below (Theorem 5.1) and $\delta > 0$ arbitrarily small. We have $\mathbf{R}^3_{+,\sigma\eta} = \mathcal{R}_1 \cup \mathcal{R}_{\mathrm{tr}} \cup \mathcal{R}^2$.

Symbol of localization $\chi(\sigma, \eta)$ **and corresponding pseudo-differential operator** $\mathcal{X} \in OPS^0$**.** With reference to the above cones in (4.6)–(4.8), let $\chi(\sigma, \eta) \in S^0$ be a homogeneous symbol of localization of order zero (i.e., a C^∞-homogeneous function of order zero in both variables $[\sigma, \eta]$) such that

$$\begin{aligned} \mathcal{X}(\sigma,\eta) &\equiv 1 \text{ in } \mathcal{R}_1; \quad \operatorname{supp}\chi \subset \mathcal{R}_1 \cup \mathcal{R}_{\mathrm{tr}}\,; \\ [1-\chi(\sigma,\eta)] &\equiv 1 \text{ in } \mathcal{R}_2; \quad \operatorname{supp}(1-\chi) \subset \mathcal{R}_2 \cup \mathcal{R}_{\mathrm{tr}}\,. \end{aligned} \tag{4.9}$$

Let $\mathcal{X} \in OPS^0$ be the pseudo-differential operator of order zero generated by the symbol χ.

Localized $(\mathcal{X}w_c)$**-problem.** With reference to the solution w_c of problem (4.3), we write:

$$w_c = \mathcal{X}w_c + (1-\mathcal{X})w_c = w_1 + w_2, \tag{4.10}$$

where, with $\bar{w}_0 = w(0,\cdot)$ and $\bar{w}_1 = w_t(0,\cdot)$:

$$\begin{aligned} &\{\bar{w}_0, \bar{w}_1, q\} \in H^2(\Omega) \times H^1(\Omega) \times L_1(0,T;[H^1(\Omega)]') \\ &\quad \Rightarrow w_1 = \mathcal{X} w_c,\ w_2 = (1-\mathcal{X}) w_c \in C(R^1_t; H^2(\Omega_c)) \cap C^1(R^1_t; H^1(\Omega_c)), \end{aligned} \tag{4.11}$$

$R^1_t = (-\infty, \infty)$ in the t-variable, where the indicated regularity follows from (1.8) of Proposition 1.0, (4.2) and $\mathcal{X} \in OPS^0$. Next, we apply $\mathcal{X}$ to the equations of problem (4.3), keep track of the commutators, and obtain

Lemma 4.2. *The new variable $w_1 = \mathcal{X} w_c$ in* (4.11) *solves the following localized mixed problem*

$$\tilde{\mathcal{P}} w_1 \equiv w_{1,tt} - \gamma \tilde{\Delta} w_{1,tt} + \tilde{\Delta}^2 w_1 = f \quad \text{in } Q_{c,\infty}, \tag{4.12a}$$

$$\tilde{\mathcal{B}}_1 w_1 \equiv \tilde{\Delta} w_1 + \tilde{B}_1 w_1 = g_1 \quad \text{in } \Sigma_{c,\infty}, \tag{4.12b}$$

$$\tilde{\mathcal{B}}_2 w_1 \equiv \frac{\partial \tilde{\Delta} w_1}{\partial \nu} + \tilde{B}_2 w_1 - \gamma \frac{\partial w_{1,tt}}{\partial \nu} = g_2 \quad \text{in } \Sigma_{c,\infty}, \tag{4.12c}$$

where, with reference to (4.4) *and* (4.5), *we have*:

$$f \equiv \mathcal{X}[\tilde{\mathcal{P}}, \phi] w + \mathcal{X}(\phi q) + [\tilde{\mathcal{P}}, \mathcal{X}] w_c \quad \text{in } Q_{c,\infty}, \tag{4.13}$$

$$g_1 = [\tilde{\mathcal{B}}_1, \mathcal{X}] w_c \quad \text{in } \Sigma_{c,\infty}, \tag{4.14}$$

$$g_2 = \mathcal{X}[\tilde{B}_2, \phi] w + [\tilde{\mathcal{B}}_2, \mathcal{X}] w_c \quad \text{in } \Sigma_{c,\infty}. \tag{4.15}$$

The following estimates will be critically used in the sequel. To this end, we set for convenience

$$E_0 \equiv \|\{\bar{w}_0, \bar{w}_1, q\}\|^2_{H^2(\Omega) \times H^1(\Omega) \times L_1(0,T;[H^1(\Omega)]')}. \tag{4.16}$$

Proposition 4.3. *With reference to the nonhomogeneous terms f, g_1, g_2 in* (4.12), *defined by* (4.13)–(4.15) *and in the notation of* (4.16), *we have*:

(i) *let $h(x)$ be the vector field $h(x) = [-1, 0]$ on Ω_c, so that $h|_{x=0} = \nu$. Then*

$$\int_{Q_{c,\infty}} fh \cdot \nabla w_1 \, dQ = \mathcal{O}_T(E_0); \tag{4.17}$$

(ii) *moreover, with $\Sigma_{c,\infty} = (-\infty, \infty) \times (\Omega_c|_{x=0})$:*

$$\int_{\Sigma_{c,\infty}} g_1^2 \, d\Sigma = \mathcal{O}_T(E_0); \quad \int_{\Sigma_{c,\infty}} g_2 \frac{\partial w_1}{\partial \nu} \, d\Sigma = \mathcal{O}_T(E_0), \tag{4.18}$$

where $a = \mathcal{O}_T(E_0)$ means $|a| \leq \text{const}_T E_0$, as usual.

Proof. See Appendix B. □

5. Proof of Theorem 1.1 in the new variables over Ω_c: Preliminary analysis of the w_1-problem (4.12). In this section we analyze the trace regularity of the localized mixed problem (4.12) for $w_1 = \mathcal{X} w_c$. Problem (4.12) for $w_1 = \mathcal{X} w_c$, except that it is defined on the collar domain Ω_c in (3.3) with Δ replaced by $\tilde{\Delta}$ as defined in (3.1), is precisely the same as problem (2.1) for v defined on the original domain Ω, with nonhomogeneous terms $f = F$, $g_1 = \beta_1$, $g_2 = \beta_2$, which satisfy the required assumptions (2.2) and (2.3) on two grounds: (i) the a-priori regularity property (4.11) for w_1 (ultimately due to Proposition 1.0, equation (1.8)); (ii) the estimates for the nonhomogeneous terms $f = F$, $g_1 = \beta_1$, $g_2 = \beta_2$ guaranteed by Proposition 4.3, equations (4.17) and (4.18) with E_0 as in (4.16). We can then appeal to Theorem 2.1 combined with the above properties, (4.11) and (4.17)–(4.18), to obtain our main result for the w_1-problem (4.12).

Theorem 5.1. *The solution $w_1 = \mathcal{X} w_c$ of the localized problem (4.12) satisfies the following estimate, where E_0 is defined in (4.16) and $\Sigma_{c,\infty} = R_t^1 \times (\Omega_c|_{x=0})$:*

$$\frac{\gamma}{2} \int_{\Sigma_{c,\infty}} |\text{grad}\, w_{1,t}|^2 \, d\Sigma - C_\mu \int_{\Sigma_{c,\infty}} |\text{grad}\, (D_y w_1)|^2 \, d\Sigma = \mathcal{O}_T(E_0), \tag{5.1}$$

where $C_\mu > 0$ is a suitable constant depending on μ, and grad $=$ $[D_x, D_y]$.

Proof. Invoke Theorem 2.1, equation (2.4), equation (4.11) and Proposition 4.3, equations (4.17) and (4.18). □

6. Proof of Theorem 1.1.: Final trace estimate for the mixed w_1-problem (4.12). Up to now the 'size' of the cone $\mathcal{R}_1$ defined in (4.6), hence of the cone $\mathcal{R}_1 \cup \mathcal{R}_{\text{tr}}$, plays no role; more precisely, the magnitude of the constant $c_1 > 0$ in the definition of $\mathcal{R}_1$ in (4.6), hence of the constant $c_2 = c_1 - \delta$, $\delta > 0$ in (4.7), may be arbitrary. In the next final result in the analysis of the w_1-problem, the constant c_1 will have to be, however, sufficiently large.

Theorem 6.1. *With reference to the cone $\mathcal{R}_1$ defined in* (4.6)*, there exists a constant $c_1 > 0$ sufficiently large as in* (6.2) *below, such that the corresponding localized problem w_1 in* (4.12) *satisfies the following trace estimate*

$$\int_0^T \int_{\Gamma_c} |\operatorname{grad} w_{1,t}|^2 \, d\Sigma \le \text{const}_T E_0, \tag{6.1}$$

with E_0 defined by (4.16)*,* $\operatorname{grad} = [D_x, D_y]$ *and* $\Gamma_c = \Omega_c|_{x=0}$.

Proof. Let $\Sigma_\infty = R_t^1 \times R_y^1$. Let $\hat{w}_1(\sigma, 0, \eta)$ be the Fourier transform of $w_1(t, 0, y) = w_1(t, \cdot)|_{x=0}$, the solution of (4.12) evaluated at the boundary Σ_∞.

Also, if $C_\mu > 0$ is the constant in estimate (5.1) and $c_2 = c_1 - \delta$ is the constant in definition (4.7), we shall select in (6.6) below a constant $\rho_0 > 0$ defined by

$$\rho_0 = \frac{\gamma}{2} - \frac{C_\mu}{c_2^2} > 0, \quad \text{which is possible with } c_1 > \sqrt{\frac{2C_\mu}{\gamma}}, \tag{6.2}$$

since $c_2 = c_1 - \delta$ with $\delta > 0$ which can be taken arbitrarily small. With reference to the lefthand side of estimate (5.1), we compute, by Plancherel theorem, recalling that $\operatorname{supp} \hat{w}_1(\sigma, 0, \eta) \subset \mathcal{R}_1 \cup \mathcal{R}_{\text{tr}}$ since $\hat{w}_1(\sigma, x, \eta) = \chi(\sigma, \eta) \hat{w}_c(\sigma, x, \eta)$ and that, in $\mathcal{R}_1 \cup \mathcal{R}_{\text{tr}}$, we have $\sigma > c_2|\eta|$

by (4.6), (4.7):

$$\frac{\gamma}{2}\int_{\Sigma\infty}\left|\frac{\partial}{\partial t}\operatorname{grad} w_1\right|^2 d\Sigma - C_\mu \int_{\Sigma_\infty} |D_y(\operatorname{grad} w_1)|^2\, d\Sigma$$

$$= \frac{\gamma}{2} 2 \int_{\mathcal{R}_1\cup\mathcal{R}_{\mathrm{tr}}} |\sigma|^2 |\widehat{\operatorname{grad} w_1}(\sigma,0,\eta)|^2\, d\sigma\, d\eta$$

$$- C_\mu 2 \int_{\mathcal{R}_1\cup\mathcal{R}_{\mathrm{tr}}} |\eta|^2 |\widehat{\operatorname{grad} w_1}(\sigma,0,\eta)|^2\, d\sigma\, d\eta \tag{6.3}$$

$$= 2\int_{\mathcal{R}_1\cup\mathcal{R}_{\mathrm{tr}}} \left[\frac{\gamma}{2}|\sigma|^2 - C_\mu|\eta|^2\right] |\widehat{\operatorname{grad} w_1}(\sigma,0,\eta)|^2\, d\sigma\, d\eta \tag{6.4}$$

(by (4.6), (4.7))

$$\geq 2\int_{\mathcal{R}_1\cup\mathcal{R}_{\mathrm{tr}}} \left[\frac{\gamma}{2}|\sigma|^2 - \frac{C_\mu}{c_2^2}|\sigma|^2\right] |\widehat{\operatorname{grad} w_1}(\sigma,0,\eta)|^2\, d\sigma\, d\eta \tag{6.5}$$

(by (6.2))

$$= 2\rho_0 \int_{\mathcal{R}_1\cup\mathcal{R}_{\mathrm{tr}}} |\sigma^2|\, |\widehat{\operatorname{grad} w_1}(\sigma,0,\eta)|^2\, d\sigma\, d\eta \tag{6.6}$$

$$= \rho_0 \int_{\Sigma_\infty} |\operatorname{grad}(w_{1,t})|^2\, d\Sigma. \tag{6.7}$$

Thus (6.7) and (5.1) together imply

$$\rho_0 \int_0^T \int_{\Gamma_c} |\operatorname{grad}(w_{1,t})|^2\, d\Gamma\, dt \leq \rho_0 \int_{\Sigma_\infty} |\operatorname{grad}(w_1,t)|^2\, d\Sigma \tag{6.8}$$

(by (6.7))

$$\leq \frac{\gamma}{2}\int_{\Sigma_\infty} |\operatorname{grad}(w_{1,t})|^2\, d\Sigma - C_\mu \int_{\Sigma_\infty} |\operatorname{grad}(D_y w_1)|^2\, d\Sigma \tag{6.9}$$

(by (5.1))

$$\leq C_T E_0, \tag{6.10}$$

and Theorem 6.1, equation (6.1) is proved via (6.10). ◻

7. Proof of Theorem 1.1: Analysis of $w_2 = (1-\mathcal{X})w_c$. As noted in Remark 1.4, a deep analysis of the solution $w_2 = (1-\mathcal{X})w_c$ of a mixed problem such as (4.12) will require the technical methods of [**6**]. Here we shall content ourselves with the following direct analysis, which is sufficient in establishing Theorem 1.1. To begin with, we have, by (4.11) and trace theory, that $w_2 = (1-\mathcal{X})w_c \in C(R_t^1; H^2(\Omega_c))$ continuously in E_0 defined in (4.16); hence, see (3.3),

$$\left\| \frac{\partial w_2}{\partial x} \right\|^2_{C(R_t^1; H^{1/2}(\Gamma_c))} \leq C_T E_0, \tag{7.1}$$

where $\Gamma_c = \Omega_c|_{x=0}$, as a consequence of the a-priori regularity of Proposition 1.0, equation (1.8). Then

Proposition 7.1. *With reference to the localized function $w_2 = (1-\mathcal{X})w_c$ we have*

$$\begin{aligned} \left\| \frac{\partial w_{2,t}}{\partial x} \right\|^2_{L_2(0,T;H^{-1/2}(\Gamma_c))} &\leq C_T \left\| \frac{\partial w_2}{\partial x} \right\|^2_{L_2(R_t^1; H^{1/2}(\Gamma_c))} \\ &\leq C_T E_0, \quad \Gamma_c = \Omega_c|_{x=0}. \end{aligned} \tag{7.2}$$

Proof. Again we can take, as in Section 4 and Appendix B, that $\Sigma_\infty = R_t^1 \times R_y^1$. As there, we let $\hat{w}_2(\sigma, 0, \eta)$ be the Fourier transform of $w_2(t,0,y) = w_2(t,\cdot)|_{x=0}$, the boundary value of the function w_2 evaluated on the boundary in the $\{x,y\}$-collar coordinates. Finally, we recall that $\operatorname{supp} w_2(\sigma,0,\eta) \subset \mathcal{R}_2 \cup \mathcal{R}_{\text{tr}}$, since $\hat{w}_2(\sigma,x,\eta) = (1 - \chi(\sigma,\eta)\hat{w}_c(\sigma,x,\eta)$ and that $0 < \sigma < c_1|\eta|$ in $\mathcal{R}_2 \cup \mathcal{R}_{\text{tr}}$ by (4.7) and (4.8). We then compute by the Plancherel theorem,

$$\left\| \frac{\partial}{\partial x} w_{2,t} \right\|^2_{L_2(0,T;H^{-1/2}(\Gamma_c))} \leq \int_{R_t^1} \left\| \frac{\partial w_{2,t}}{\partial x} \right\|^2_{H^{-1/2}(\Gamma_c)} dt \tag{7.3}$$

$$= 2 \int_{\mathcal{R}_2 \cup \mathcal{R}_{\text{tr}}} \left| \frac{\sigma}{|\eta|^{1/2}} \frac{\partial \hat{w}_2}{\partial x}(\sigma, 0, \eta) \right|^2 d\sigma \, d\eta \tag{7.4}$$

(by (4.7) and (4.8))

$$\leq 2c_1^2 \int_{\mathcal{R}_2 \cup \mathcal{R}_{\mathrm{tr}}} \left| |\eta|^{1/2} \frac{\partial \hat{w}_2}{\partial x}(\sigma, 0, \eta) \right|^2 d\sigma\, d\eta \tag{7.5}$$

$$= c_1^2 \int_{R_t^1} \left\| \frac{\partial w_2}{\partial x} \right\|^2_{H^{1/2}(\Gamma_c)} dt \leq C_T E_0, \tag{7.6}$$

where in the last step we have invoked (7.1). Then (7.6) proves (7.2), as desired. □

The above result is all that is needed to complete the proof of Theorem 1.1, see Section 8. However, in order to prove Theorem 9.1 in Section 9, we shall need the following companion result for $w_2 = (1 - \mathcal{X})w_c$.

Proposition 7.2. *With reference to the localized function $w_2 = (1 - \mathcal{X})w_c$, we have*

$$\begin{aligned} \left\| \frac{\partial w_{2,t}}{\partial x} \right\|^2_{H^1(0,T;H^{-3/2}(\Gamma_c))} &\leq C_T \left\| \frac{\partial w_2}{\partial x} \right\|^2_{L_2(R_t^1;H^{1/2}(\Gamma_c))} \\ &\leq C_T E_0, \quad \Gamma_c = \Omega_c|_{x=0}. \end{aligned} \tag{7.7}$$

Proof. The proof is similar to that of Proposition 7.1, except that it trades, in the cone $\mathcal{R}_2 \cup \mathcal{R}_{\mathrm{tr}}$, a loss of regularity in the tangential space variable with a corresponding gain in the time variable. With the same notation used in the proof of Proposition 7.1, we compute, again by the Plancherel Theorem,

$$\left\| \frac{\partial w_{2,t}}{\partial x} \right\|^2_{H^1(0,T;H^{-3/2}(\Gamma_c))} \leq C_T \int_{R_t^1} \left\| \frac{\partial w_{2,tt}}{\partial x} \right\|^2_{H^{-3/2}(\Gamma_c))} dt \tag{7.8}$$

$$= 2 \int_{\mathcal{R}_2 \cup \mathcal{R}_{\mathrm{tr}}} \left| \frac{\sigma^2}{|\eta|^{3/2}} \frac{\partial \hat{w}_2}{\partial x}(\sigma, 0, \eta) \right|^2 d\sigma\, d\eta \tag{7.9}$$

(by (4.7), (4.8))

$$\leq 2c_1^4 \int_{\mathcal{R}_2 \cup \mathcal{R}_{\mathrm{tr}}} \left| \frac{|\eta|^2}{|\eta|^{3/2}} \frac{\partial \hat{w}^2}{\partial x}(\sigma, 0, \eta) \right|^2 d\sigma\, d\eta \tag{7.10}$$

$$= 2c_1^4 \int_{\mathcal{R}_2 \cup \mathcal{R}_{\text{tr}}} \left| |\eta|^{1/2} \frac{\partial \hat{w}_2}{\partial x}(\sigma, 0, \eta) \right|^2 d\sigma\, d\eta \tag{7.11}$$

$$= c_1^4 \int_{R_t^1} \left\| \frac{\partial w_2}{\partial x} \right\|^2_{H^{1/2}(\Gamma_c)} dt \le C_T E_0, \tag{7.12}$$

where in the last step we have again invoked (7.1). Then (7.12) proves (7.7), as desired.

Remark 7.1. In the results expressed by (7.2) and (7.7), the sum of Sobolev indices in time and space remains the same: $0 - (1/2) = 1 - (3/2)$. □

8. Completion of the proof of Theorem 1.1. Returning to equation (4.10) and (4.1), (4.2), we obtain, since $\phi \equiv 1$ on $[0, T]$:

$$w \equiv w_c \equiv w_1 + w_2, \quad \text{hence } \frac{\partial w_t}{\partial \nu} = \frac{\partial w_{1,t}}{\partial \nu} + \frac{\partial w_{2,t}}{\partial \nu} \quad \text{on } [0, T], \tag{8.1}$$

where, by Theorem 6.1, equation (6.1) on $w_{1,t}$, and by Propositions 7.1 and 7.2, equations (7.2) and (7.7), on $w_{2,t}$ we have:

$$\begin{aligned} &\frac{\partial w_{1,t}}{\partial \nu} \in L_2(0, T; L_2(\Gamma_c)); \\ &\frac{\partial w_{2,t}}{\partial \nu} \in L_2(0, T; H^{-1/2}(\Gamma_c)) \cap H^1(0, T; H^{-3/2}(\Gamma_c)). \end{aligned} \tag{8.2}$$

Thus, (8.2) used in (8.1) shows in particular that $\partial w_t / \partial \nu \in L_2(0, T; H^{-1/2}(\Gamma_c))$, and this then establishes the conclusion (1.9) of Theorem 1.1, via Section 3. Theorem 1.1 is proved. □

9. Further regularity results of the thermoelastic system. Armed with the technical background of the preceding sections, we can now return to the boundary nonhomogeneous thermoelastic problem (1.19)—where we now switch to the more appealing variables $\{z, \theta\}$, rather than $\{y, \alpha\}$.

$$z_{tt} - \gamma \Delta z_{tt} + \Delta^2 z + \Delta \theta = 0 \quad \text{in } (0, T] \times \Omega = Q; \tag{9.1a}$$

$$\theta_t - \Delta \theta - \Delta z_t = 0 \quad \text{in } Q; \tag{9.1b}$$

(9.1c)
$$z(0,\cdot)=0;\ z_t(0,\cdot)=0;\ \theta(0,\cdot)=0 \quad \text{in } \Omega;$$

(9.1d)
$$\Delta z + B_1 z + \theta = g_1 \quad \text{in } (0,T]\times\Gamma = \Sigma;$$

(9.1e)
$$\frac{\partial \Delta z}{\partial \nu} + B_2 z - \gamma \frac{\partial z_{tt}}{\partial \nu} + \frac{\partial \theta}{\partial \nu} = g_2 \quad \text{in } \Sigma;$$

(9.1f)
$$\frac{\partial \theta}{\partial \nu} + b\theta = 0, \quad b \geq 0 \quad \text{in } \Sigma.$$

Theorem 9.1. *With reference to problem* (9.1)*, let*

(9.2)
$$g_1 \in L_2(0,T;H^{1/2}(\Gamma)); \quad g_2 \in L_2(0,T;H^{-1/2}(\Gamma)).$$

Then, the following regularity results hold true, continuously in g_1, g_2, assumed as in (9.2):

(a)

(9.3)
$$\{z, z_t, \theta\} \in C([0,T]; H^2(\Omega)\times H^1(\Omega)\times H^{1/2}(\Omega))$$

(*this refines, for the variable θ, the regularity of Theorem* 1.6*, equation* (1.20)*, by boosting the regularity of θ from $L_2(\Omega)$ to $H^{1/2}(\Omega)$*):

(b)

(9.4)
$$\theta \in L_2(0,T;H^1(\Omega));\ \theta|_\Gamma \in L_2(0,T;H^{1/2}(\Gamma));$$

(c)

(9.5)
$$z_{tt} \in L_2(0,T;L_2(\Omega));\ \Delta z_t \in L_2(0,T;[H^1(\Omega)]');$$

(d)

(9.6)
$$\frac{\partial z_t}{\partial \nu} \in L_2(0,T;H^{-1/2}(\Gamma))$$

(*a more refined result for* $(\partial z_t/\partial\nu)$ *will be given in Proposition* 9.3 *below*).

Proof. Step 1. By Theorem 1.6, we have the following preliminary interior regularity result (in the new notation)

$$\{z, z_t, \theta\} \in C([0,T]; H^2(\Omega) \times H^1(\Omega) \times L_2(\Omega)) \tag{9.7}$$

continuously in g_1, g_2 assumed as in (9.2).

Step 2. We rewrite problem (9.1) as

$$z_{tt} - \gamma \Delta z_{tt} + \Delta^2 z = -\theta_t + \Delta z_t \quad \text{in } Q; \tag{9.8a}$$

(9.8b)

$$z(0,\cdot) = 0;\ z_t(0,\cdot) = 0;\ \theta(0,\cdot) = 0 \quad \text{in } \Omega;$$

$$\Delta z + B_1 z = -\theta|_\Gamma + g_1 \quad \text{in } \Sigma; \tag{9.8c}$$

$$\frac{\partial \Delta z}{\partial \nu} + B_2 z - \gamma \frac{\partial z_{tt}}{\partial \nu} = b\theta|_\Gamma + g_2 \quad \text{in } \Sigma, \tag{9.8d}$$

after substituting $\Delta\theta$ from (9.1b) into (9.1a) to get (9.8a), and after using (9.1f) to get (9.8d). By the a-priori regularity (9.7), we have that: $f_t \equiv -\theta_t - \Delta z_t$ satisfies $f \in C([0,T]; L_2(\Omega))$. Recalling the claim in Remark 2.1, as well as (9.7) for $\{z, z_t\}$, we see that we can invoke Theorem 2.1 for the Kirchoff problem (9.8), with the boundary terms in (9.8c) and (9.8d) penalized as

$$\begin{aligned} &\beta_1 \equiv -\theta|_\Gamma + g_1 \in L_2(0,T; L_2(\Gamma)); \\ &\beta_2 \equiv b\theta|_\Gamma + g_2 \in L_2(0,T; H^{-1/2}(\Gamma)), \end{aligned} \tag{9.9}$$

see (2.20), where the quantities in (9.9) will be shown to be well defined below. The proof of Theorem 1.5, given in Sections 3 through 7, rests on Theorem 2.1, with β_1 and β_2 as in (9.9) and $\{z, z_t\}$ as in (9.7). Thus, that proof yields in our present case of problem (9.8) the following estimate (see equation (8.2) of Section 8)

$$\begin{aligned} &\left\| \frac{\partial z_t}{\partial \nu} \right\|^2_{L_2(0,T;H^{-1/2}(\Gamma))} \\ &\leq C_T \big\{ \|\{z, z_t\}\|^2_{C([0,T];H^2(\Omega)\times H^1(\Omega))} + \| -\theta|_\Gamma + g_1 \|^2_{L_2(\Sigma)} \\ &\qquad + \|b\theta|_\Gamma + g_2\|^2_{L_2(0,T;H^{-1/2}(\Gamma))} \big\}; \end{aligned} \tag{9.10}$$

hence

$$\begin{aligned}(9.11)\qquad &\left\|\frac{\partial z_t}{\partial \nu}\right\|^2_{L_2(0,T;H^{-1/2}(\Gamma))}\\ &\leq C_T\big\{\|\{z,z_t\}\|^2_{C([0,T];H^2(\Omega)\times H^1(\Omega))}+\|\theta|_\Gamma\|^2_{L_2(\Sigma)}+\|g_1\|^2_{L_2(\Sigma)}\\ &\qquad +\|g_2\|^2_{L_2(0,T;H^{-1/2}(\Gamma))}\big\}.\end{aligned}$$

Step 3.

Proposition 9.2. *With reference to problem* (9.1), *we have*:

(a) *the following 'dissipation identity/inequality' holds true for all* $t>0$.

$$\begin{aligned}&E(t)+2b\int_0^t\int_\Gamma \theta^2(\tau)|_\Gamma\,d\Gamma\,d\tau+2\int_0^t\int_\Omega|\nabla\theta(\tau)|^2\,d\Omega\,d\tau\\ (9.12)\qquad &\quad =2\int_0^t\left(g_1,\frac{\partial z_t}{\partial\nu}\right)_{L_2(\Gamma)}d\tau-2\int_0^t(g_2,z_t|_\Gamma)_{L_2(\Gamma)}\,d\tau\\ &\quad\leq \varepsilon\left\|\frac{\partial z_t}{\partial\nu}\right\|^2_{L_2(0,T;H^{-1/2}(\Gamma))}+\frac{1}{\varepsilon}\|g_1\|^2_{L_2(0,T;H^{1/2}(\Gamma))}\\ (9.13)\qquad &\qquad +\|z_t|_\Gamma\|^2_{L_2(0,T;H^{1/2}(\Gamma))}+\|g_2\|^2_{L_2(0,T;H^{-1/2}(\Gamma))},\end{aligned}$$

for any $\varepsilon>0$, *where*

$$(9.14)\qquad \begin{aligned}E(t)&\equiv\|\{z(t),z_t(t),\theta(t)\}\|^2_{Y_\gamma};\\ Y_\gamma&\equiv\mathcal{D}(\mathcal{A}^{1/2})\times\mathcal{D}((I+\gamma\mathcal{A}_N)^{1/2})\times L_2(\Omega)\end{aligned}$$

norm-equivalent to $H^2(\Omega)\times H^1(\Omega)\times L_2(\Omega)$.

(b) *Consequently, by* (9.11) *used in* (9.13), *and trace theory*

(9.15)

$$\begin{aligned}&E(t)+2b\int_0^t\int_\Gamma\theta^2(\tau)|_\Gamma\,d\Gamma\,d\tau+(2-\varepsilon C_T)\int_0^t\int_\Omega|\nabla\theta(\tau)|^2\,d\Omega\,d\tau\\ &\quad\leq C_{T,\varepsilon}\big\{\|\{z,z_t,\theta\}\|^2_{C([0,t];H^2(\Omega)\times H^1(\Omega)\times L_2(\Omega))}+\|g_1\|^2_{L_2(0,T;H^{1/2}(\Gamma))}\\ &\qquad\quad+\|g_2\|^2_{L_2(0,T;H^{-1/2}(\Gamma))}\big\}.\end{aligned}$$

(c) *Hence, by* (9.15) *and* (9.7),

$$\text{(9.16)}\qquad \begin{aligned}&\|\theta\|^2_{L_2(0,T;H^1(\Omega))} + \|\theta|_\Gamma\|^2_{L_2(0,T;H^{1/2}(\Gamma))}\\ &\qquad \leq C_T\big\{\|g_1\|^2_{L_2(0,T;H^{1/2}(\Gamma))} + \|g_2\|^2_{L_2(0,T;H^{-1/2}(\Gamma))}\big\},\end{aligned}$$

and by (9.16) *used in* (9.11), *and by* (9.7),

$$\text{(9.17)}\qquad \begin{aligned}&\left\|\frac{\partial z_t}{\partial \nu}\right\|^2_{L_2(0,T;H^{-1/2}(\Gamma))}\\ &\qquad \leq C_T\big\{\|g_1\|^2_{L_2(0,T;H^{1/2}(\Gamma))} + \|g_2\|^2_{L_2(0,T:H^{-1/2}(\Gamma))}\big\}.\end{aligned}$$

(d)

$$\text{(9.18)}\qquad \begin{aligned}&\|z_{tt}\|^2_{L_2(0,T;L_2(\Omega))} + \|\Delta z_t\|^2_{L_2(0,T;[H^1(\Omega)]')}\\ &\qquad = C_T\big\{\|g_1\|^2_{L_2(0,T;H^{1/2}(\Gamma))} + \|g_2\|^2_{L_2(0,T;H^{-1/2}(\Gamma))}\big\}.\end{aligned}$$

Proof. First, the following facts are taken from [**12**, Sect. 1.3]. Problem (9.1) may be written abstractly, with $y(t) = \{z(t), z_t(t), \theta(t)\}$ and $g = \{g_1, g_2\}$, as

$$\text{(9.19)}\qquad \dot{y} = \mathbf{A}_\gamma y + \mathbf{B}g \quad \text{on } [\mathcal{D}(\mathbf{A}^*_\gamma)]';\ y(0) = 0,$$

where the indicated duality is with respect to Y_γ, and where the operator $\mathbf{A}_\gamma$ (which is explicitly identified in [**12**, Eqn. (1.3.22a), p. 25] generates a s.c. contraction semigroup $e^{\mathbf{A}_\gamma t}$ on the space Y_γ defined by (9.14); moreover, $\mathbf{A}_\gamma$ satisfies

$$\text{(9.20)}\qquad \operatorname{Re}(\mathbf{A}_\gamma x, x)_{Y_\gamma} = -(\mathcal{A}_R x_3, x_3)_{L_2(\Omega)}, \quad x = [x_1, x_2, x_3] \in \mathcal{D}(\mathbf{A}_\gamma),$$

see [**12**], Proposition 1.3.1, p. 25]. Furthermore, the operator $\mathbf{A}$ is given by [**12**, Eqn. (1.3.24), p. 26]

$$\text{(9.21)}\qquad \mathbf{B}g = \begin{bmatrix} 0 \\ (I+\gamma\mathcal{A}_N)^{-1}[\mathcal{A}G_1 g_1 + \mathcal{A}G_2 g_2] \\ 0 \end{bmatrix},$$

where the G_i are the appropriate Green maps, so that by (9.14) and (9.20) and $y = [z, z_t, \theta]$, we have

$$(\mathbf{B}g, y)_{Y_\gamma} = ([\mathcal{A}G_1 g_1 + \mathcal{A}G_2 g_2], z_t)_{L_2(\Omega)}$$

$$= (g_1, G_1^* \mathcal{A} z_t)_{L_2(\Gamma)} + (g_2, G_2^* \mathcal{A} z_t)_{L_2(\Gamma)} \tag{9.22}$$

$$= \left(g_1, \frac{\partial z_t}{\partial \nu}\right)_{L_2(\Gamma)} + (g_2, -z_t|_\Gamma)_{L_2(\Gamma)}, \tag{9.23}$$

using [**12**, Eqn. (1.3.18)] for the traces.

(a) Thus, taking the Y_γ-inner product of (9.19) with $y = [z, z_t, \theta]$, using (9.14), (9.20) and (9.23), yields

$$\frac{1}{2}\frac{d}{dt} E(t) = (\Delta\theta, \theta)_{L_2(\Omega)} + (\mathbf{B}g, y)_{Y_\gamma} \tag{9.24}$$

$$= \int_\Gamma \frac{\partial \theta}{\partial \nu}\, \theta \, d\Gamma - \int_\Omega |\nabla \theta|^2 d\Omega + \left(g_1, \frac{\partial z_t}{\partial \nu}\right)_{L_2(\Gamma)} - (g_2, z_t|_\Gamma)_{L_2(\Gamma)}, \tag{9.25}$$

by Green's Theorem. Integrating (9.25) in t, using $E(0) = 0$ by (9.1c) and using (9.1f) yields (9.12), from which (9.13) follows at once.

Remark 9.1. One could likewise obtain identity (9.25) by multiplying (9.1a) by w_t, (9.1b) by θ and integrating by parts using the BC; see [**1**]. □

(b) Furthermore, using estimate (9.11) on the right side of (9.13), along with trace theory for $\theta|_\Gamma$ and $z_t|_\Gamma$, we readily find (9.15), recalling the a-priori regularity of $\{z, z_t, \theta\}$ in (9.7).

(c) Equation (9.15) readily implies (9.16) for θ and, via trace theory, for $\theta|_\Gamma$. Using estimate (9.16) for $\theta|_\Gamma$ back in (9.11), along with the a-priority regularity in (9.7), readily implies (9.17) for $\partial z_t/\partial \nu$.

(d) Next, we show (9.18) for z_{tt}. We refer once more to [**12**, Eqn. (1.3.9), p. 26] for the abstract model of equation (9.1a): this is given by

$$\begin{aligned} z_{tt} + \gamma \mathcal{A}_N z_{tt} + \mathcal{A} z - \mathcal{A}_R \theta \\ = -\mathcal{A}G_1(\theta|_\Gamma) + b\mathcal{A}G_2(\theta|_\Gamma) + \mathcal{A}G_1 g_1 + \mathcal{A}G_2 g_2, \end{aligned} \tag{9.26}$$

where we recall from [**12**, Sect. 1.3] some of the following properties:

(i) $\mathcal{D}(\mathcal{A}_N) \subset \mathcal{D}(\mathcal{A}^{1/2}) \equiv H^2(\Omega)$, so that

$$\text{(9.27)} \qquad \begin{array}{l} \mathcal{A}^{1/2}\mathcal{A}_N^{-1} \in \mathcal{L}(L_2(\Omega)) \quad \text{and} \quad \mathcal{A}_N^{-1}\mathcal{A}^{1/2} \quad \text{has} \\ \text{a bounded extension in } \mathcal{L}(L_2(\Omega)). \end{array}$$

Hence, by (9.7) on z and (9.27), we have:

$$\text{(9.28)} \qquad [(I+\gamma\mathcal{A}_N)^{-1}\mathcal{A}^{1/2}]\mathcal{A}^{1/2}z \in C([0,T];L_2(\Omega)).$$

(ii) By (9.7), (9.16) on θ, we likewise have, since $\mathcal{A}_R\theta = \mathcal{A}_N[\theta + bN(\theta|_\Gamma)]$ by [**12**, Eqn. (5.1.1), p. 14]:

$$\text{(9.29)} \qquad (I+\gamma\mathcal{A}_N)^{-1}\mathcal{A}_R\theta \in L_2(0,T;L_2(\Omega)).$$

(iii) Since $\mathcal{A}^{(5/8)-\varepsilon}G_1 \in \mathcal{L}(L_2(\Gamma);L_2(\Omega))$ [**12**, Eqn. (1.3.17)], and recalling (9.27), we have, a fortiori from (9.16):

$$\text{(9.30)} \qquad [(I+\gamma\mathcal{A}_N)^{-1}\mathcal{A}^{(3/8)+\varepsilon}]\mathcal{A}^{(5/8)-\varepsilon}G_1(\theta|_\Gamma) \in L_2(0,T;L_2(\Omega));$$

$$\text{(9.31)} \qquad \left\| [(I+\gamma\mathcal{A}_N)^{-1}\mathcal{A}^{(3/8)+\varepsilon}]\mathcal{A}^{(5/8)-\varepsilon}G_1 g_1 \right\|_{L_2(0,T;L_2(\Omega))} \le C_T \|g_1\|_{L_2(\Sigma)}.$$

(iv) Finally,

$$\text{(9.32)} \qquad \|(I+\gamma\mathcal{A}_N)^{-1}\mathcal{A}G_2 g_2\|_{L_2(0,T;L_2(\Omega))} \le C_T\|g_2\|_{L_2(0,T;H^{-1/2}(\Gamma))},$$

since $G_2 : H^{-1/2}(\Gamma) \to H^{-(1/2)+(7/2)}(\Omega) = H^3(\Omega)$ continuously and $\mathcal{B}_1(G_2 g_2) = 0$ [**12**, Eqns. (1.3.16), (1.3.17)], so that $\mathcal{A}^{3/4}G_2 : H^{-1/2}(\Gamma) \to L_2(\Omega)$, and (9.32) follows via (9.27).

Then, (9.27)–(9.32), used in (9.26), yield $z_{tt} \in L_2(0,T;L_2(\Omega))$ continuously in g_1 and g_2, and (9.18) is established for z_{tt}. The above argument in (d) proceeds unchanged and yields $z_{tt} \in L_2(0,T;L_2(\Omega))$, even in the presence of $\{z_0, z_1, \theta_0\} \in H^2(\Omega)\times H^1(\Omega)\times L_2(\Omega)$, by (1.13).

We finally prove (9.18) for Δz_t. To this end, proceeding as in [**12**, Sect. 1.3], we write

$$\Delta z_t = \Delta\left(z_t - N\frac{\partial z_t}{\partial \nu}\right) = -\mathcal{A}_N\left(z_t - N\frac{\partial z_t}{\partial \nu}\right) = -\mathcal{A}_N z_t + \mathcal{A}_N N\frac{\partial z_t}{\partial \nu}, \tag{9.33}$$

where N is the Neumann map [**12**, Eqn. (1.3.14)]. Then:

(i) the a-priori regularity $z_t \in C([0,T]; H^1(\Omega) = \mathcal{D}(\mathcal{A}_N^{1/2}))$ from (9.7) yields

$$\mathcal{A}_N z_t \in C([0,T]; [\mathcal{D}(\mathcal{A}_N^{1/2})]' = [H^1(\Omega)]'). \tag{9.34}$$

(ii) The regularity $(\partial z_t/\partial\nu) \in L_2(0,T; H^{-1/2}(\Gamma))$ already established in (9.17), together with the elliptic property $N : H^s(\Gamma) \to H^{s+(3/2)}(\Omega)$ of the Neumann map, here specialized for $s = -(1/2)$, yields

$$N\frac{\partial z_t}{\partial \nu} \in L_2(0,T; H^1(\Omega) = \mathcal{D}(\mathcal{A}_N^{1/2})),$$

hence

$$\mathcal{A}_N N\frac{\partial z_t}{\partial \nu} \in L_2(0,T; [\mathcal{D}(\mathcal{A}_N^{1/2})]' = [H^1(\Omega)]'). \tag{9.35}$$

Then, (9.34) and (9.35) used in (9.33) yield $\Delta z_t \in L_2(0,T; [H^1(\Omega)]')$, continuously in g_1 and g_2, as in (9.2); and (9.18) is fully proved.

Remark 9.2. The *companion* result $\Delta z_t \in C([0,T]; H^{-1}(\Omega))$ follows at once from the a-priori regularity of z_t in (9.7) and [**15**, p. 85].

The proof of Proposition 9.2. is complete. □

Step 4. Complementing the regularity of $(\partial z_t/\partial\nu)$ in (9.6)—proved in (9.17)—we have a more refined result.

Proposition 9.3. *With reference to problem* (9.1)*, under the assumptions in* (9.2) *for* g_1 *and* g_2*, we have*

$$\frac{\partial z_t}{\partial \nu} \equiv \frac{\partial z_{1,t}}{\partial \nu} + \frac{\partial z_{2,t}}{\partial \nu}, \tag{9.36}$$

where

$$\frac{\partial z_{1,t}}{\partial \nu} \in L_2(0,T;L_2(\Gamma));$$

(9.37a)

$$\frac{\partial z_{2,t}}{\partial \nu} \in L_2(0,T;H^{-1/2}(\Gamma)) \cap H^1(0,T;H^{-3/2}(\Gamma));$$

(9.37b)

$$\frac{\partial z_{2,t}}{\partial \nu} \in C([0,T];H^{-1}(\Gamma)).$$

Proof. In Step 2, in connection with problem (9.1) rewritten as in (9.8), we have already noted that the Claim of Remark 2.1 applies. This, combined with the a-priori regularity in (9.7) for $\{z, z_t\}$, guarantees that the results in Section 2 through 7 hold for the Kirchoff problem (9.8). Thus, in the present new notation, writing as in (4.10),

$$z_t = \phi z = \mathcal{X} z_c + (1-\mathcal{X}) z_c = z_1 + z_2, \tag{9.38}$$

we have that

(i)

$$\nabla z_{1,t} \in L_2(\Sigma), \quad \text{in particular } \frac{\partial z_{1,t}}{\partial \nu} \in L_2(0,T;L_2(\Gamma)), \tag{9.39}$$

by Theorem 6.1, equation (6.1);

(ii)

$$\frac{\partial z_{2,t}}{\partial \nu} \in L_2(0,T;H^{-1/2}(\Gamma)) \cap H^1(0,T;H^{-3/2}(\Gamma)), \tag{9.40}$$

by Proposition 7.1, equation (7.2), and Proposition 7.2, equation (7.7). Both memberships (9.39) and (9.40) are continuous with respect to g_1 and g_2 as in assumption (9.2). But then, (9.40) implies [**15**, p. 19] that

$$\frac{\partial z_{2,t}}{\partial \nu} \in C([0,T];H^{-1}(\Gamma)),$$

since

$$[H^{-1/2}(\Gamma), H^{-3/2}(\Gamma)]_{1/2} = H^{-1}(\Gamma), \tag{9.41}$$

as well. Thus Proposition 9.3 is proved. $\square$

Step 5. To complete the proof of Theorem 9.1, it remains to show that, in fact,

$$\theta \in C([0,T]; H^{1/2}(\Omega)), \tag{9.42}$$

a (1/2)-improvement over the a-priori regularity of θ in (9.7). To this end, we return to equation (9.1b): property (9.5) for Δz_t (proved in (9.18) above) is not enough.

Case $b = 0$. Initially, we take $b = 0$ and rewrite (9.1b) abstractly, via (9.33), as

$$\theta_t + \mathcal{A}_N \theta + \mathcal{A}_N z_t - \mathcal{A}_N N \frac{\partial z_t}{\partial \nu} = 0 \tag{9.43}$$

in agreement with [**12**, Eqn. (1.3.10)]. Since $\theta(0) = 0$ by (9.1c), the solution of (9.43) is given by

$$\begin{aligned} \theta(t) = \theta_1(t) + \theta_2(t) = -\int_0^t e^{-\mathcal{A}_N(t-\tau)} \mathcal{A}_N z_t(\tau)\, d\tau \\ + \int_0^t e^{-\mathcal{A}_N(t-\tau)} \mathcal{A}_N N \frac{\partial z_t}{\partial \nu}(\tau)\, d\tau, \end{aligned} \tag{9.44}$$

where:

(i)

(9.45)
$$\theta_1 = -\int_0^t e^{-\mathcal{A}_N(t-\tau)} \mathcal{A}_N z_t(\tau)\, d\tau = -\int_0^t \frac{de^{-\mathcal{A}_N(t-\tau)}}{d\tau} z_t(\tau)\, d\tau$$
(9.46)
$$= z_t(t) + \int_0^t e^{-\mathcal{A}_N(t-\tau)} z_{tt}(\tau)\, d\tau \in C([0,T]; \mathcal{D}(\mathcal{A}_N^{1/2}) = H^1(\Omega)),$$

after integrating by parts in t and using $z_t(0) = 0$ by (9.1c). It remains to justify the regularity noted in (9.46). First we recall that $z_t \in C([0,T]; H^1(\Omega))$ by the a-priori regularity in (9.7). Next, as to the integral term in (9.46), we invoke critically that $z_{tt} \in L_2(0,T;L_2(\Omega))$ by (9.18); then standard analytic semigroup theory allows the integral term to absorb $\mathcal{A}_N^{1/2}$ in front and produce the regularity result in (9.46).

(ii) Recalling critically Proposition 9.3, we write

(9.47)
$$\theta_2(t) = \int_0^t e^{-\mathcal{A}_N(t-\tau)} \mathcal{A}_N N \frac{\partial z_t}{\partial \nu}(\tau)\, d\tau = \theta_{2,A}(t) + \theta_{2,B}(t)$$

(9.48)
$$\theta_{2,A}(t) = \int_0^t e^{-\mathcal{A}_N(t-\tau)} \mathcal{A}_N N \frac{\partial z_{1,t}}{\partial \nu}(\tau)\, d\tau$$

(9.49)
$$\theta_{2,B}(t) = \int_0^t e^{-\mathcal{A}_N(t-\tau)} \mathcal{A}_N N \frac{\partial z_{2,t}}{\partial \nu}(\tau)\, d\tau$$

(9.50)
$$= N\frac{\partial z_{2,t}(t)}{\partial \nu} - \int_0^t e^{-\mathcal{A}_N(t-\tau)} N \frac{\partial z_{2,tt}}{\partial \nu}(\tau)\, d\tau.$$

First we claim that

(9.51)
$$\theta_{2,A}(t) \in C([0,T];H^{1/2}(\Omega)).$$

In fact, $\theta_{2,A}$ is, by its own definition, the solution of a heat equation with Neumann boundary datum $(\partial z_{1,t}/\partial\nu) \in L_2(0,T;L_2(\Gamma))$ via (9.37) and zero initial conditions; i.e., $\theta_{2,A} \equiv \rho$ where

$$\begin{cases} \rho_{tt} - \Delta\rho = 0 & \text{in } Q, \\ \dfrac{\partial \rho}{\partial \nu} = f, \quad f = \dfrac{\partial z_{1,t}}{\partial \nu} \in L_2(0,T;L_2(\Gamma)), \end{cases}$$

and zero I.C. $\rho(0,\cdot) = 0$. Then, it is well known [**15**, Vol. 2, p. 81], [**13**, Eqn. (3.3.1.3), p. 194] that $\rho = \theta_{2,A} \in C([0,T];H^{1/2}(\Omega))$ and (9.51) is proved.

Finally, we claim that

(9.53)
$$\theta_{2,B}(t) \in C([0,T];H^{1/2}(\Omega)),$$

as well. To this end, we use (9.50) along with $(\partial z_{2,t}/\partial\nu) \in H^1(0,T;H^{-3/2}(\Gamma))$, see (9.37) of Proposition 9.3; consequently,

$$\frac{\partial z_{2,tt}}{\partial \nu} \in L_2(0,T;H^{-3/2}(\Gamma)),$$

hence

$$N\frac{\partial z_{2,tt}}{\partial \nu} \in L_2(0,T;L_2(\Omega)), \tag{9.54}$$

by the elliptic property of the Neumann map. Then, by the already invoked standard semigroup theory, property (9.53) yields

$$\int_0^t e^{-\mathcal{A}_N(t-\tau)} N\frac{\partial z_{2,tt}}{\partial \nu}(\tau)\, d\tau \in C([0,T];\mathcal{D}(\mathcal{A}_N^{1/2}) = H^1(\Omega)). \tag{9.55}$$

Moreover, recalling the regularity (9.37b) for $(\partial z_{2,t}/\partial \nu)$, and using again the elliptic property of the Neumann map, one obtains

$$\frac{\partial z_{2,t}}{\partial \nu} \in C([0,T];H^{-1}(\Gamma)),$$

hence

$$N\frac{\partial z_{2,t}}{\partial \nu} \in C([0,T];H^{1/2}(\Omega)). \tag{9.56}$$

Thus, (9.55) and (9.56), used in (9.50), prove (9.53) as desired. Then, (9.51) and (9.53), used in (9.47), establish that

$$\theta_2 \in C([0,T];H^{1/2}(\Omega)). \tag{9.57}$$

Then, (9.46) and (9.57) finally prove that $\theta \in C([0,T];H^{1/2}(\Omega))$, as claimed in (9.42). The case $b = 0$ is proved.

Case $b \neq 0$. Here we use $\mathcal{A}_R f = \mathcal{A}_N[f + bN(f|_\Gamma)]$, see [**12**, Eqn. (5.1.1), p. 44]. The proof of Theorem 9.1 is complete. □

Final remark. Theorem 9.1 may serve as a starting point for further regularity results of problem (9.1) under differently assumed regularity of the boundary data. An example is the following: with reference to problem (9.1), let

$$g_1 \in L_2(0,T;L_2(\Gamma)), \quad g_2 \in L_2(0,T;H^{-1}(\Gamma)).$$

Then

$$\{z, z_t, \theta\} \in C([0,T];H^{3/2}(\Omega) \times H^{1/2}(\Omega) \times L_2(\Omega)).$$

This result could be derived from Theorem 9.1 by applying the tangential (space) pseudo-differential operator $\left\{1/\sqrt{D_\eta^2+1}\right\}^{1/2}$ to problem (9.1), where $y \to \eta$ as in Section 4.

APPENDIX

A. Proof of Proposition 2.2. The following result is essentially contained in the literature [**2**], [**3**], [**10**].

Proposition A.1. *With Γ of class C^2, let $h(x) \in C^2(\overline{\Omega})$ be a vector field such that $h|_\Gamma = \nu$. Let w be a smooth solution of equation* (1.1a) *only* (*no BC*).

(i) *Then the following identity holds true*:

$$
\begin{aligned}
\text{(A.1)}\qquad & \int_\Sigma \Delta w \frac{\partial^2 w}{\partial \nu^2}\, d\Sigma - \frac{1}{2}\int_\Sigma |\Delta w|^2\, d\Sigma + \frac{1}{2}\int_\Sigma w_t^2\, d\Sigma \\
& - \int_\Sigma \left(\frac{\partial \Delta w}{\partial \nu} - \gamma \frac{\partial w_{tt}}{\partial \nu}\right) \frac{\partial w}{\partial \nu}\, d\Sigma + \frac{\gamma}{2}\int_\Sigma |\nabla w_t|^2\, d\Sigma \\
&= 2\int_Q \Delta w \left(\sum_{i=1}^2 \nabla h_i \cdot \nabla w_{x_i}\right) dQ + \frac{1}{2}\int_Q [w_t^2 - (\Delta w)^2] \operatorname{div} h\, dQ \\
& + \int_Q \Delta w [\Delta h_1, \Delta h_2] \cdot \nabla w\, dQ - \gamma \int_Q H \nabla w_t \cdot \nabla w_t\, dQ \\
& + \frac{\gamma}{2}\int_Q |\nabla w_t|^2 \operatorname{div} h\, dQ - \int_Q qh \cdot \nabla w\, dQ \\
& + [(w_t, h\cdot \nabla w)_{L_2(\Omega)}]_0^T + \gamma \left[\int_\Omega \nabla w_t \cdot \nabla (h \cdot \nabla w)\, d\Omega \right]_0^T .
\end{aligned}
$$

(ii) *More specifically, let $\{w, w_t\} \in C([0,T]; H^2(\Omega) \times H^1(\Omega))$ continuously with respect to*

$$
\text{(A.2)}\qquad E_0 = \| \{w_0, w_1, q\} \|^2_{H^2(\Omega)\times H^1(\Omega) \times L_1(0,T;[H^1(\Omega)]')}.
$$

Then, for the righthand side (*RHS*) *of identity* (A.1), *we have*

$$
\text{(A.3)}\qquad \text{RHS of (A.1)} = \mathcal{O}_T(E_0) - \int_Q qh \cdot \nabla w\, dQ = \mathcal{O}_T(E_0).
$$

Proof. As noted already, the above identity (A.1) is known. One multiplies equation (1.1a) (left) for w by the multiplier $h \cdot \nabla w$ and integrates by parts. □

B. Proof of Proposition 4.3.

Proof of part (i), *equation* (4.17). From (4.13) we have that the righthand side commutator term f in (4.12a) is given by $f = f_1 + f_2 + f_3$, where

$$f_1 \equiv \mathcal{X}[\tilde{\mathcal{P}}, \phi]w; \qquad f_2 = \mathcal{X}(\phi q); \qquad f_3 = [\tilde{\mathcal{P}}, \mathcal{X}]w_c. \tag{B.1}$$

Terms f_1 and f_2.

Lemma B.1. *The following estimates hold true*:

$$\int_{Q_{c,\infty}} f_1 h \cdot \nabla w_1 \, dQ = \mathcal{O}_T(E_0); \int_{Q_{c,\infty}} f_2 h \cdot \nabla w_1 \, dQ = \mathcal{O}_T(E_0), \tag{B.2}$$

where $h(x) = (-1, 0)$, so that $h|_{x=0} = \nu$.

Proof. We use the a-priori interior regularity that $\{w, w_t\}$, hence $\{w_1, w_{1,t}\}$ by (4.11), is in $C(R_t^1; H^2(\Omega_c) \times H^1(\Omega_c))$. Indeed, this information plus the assumption $(\phi q) \in L_1(R_t^1; [H^1(\Omega)]')$ establish at once the validity of the second estimate in (B.2) via f_2 in (B.1). As to the first integral in (B.2), we see from the explicit expression of $[\mathcal{P}, \phi]w$ in (4.4) that it suffices to estimate its worst term, i.e., $\phi_t \Delta w_t$ (under the action of $\mathcal{X}$, see f_1 in (B.1)). Thus, integrating by parts in t, we readily obtain since ϕ is compactly supported:

$$\begin{aligned}
\int_{Q_{c,\infty}} \mathcal{X}(\phi_t \Delta w_t) h \cdot \nabla w_1 \, dQ &= \int_{\Omega_c} \int_{-\infty}^{\infty} \mathcal{X}(\phi_t \Delta w_t) h \cdot \nabla w_1 \, dt \, d\Omega \\
&= - \int_{\Omega_c} \int_{-\infty}^{\infty} \mathcal{X}(\phi_t \Delta w) h \cdot \nabla w_{1,t} \, dt \, d\Omega \\
&\quad - \int_{\Omega_c} \int_{-\infty}^{\infty} \mathcal{X}(\phi_{tt} \Delta w) h \cdot \nabla w_1 \, dt \, d\Omega \qquad \text{(B.3)} \\
&= \mathcal{O}_T(E_0), \qquad \text{(B.4)}
\end{aligned}$$

see (4.16) or (A.2), by the a-priori regularity of w and $w_{1,t}$. Thus (B.2) is established. □

Term f_3. With reference to f_3 in (B.1), we finally seek to establish

$$\int_{Q_{c,\infty}} f_3 h \cdot \nabla w_1 \, dQ = \mathcal{O}_T(E_0). \tag{B.5}$$

The analysis to prove (B.5) is more elaborate.

Step 1.

Lemma B.2. *With reference to* f_3 *in* (B.1)*, we have with* $D_t = \partial/\partial t$*,* $D_x = \partial/\partial x$*,* $D_y = \partial/\partial y$:

$$\begin{aligned} f_3 = [\tilde{\mathcal{P}}, \mathcal{X}]w_c &= -\gamma D_t^2 [R(x,y,D_y), \mathcal{X}]w_c + D_x^2 [R(x,y,D_y), \mathcal{X}]w_c \\ &\quad + [R(x,y,D_y), \mathcal{X}] D_x^2 w_c + [R^2(x,y,D_y), \mathcal{X}]w_c. \end{aligned} \tag{B.6}$$

Proof. By (4.3a) we have $\tilde{\mathcal{P}} = D_t^2 - \gamma D_t^2 \tilde{\Delta} + \tilde{\Delta}^2$, with $\tilde{\Delta}$ given explicitly by (3.1) and (3.2), yielding the expansion

$$\begin{aligned} [\tilde{\mathcal{P}}, \mathcal{X}]w_c = [D_t^2, \mathcal{X}]w_c &- \gamma [D_t^2 (D_x^2 + R(x,y,D_y)), \mathcal{X}]w_c \\ &+ [(D_x^4 + D_x^2 R(x,y,D_y) + R(x,y,D_y) D_x^2 \\ &\quad + R^2(x,y,D_y)), \mathcal{X}]w_c. \end{aligned} \tag{B.7}$$

In (B.7), we first use (D_t and D_x commute with time-independent $\mathcal{X}$)

$$[D_t^2, \mathcal{X}] = 0; \qquad [D_x^2, \mathcal{X}] = 0; \qquad [D_x^4, \mathcal{X}] = 0, \tag{B.8}$$

and hence by the first identity of (B.8),

$$\begin{aligned} [D_t^2 D_x^2, \mathcal{X}] = D_t^2 D_x^2 \mathcal{X} - \mathcal{X} D_t^2 D_x^2 &= D_t^2 D_x^2 \mathcal{X} - D_t^2 \mathcal{X} D_x^2 \\ = D_t^2 [D_x^2, \mathcal{X}] &= 0, \end{aligned} \tag{B.9}$$

recalling, in the last step, the second identity in (B.8). Using again the second identity in (B.8), we obtain next

$$\begin{aligned}[D_x^2R(x,y,D_y),\mathcal{X}] &= D_x^2R(x,y,D_y)\mathcal{X} - \mathcal{X}D_x^2R(x,y,D_y)\\ &= D_x^2R(x,y,D_y)\mathcal{X} - D_x^2\mathcal{X}R(x,y,D_y)\\ &= D_x^2[R(x,y,D_y),\mathcal{X}],\end{aligned} \tag{B.10}$$

and, similarly, that

$$\begin{aligned}[R(x,y,D_y)D_x^2,\mathcal{X}] &= R(x,y,D_y)D_x^2\mathcal{X} - \mathcal{X}R(x,y,D_y)D_x^2\\ &= R(x,y,D_y)D_x^2\mathcal{X} - R(x,y,D_y)\mathcal{X}D_x^2\\ &\quad + [R(x,y,D_y),\mathcal{X}]D_x^2\end{aligned}$$

(by (B.8))

$$\begin{aligned}&= R(x,y,D_y)[D_x^2,\mathcal{X}] + [R(x,y,D_y),\mathcal{X}]D_x^2\\ &= [R(x,y,D_y),\mathcal{X}]D_x^2.\end{aligned} \tag{B.11}$$

Thus, (B.8)–(B.11), used in (B.7), produce (B.6). □

Step 2.

Lemma B.2. *With reference to the last term on the RHS of* (B.6)*, we have, recalling* (3.2)*,*

$$[R^2(x,y,D_y),\mathcal{X}] = D_y(\rho(x,y)D_y)[\rho(x,y)D_y^2,\mathcal{X}] + \text{l.o.t.} \tag{B.12}$$

Proof. Recalling (3.2), we write

$$\begin{aligned}[R^2(x,y,D_y),\mathcal{X}] &= [(\rho(x,y)D_y^2)^2,\mathcal{X}] + \text{l.o.t.}\\ &= (\rho(x,y)D_y^2)(\rho(x,y)D_y^2)\mathcal{X}\\ &\quad - \mathcal{X}(\rho(x,y)D_y^2)(\rho(x,y)D_y^2) + \text{l.o.t.}\\ &= (\rho(x,y)D_y^2)[\rho(x,y)D_y^2,\mathcal{X}] + \text{l.o.t.}\\ &= \{D_y(\rho(x,y)D_y) - \rho_y D_y\}[\rho(x,y)D_y^2,\mathcal{X}] + \text{l.o.t.}\\ &= D_y(\rho(x,y)D_y)[\rho(x,y)D_y^2,\mathcal{X}] + \text{l.o.t.},\end{aligned} \tag{B.13}$$

and (B.13) proves (B.12). □

Step 3. A more convenient expression for $f_3 = [\tilde{\mathcal{P}}, \mathcal{X}]w_c$ is given next.

Proposition B.3. *Identity* (B.6) *may be rewritten as*

$$\begin{aligned} f_3 = [\tilde{P}, \mathcal{X}]w_c &= -\gamma D_t[R(x,y,D_y), \mathcal{X}]D_t w_c \\ &\quad + 2D_x[R(x,y,D_y), \mathcal{X}]D_x w_c \\ &\quad + D_y(\rho(x,y)D_y)[\rho(x,y)D_y^2, \mathcal{X}]w_c + \text{l.o.t.} \end{aligned} \tag{B.14}$$

Proof. First D_t commutes with $[R(x,y,D_y), \mathcal{X}]$; next the second and third terms on the RHS of (B.6) may be replaced by the second term in (B.14) modulo an l.o.t. commutator; finally, we use Lemma B.2, equation (B.12), for the last term in (B.6). This way (B.14) is obtained. □

We next verify condition (B.5) by using the form (B.14) for f_3. We recall that $Q_{c,\infty} = R_t^1 \times \Omega_c$.

Step 4. (First term of f_3 in (B.14)). With reference to (B.5), integrating by parts on t, we see that the second integral below is finite ($\mathcal{O}_T(E_0)$, see (4.16) or (A.2)):

$$\begin{aligned} &\int_{Q_{c,\infty}} (D_t[R(x,y,D_y), \mathcal{X}]D_t w_c) h \cdot \nabla w_1 \, dt \, d\Omega_c \\ &= -\int_{\Omega_c} \int_{R_t^1} ([R(x,y,D_y), \mathcal{X}]D_t w_c)(h \cdot \nabla w_{1t}) \, dt \, d\Omega_c \\ &= \mathcal{O}_T(E_0). \end{aligned} \tag{B.15}$$

This is so since: (i) the a-priori regularity of $\{w, w_t\}$ in (1.8) yields $D_t w_c \in C(R_t^1; H^1(\Omega_c))$, $\nabla w_{1,t} \in C(R_t^1; L_2(\Omega_c))$, and (ii) the commutator $[R(x,y,D_y), \mathcal{X}]$ is a first-order $(2+0-1)$ tangential operator.

Step 5. (Second term of f_3 in (B.14)). Integrating by parts in D_x, we likewise see that the following integral is finite ($\mathcal{O}_T(E_0)$, see (4.16) or (A.2)):

$$
\begin{aligned}
&\int_{Q_{c,\infty}} (D_x[R(x,y,D_y),\mathcal{X}]D_x w_c)(h\cdot\nabla w_1)\,dx\,dy\,dt \\
&\quad = \int_{R_t^1}\int_{R_y} [([R(x,y,D_y),\mathcal{X}]D_x w_c)(h\cdot\nabla w_1)]_{[x=0]}^{x=1}\,dy\,dt \\
&\qquad - \int_{\Omega_c}\int_{R_t^1} ([R(x,y,D_y),\mathcal{X}]D_x w_c)(D_x(h\cdot\nabla w_1))\,dt\,d\Omega_c \\
&\quad = \mathcal{O}_T(E_0).
\end{aligned} \tag{B.16}
$$

In fact, with $w_c \in C(R_t^1; H^2(\Omega_c))$, $w_1 \in C(R_t^1; H^2(\Omega_c))$ by a-priori regularity, we have that the trace on $x = 0$ satisfies $D_x w_c \in C(R_t^1; H^{1/2}(\Gamma_c))$, $D_x(h\cdot\nabla w_1) \in C(R_t^1; H^{1/2}(\Gamma_c))$, while the trace on $x = 1$ is zero by (3.4); moreover, as a consequence, $[R,\mathcal{X}]D_x w_c \in C(R_t^1; H^{-1/2}(\Gamma_c))$ since $[R,\mathcal{X}]$ is a tangential first-order operator. Hence, (B.16) follows.

Step 6. (Third term of f_3 in (B.14)). Integrating by parts in D_y, we likewise see that, since $w \equiv 0$ near $|y| = 1$ by (3.4) the following integral is finite:

$$
\begin{aligned}
&\int_{Q_{c,\infty}} (D_y(\rho(x,y)D_y)[\rho(x,y)D_y^2,\mathcal{X}]w_c)(h\cdot\nabla w_1)dQ_{c,\infty} \\
&\quad = -\int_{Q_{c,\infty}} ((\rho(x,y)D_y)[\rho(x,y)D_y^2,\mathcal{X}]w_c)(D_y(h\cdot\nabla w_1))dQ_{c,\infty} \\
&\quad = \mathcal{O}_T(E_0).
\end{aligned} \tag{B.17}
$$

Step 7. (Conclusion). Recalling (B.14) and using (B.15)–(B.17) prove (B.5) as desired. The proof of Proposition 4.3 is complete. ◻

C. Duality of thermoelastic problem (1.19). We return to problem (1.19)—rewritten for convenience in the variables $\{z,\theta\}$ as in (9.1). By (9.21), the semigroup solution of problem (9.19) with zero initial condition may be written for $g = [g_1, g_2]$ as

$$
\begin{bmatrix} z(t) \\ z_t(t) \\ \theta(t) \end{bmatrix} = (Lg)(t) = \int_0^t e^{\mathbf{A}_\gamma(t-\tau)}\mathbf{B}g(\tau)\,d\tau, \tag{C.1}
$$

where

$$L : L_2\left(0, T; H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)) \Rightarrow C([0,T]; Y_\gamma\right), \tag{C.2}$$

with $Y_\gamma \equiv H^2(\Omega) \times H^1(\Omega) \times L_2(\Omega)$ (norm-equivalence), see (9.14), if and only if [**13**],

$$\mathbf{B}^* e^{\mathbf{A}_\gamma^* t} : Y_\gamma \to L_2\left(0, T; H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)\right) \tag{C.3}$$

Let ($\bar{y}_0 = \{\psi_0, \psi_1, \eta_0\} \in Y_\gamma$, and let $\{\psi(t), -\psi_t(t), \eta(t)\} = e^{\mathbf{A}_\gamma^* t}\bar{\psi}_0$ be the solution of the adjoint thermoelastic problem

$$\psi_{tt} - \gamma\Delta\psi_{tt} + \Delta^2\psi - \Delta\eta = 0 \quad \text{in } Q; \tag{C.4a}$$

$$\eta_t - \Delta\eta + \Delta\psi_t = 0 \quad \text{in } Q; \tag{C.4b}$$

$$\psi(0,\cdot) = \psi_0,\ \psi_t(0,\cdot) = \psi_1,\ \eta(0,\cdot) = \eta_0 \quad \text{in } \Omega \tag{C.4c}$$

(which interchanges the sign of the coupling terms with respect to (9.1)) plus free homogeneous boundary conditions. Then, by (9.21)–(9.23), we obtain

$$\mathbf{B}^* e^{\mathbf{A}_\gamma^* t}\bar{y}_0 = \begin{bmatrix} \dfrac{\partial\psi_t}{\partial\nu}(t;\bar{y}_0)|_\Gamma \\ \psi_t(t;\bar{y}_0)|_\Gamma \end{bmatrix}. \tag{C.5}$$

Thus, explicitly, (C.3) means by (C.5):

$$\begin{aligned} &\{\psi_0, \psi_1, \eta_0\} \in H^2(\Omega) \times H^1(\Omega) \times L_2(\Omega) \\ &\qquad\qquad \Rightarrow \begin{cases} \dfrac{\partial\psi_t}{\partial\nu} \in L_2(0,T; H^{-1/2}(\Gamma)) \\ \psi_t|_\Gamma \in L_2(0,T; H^{1/2}(\Gamma)) \end{cases} \end{aligned} \tag{C.6}$$

for the solution of problem (C.4) with free homogeneous boundary conditions (the result on $\psi_t|_\Gamma$ followed already by trace theory on the interior regularity), while (C.2) means

$$\begin{aligned} &\begin{cases} g_1 \in L_2(0,T; H^{1/2}(\Gamma)) \\ g_2 \in L_2(0,T; H^{-1/2}(\Gamma)) \end{cases} \\ &\qquad\qquad \Rightarrow \{z, z_t, \theta\} \in C([0,T]; H^2(\Omega) \times H^1(\Omega) \times L_2(\Omega)) \end{aligned} \tag{C.7}$$

for the nonhomogeneous problem (9.1). Implication (C.7) is the content of Theorem 1.5 (in the new notation). Implication (C.7) is the content of Theorem 1.6 (in the new notation). They are the dual of each other.

REFERENCES

1. G. Avalos and I. Lasiecka, *Exponential stability of a thermo-elastic system with free boundary conditions without mechanical dissipation*, SIAM J. Math. Anal. **29** (1998), 155–182.

2. J. Lagnese, *Boundary stabilization of thin plates*, SIAM, Philadelphia, 1989.

3. ———, *Uniform boundary stabilization of homogeneous isotropic plates*, Lecture Notes in Control and Inform. Sci., Vol. 12, Proc. 3rd Internat. Conf., Vorau, Austria, July 1986.

4. ———, *The reachability problem for thermo-elastic plates*, Arch. Rational Mech. Anal. **112** (1990), 223–267.

5. J. Lagnese and J.L. Lions, *Modelling, analysis and control of thin plates*, Masson, Paris, 1988.

6. I. Lasiecka and R. Triggiani, *Sharp regularity for mixed second order hyperbolic equations of Neumann type.* Part I: *The L_2-boundary case*, Ann. Mat. Pura Appl. **157** (1990), 285–367.

7. ———, *Exact controllability and uniform stabilization of Kirchoff plates with boundary controls only in* $\Delta w|_{\Sigma}$, J. Differential Equations **93** (1991), 62–101.

8. ———, *Uniform stabilization of the wave equation with Dirichlet of Neumann feedback control without geometrical conditions*, Appl. Math. Optim. **25** (1992), 189–224.

9. ———, *Sharp trace estimates of solutions to Kirchoff and Euler-Bernoulli equations*, Appl. Math. Optim. **28** (1993), 277–306.

10. ———, *Exact controllability and uniform stabilization of the Euler-Bernoulli equation with control only in* $\Delta w|_{\Sigma}$, Boll. Un. Mat. Ital. B (7) **5** (1991), 665–702.

11. ———, *Analyticity of thermo-elastic semigroups with free BC*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **XXVII** (1998), 457–482. Presented at workshop on Deterministic and Stochastic Evolutionary Systems, Scuola Norm. Sup., Pisa, July 1997, and at IFIP Conf., Detroit, July 1997.

12. ———, *Structural decomposition of thermo-elastic semigroups with rotational forces*, Semigroup Forum **60** (2000), 16–66.

13. ———, *Control theory for partial differential equations*: *Continuous and approximation theories*, Encyclopedia Math Appl., Vols. 1 (500 pp.) and 2 (400 pp.), Cambridge University Press, January, 2000.

14. ———, *A sharp trace result on thermo-elastic plate equation with coupled hinged/Neumann boundary conditions*, Discrete Continuous Dynamical Systems **5** (1999), 585–598.

15. J.L. Lions and E. Magenes, *Non-homogeneous boundary value problems*, Vols. 1 and 2, Springer-Verlag, Berlin, New York, 1972.

16. R.B. Melrose and J. Sjöstrand, *Singularities of boundary value problems*, I, Comm. Pure Appl. Math. **31** (1978), 593–617.

17. S. Miyatake, *Mixed problems for hyperbolic equations of second order*, J. Math. Kyoto Univ. **13** (1973), 435–487.

18. A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer, 1983.

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