ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 30, Number 3, Fall 2000

SOME RESULTS ON MEAN LIPSCHITZ SPACES OF ANALYTIC FUNCTIONS

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ABSTRACT. If f is a function which is analytic in the unit disk Δ and has a nontangential limit $f(e^{i\theta})$ at almost every $e^{i\theta} \in \partial\Delta$ and $1 \le p \le \infty$, then $\omega_p(\cdot, f)$ denotes the integral modulus of continuity of order p of the boundary values $f(e^{i\theta})$ of f. If $\omega : [0, \pi] \to [0, \infty)$ is a continuous and increasing function with $\omega(0) = 0$ and $\omega(t) > 0$ if $t > 0$ then, for $1 \le p \le \infty$, the mean Lipschitz space $\Lambda(p, \omega)$ consists of those functions f which belong to the classical Hardy space H^p and satisfy $\omega_p(\delta, f) = O(\omega(\delta))$ as $\delta \to 0$. If, in addition, ω satisfies the so-called Dini condition and the condition b_1, we say that ω is an admissible weight. If $0 < \alpha \le 1$ and $\omega(\delta) = \delta^\alpha$, we shall write Λ^p_α instead of $\Lambda(p, \omega)$, that is, we set $\Lambda^p_\alpha = \Lambda(p, \delta^\alpha)$.

In this paper we obtain several results about the Taylor coefficients and the radial variation of the elements of the spaces $\Lambda(p, \omega)$. In particular, if ω is an admissible weight, then we give a complete characterization of the power series with Hadamard gaps which belong to $\Lambda(p, \omega)$.

If f is an analytic function in Δ and $\theta \in [-\pi, \pi)$, we let $V(f, \theta)$ denote the radial variation of f along the radius $[0, e^{i\theta})$. We also define the exceptional set $E(f)$ associated to f as $E(f) = \{e^{i\theta} \in \mathbf{T} : V(f, \theta) = \infty\}$. For any given $p \in [1, \infty]$, we obtain a characterization of those admissible weights ω for which the implication

$$f \in \Lambda(p, \omega) \Longrightarrow E(f) = \varnothing,$$

holds. We also obtain a number of results about the "size" of the exceptional set $E(f)$ for $f \in \Lambda^p_\alpha$.

1. Introduction. Let Δ denote the unit disk $\{z \in \mathbf{C} : |z| < 1\}$ and $\mathbf{T}$ the unit circle $\{\xi \in \mathbf{C} : |\xi| = 1\}$. If $0 < r < 1$ and g is a function

Received by the editors on January 20, 1999, and in revised form on June 21, 1999.
1991 AMS *Mathematics Subject Classification.* 30D55, 30D50.
This research has been supported in part by a grant from "El Ministerio de Educación y Cultura, Spain," (PB97-1081) and by a grant from "La Junta de Andalucía."

which is analytic in Δ, we set

$$M_p(r,g) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(re^{i\theta})|^p \, d\theta\right)^{1/p}, \quad 0 < p < \infty,$$

$$M_\infty(r,g) = \max_{|z|=r} |g(z)|.$$

For $0 < p \le \infty$, the Hardy space H^p consists of those functions g, analytic in Δ, for which

$$\|g\|_{H^p} = \sup_{0<r<1} M_p(r,g) < \infty.$$

The space BMOA consists of those functions $f \in H^1$ whose boundary values have bounded mean oscillation on $\mathbf{T}$. We refer to [**3**] and [**18**] for the main properties of BMOA-functions.

If f is a function which is analytic in Δ and has a nontangential limit $f(e^{i\theta})$ at almost every $e^{i\theta} \in \mathbf{T}$, we define

$$\omega_p(\delta,f) = \sup_{0<|t|\le\delta} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i(\theta+t)}) - f(e^{i\theta})|^p \, d\theta\right)^{1/p},$$

$$\delta > 0, \quad \text{if } 1 \le p < \infty,$$

$$\omega_\infty(\delta,f) = \sup_{0<|t|\le\delta} \Big(\text{ess sup}_{\theta\in[-\pi,\pi]} |f(e^{i(\theta+t)}) - f(e^{i\theta})|\Big), \quad \delta > 0.$$

Then $\omega_p(\cdot, f)$ is the integral modulus of continuity of order p of the boundary values $f(e^{i\theta})$ of f.

Given $1 \le p \le \infty$ and $0 < \alpha \le 1$, the mean Lipschitz space Λ^p_α consists of those functions f analytic in Δ having a nontangential limit almost everywhere for which $\omega_p(\delta,f) = O(\delta^\alpha)$, as $\delta \to 0$. If $p = \infty$ we write Λ_α instead of Λ^∞_α. This is the usual Lipschitz space of order α. More precisely, a function f analytic in Δ belongs to Λ_α if and only if it has a continuous extension to the closed unit disk $\overline{\Delta}$ and its boundary values satisfy a Lipschitz condition of order α.

A classical result of Hardy and Littlewood [**21**] (see also Chapter 5 of [**14**]), asserts that for $1 \le p \le \infty$ and $0 < \alpha \le 1$, we have that $\Lambda^p_\alpha \subset H^p$ and

(1.1)

$$\Lambda^p_\alpha = \{f \text{ analytic in } \Delta : M_p(r,f') = O(1/(1-r)^{1-\alpha}), \text{ as } r \to 1\}.$$

Of special interest are the spaces $\Lambda^p_{1/p}$ since they lie in the border of continuity. Indeed, using (1.1) with $p = \alpha = 1$, we see that a classical result of Privalov [**14**, Theorem 3.11] can be stated saying that if f is analytic in Δ then f has a continuous extension to the closed unit disk $\overline{\Delta}$ whose boundary values are absolutely continuous on $\partial\Delta$ if and only if $f \in \Lambda^1_1$. If $1 < p < \infty$ and $\alpha > (1/p)$, then $\Lambda^p_\alpha \subset \Lambda_{\alpha-(1/p)}$, and hence each $f \in \Lambda^p_\alpha$ has a continuous extension to the closed unit disk (see [**12**, p. 88]). This is not true for $\alpha = (1/p)$. This follows easily noticing that the function $f(z) = \log(1-z)$ belongs to $f \in \Lambda^p_{1/p}$ for all $p \in (1,\infty)$. Cima and Petersen proved in [**13**] that $\Lambda^2_{1/2} \subset \mathrm{BMOA}$, and this result was generalized by Bourdon, Shapiro and Sledd who proved in [**12**, Theorem 2.5] the following result.

Theorem A. *For* $1 < p < \infty$, $\Lambda^p_{1/p} \subset \mathrm{BMOA}$.

These results have been shown to be sharp in a very strong sense in [**19**], [**20**] and [**9**] using the following generalization of the spaces Λ^p_α which occurs frequently in the literature. Let $\omega : [0,\pi] \to [0,\infty)$ be a continuous and increasing function with $\omega(0) = 0$ and $\omega(t) > 0$ if $t > 0$. Then, for $1 \le p \le \infty$, the mean Lipschitz space $\Lambda(p,\omega)$ consists of those functions $f \in H^p$ such that

$$\omega_p(\delta, f) = O(\omega(\delta)), \quad \text{as } \delta \to 0.$$

With this notation we have $\Lambda^p_\alpha = \Lambda(p, \delta^\alpha)$.

The question of finding conditions on ω so that it is possible to obtain results on the spaces $\Lambda(p,\omega)$ analogous to those proved by Hardy and Littlewood for the spaces Λ^p_α has been studied by several authors (see, e.g., [**10**], [**11**] and [**23**]). We shall say that ω satisfies the Dini condition or that ω is a Dini-weight if a positive constant C exists such that

$$\int_0^\delta \frac{\omega(t)}{t}\,dt \le C\omega(\delta), \quad 0 < \delta < 1. \tag{1.2}$$

Given $0 < q < \infty$, we shall say that ω satisfies the condition b_q or that $\omega \in b_q$ if a positive constant C exists such that

$$\int_\delta^\pi \frac{\omega(t)}{t^{q+1}}\,dt \le C\frac{\omega(\delta)}{\delta^q}, \quad 0 < \delta < 1. \tag{1.3}$$

In order to simplify our notation, let $\mathcal{AW}$ denote the family of all functions $\omega : [0, \pi] \to [0, \infty)$ which satisfy the following conditions:

(i) ω is continuous and increasing in $[0, \pi]$.

(ii) $\omega(0) = 0$ and $\omega(t) > 0$ if $t > 0$.

(iii) ω is a Dini-weight.

(iv) ω satisfies the condition b_1.

The elements of $\mathcal{AW}$ will be called admissible weights.

Typical elements of admissible weights are the functions

$$\omega(t) = t^{\alpha}\left(\log \frac{A}{t}\right)^{\beta}, \quad t \in [0, \pi],$$

where $0 < \alpha < 1$ and $\beta \in \mathbf{R}$ and A is an appropriate positive constant.

Blasco and de Souza proved in [**10**, Theorem 2.1] the following extension of (1.1).

Theorem B. *If $\omega \in \mathcal{AW}$, then*

(1.4)
$$\Lambda(p, \omega) = \left\{ f \text{ analytic in } \Delta : M_p(r, f') = O\left(\frac{\omega(1-r)}{1-r}\right), \text{ as } r \to 1 \right\}.$$

In this paper we shall be concerned with the spaces $\Lambda(p, \omega)$ with $\omega \in \mathcal{AW}$. In view of Theorem B, these are the most interesting ones among all generalized mean Lipschitz spaces $\Lambda(p, \omega)$. In Section 2 we shall obtain certain results about the Taylor coefficients of functions $f \in \Lambda(p, \omega)$ and Section 3 will be devoted to the study of the radial variation of these functions.

Let us finish this section by saying that from now on we shall be using the convention that C will denote a positive constant (which may depend on $\omega, \lambda, p, f, \ldots$ but not on $s, t, \delta, n, \ldots$) and which may be different at each occurrence.

2. Taylor coefficients of mean Lipschitz functions. We start with the following simple result.

Lemma 1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function in Δ. Let $\omega \in \mathcal{AW}$ and $1 \le p \le \infty$. If $f \in \Lambda(p, \omega)$, then*

$$a_n = O\left(\omega\left(\frac{1}{n}\right)\right), \quad \text{as } n \to \infty. \tag{2.1}$$

Proof. We have

$$na_n = \frac{1}{2\pi i}\int_{|z|=r} \frac{f'(z)}{z^n}\, dz, \quad 0 < r < 1.$$

Hence, using Hölder's inequality and Theorem B, we obtain

$$n|a_n| \le r^{1-n} M_1(r, f') \le r^{1-n} M_p(r, f') \le C r^{1-n} \frac{\omega(1-r)}{1-r},$$
$$0 < r < 1.$$

If we take $r = 1 - (1/n)$ with $n > 1$, we deduce that

$$n|a_n| \le Cn\omega\left(\frac{1}{n}\right), \quad n > 1.$$

This gives (2.1). $\square$

It is easy to see that the converse of Lemma 1 is not true, that is, condition (2.1) does not imply that $f \in \Lambda(p, \omega)$. However, we can show that this is true if f is given by a power series with Hadamard gaps, i.e., a power series of the form

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$$

analytic in Δ such that $n_{k+1} \ge \lambda n_k$ for all k with λ being a constant bigger than 1. We can prove the following result.

Theorem 1. *Let f be an analytic function in Δ given by a power series with Hadamard gaps*

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k} \quad \text{with } \frac{n_{k+1}}{n_k} \ge \lambda > 1 \quad \text{for all } k. \tag{2.2}$$

Let $1 \le p \le \infty$ and $\omega \in \mathcal{AW}$. Then the following two conditions are equivalent.

$$\text{(2.3)} \qquad \begin{aligned} &\text{(i)} \quad f \in \Lambda(p,\omega), \\ &\text{(ii)} \quad a_k = O\left(\omega\left(\frac{1}{n_k}\right)\right), \quad \textit{as } k \to \infty. \end{aligned}$$

Remark 1. In the case of the spaces Λ^p_α this result was proved by Essén and Xiao in [**16**] (see also [**27**] for the spaces Λ_α and [**2**] for the spaces $\Lambda^p_{1/p}$).

Proof of Theorem 1. The implication (i) $\Rightarrow$ (ii) follows from Lemma 1.

To prove the other implication, suppose that f is as in (2.2) and the a_k's satisfy (2.3). Define

$$\text{(2.4)} \qquad I(k) = \{n \in \mathbf{N} : 2^k \le n < 2^{k+1}\}, \quad k = 0,1,2,\dots .$$

We have $f'(z) = \sum_{k=0}^\infty n_k a_k z^{n_k-1}$ and then, using (2.3), we see that, for $0 < r < 1$,

$$\text{(2.5)} \qquad \begin{aligned} M_p(r,f') \le M_\infty(r,f') &\le \sum_{k=0}^\infty n_k |a_k| r^{n_k-1} \\ &\le C r^{-1} \sum_{k=0}^\infty \sum_{n_j \in I(k)} n_j \omega\left(\frac{1}{n_j}\right) r^{n_j}. \end{aligned}$$

Using the gap condition in (2.2), we see that there are at most $C_\lambda = (2/\log\lambda) + 1$ of the n_j's in the set $I(k)$. This, (2.5) and the fact that ω is increasing, give

$$\text{(2.6)} \qquad M_p(r,f') \le C r^{-1} \sum_{k=0}^\infty 2^k \omega(2^{-k}) r^{2^k}, \quad 0 < r < 1.$$

Now from the fact that $\omega(\delta)/\delta$ is "essentially" a decreasing function in $(0,1)$ when $\omega \in \mathcal{AW}$ (see [**20**, Lemma 1]), it is clear that

$$\begin{aligned} \omega(2^{-k}) &\le C \int_{1-2^{-k}}^{1-2^{-k-1}} \frac{\omega(1-s)}{1-s} s^{2^k}\, ds \\ &\le C \int_0^1 \frac{\omega(1-s)}{1-s} s^{2^k}\, ds, \quad k = 0,1,2,\dots , \end{aligned}$$

and then, using (2.6), we obtain

$$\begin{aligned} M_p(r,f') &\le Cr^{-1}\sum_{k=0}^{\infty} 2^k\left(\int_0^1 \frac{\omega(1-s)}{1-s}s^{2^k}\,ds\right)r^{2^k} \\ &= Cr^{-1}\int_0^1 \frac{\omega(1-s)}{1-s}\sum_{k=0}^{\infty}2^k(rs)^{2^k}\,ds \qquad (2.7)\\ &= Cr^{-1}\int_0^1 \frac{\omega(1-s)}{1-s}G(rs)\,ds, \end{aligned}$$

where

$$G(z)=\sum_{k=0}^{\infty}2^k z^{2^k}, \quad z\in\Delta.$$

Notice that $G(z)=zF'(z)$ where $F(z)=\sum_{k=0}^{\infty} z^{2^k}$. It is well known (see, e.g., [**1**]) that F is a Bloch function, i.e., $|F'(z)|\le C(1-|z|)^{-1}$. Then (2.7) gives

$$M_p(r,f')\le Cr^{-1}\int_0^1 \frac{\omega(1-s)}{(1-s)(1-rs)}\,ds$$

which, using the fact that $\omega\in\mathcal{AW}$ and Lemma 1.1 of [**8**], implies

$$M_p(r,f')\le C\frac{\omega(1-r)}{1-r}, \quad 0<r<1.$$

This shows that $f\in\Lambda(p,\omega)$ and finishes the proof. □

Next we shall relate the membership of an analytic function f to the space $\Lambda(p,\omega)$ ($1<p<\infty$ and $\omega\in\mathcal{AW}$) with the speed at which it is approached by its sequence of Taylor polynomials.

We start by introducing some notation. If m and n are two nonnegative integers with $m\le n$ and $f(z)=\sum_{n=0}^{\infty}a_n z^n$ is an analytic function in Δ, we define

$$\mathbf{S}_m^n f(z)=\sum_{k=m}^{n}a_k z^k, \quad z\in\Delta.$$

The usual polynomial approximation of the function f is, in this case, $\mathbf{S}_n f \equiv \mathbf{S}_0^n f$. Denote also by Δ_k, $k = 0, 1, 2, \dots$, the operators constructed from the dyadic sequence $\{2^k\}$ as follows

$$\Delta_k f = \mathbf{S}_{2^k}^{2^{k+1}-1} f, \quad k = 0, 1, 2, \dots .$$

Notice that, if $1 < p < \infty$, then by the Riesz theorem [**18**, pp. 108–109], a constant C exists, depending only on p, such that

$$M_p(r, \mathbf{S}_m^n f) \le C M_p(r, f), \quad 0 < r < 1. \tag{2.8}$$

We start by estimating the H^p-norm of $\mathbf{S}_m^n f$ for f in $\Lambda(p, \omega)$.

Lemma 2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function in Δ. Let $\omega \in \mathcal{AW}$ and $1 < p < \infty$. If $f \in \Lambda(p, \omega)$ and $1 \le m \le n$, $n > 1$, then there is a constant C depending only on p and ω such that*

$$\|\mathbf{S}_m^n f\|_{H^p} \le C \frac{n}{m} \omega\left(\frac{1}{n}\right). \tag{2.9}$$

Proof. Using Lemma 3.4 of [**12**], (2.8) and the fact that f belongs to $\Lambda(p, \omega)$, we obtain for $0 < r < 1$,

$$\begin{aligned} \|\mathbf{S}_m^n f\|_{H^p} &\le \frac{1}{m} \|(\mathbf{S}_m^n f)'\|_{H^p} \le \frac{1}{m r^{n-1}} M_p(r, (\mathbf{S}_m^n f)') \\ &= \frac{1}{m r^{n-1}} M_p(r, \mathbf{S}_{m-1}^{n-1} f') \le C \frac{1}{m r^{n-1}} \frac{\omega(1-r)}{1-r}. \end{aligned}$$

Taking $r = 1 - (1/n)$, $n > 1$, we deduce (2.9) as desired. □

The following theorem characterizes the membership of a function f in the space $\Lambda(p, \omega)$ both in terms of the H^p-norm of the functions $\Delta_k f$ and in terms of the speed at which the partial sums $\mathbf{S}_n f$ approach (in H^p-norm) to f.

Theorem 2. *Let $1 < p < \infty$ and $\omega \in \mathcal{AW}$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function in Δ. Then the following three conditions are equivalent*:

(i) $f \in \Lambda(p,\omega)$,

(ii) $\|\Delta_k f\|_{H^p} = O(\omega(1/2^k))$, *as* $k \to \infty$.

(iii) $\|f - \mathbf{S}_n f\|_{H^p} = O(\omega(1/n))$, *as* $n \to \infty$.

Proof of Theorem 2. The implication (i) $\Rightarrow$ (ii) follows trivially from Lemma 2 and the fact that ω is increasing.

(ii) $\Rightarrow$ (iii). Suppose that f satisfies (ii). Take a positive integer n and choose j such that $2^j \le n < 2^{j+1}$. Using the Riesz theorem, (ii), and bearing in mind that $\|\cdot\|_{H^p}$ is a norm, we obtain

$$\begin{aligned}\|f - \mathbf{S}_n f\|_{H^p} &\le C\|f - \mathbf{S}_{2^j} f\|_{H^p} \\ &\le C\sum_{k\ge j}\|\Delta_k f\|_{H^p} \le C\sum_{k\ge j}\omega\left(\frac{1}{2^k}\right).\end{aligned} \tag{2.10}$$

Now, since $\omega(\delta)/\delta$ is "essentially decreasing" ([**20**, Lemma 1])

$$\omega\left(\frac{1}{2^k}\right) \le C\int_{2^{-(k+1)}}^{2^{-k}} \frac{\omega(s)}{s}\,ds.$$

Using this in (2.10) and bearing in mind that $\omega \in \mathcal{AW}$ and the definition of j, we get

$$\begin{aligned}\|f - \mathbf{S}_n f\|_{H^p} &\le C\sum_{k\ge j}\int_{2^{-(k+1)}}^{2^{-k}} \frac{\omega(s)}{s}\,ds \le C\int_0^{2^{-j}} \frac{\omega(s)}{s}\,ds \\ &\le C\omega\left(\frac{1}{2^j}\right) \le C\omega\left(\frac{1}{2^{j+1}}\right) \le C\omega\left(\frac{1}{n}\right).\end{aligned}$$

This gives (iii).

The implication (iii) $\Rightarrow$ (ii) follows trivially from the fact that, by the Riesz theorem,

$$\|\Delta_k f\|_{H^p} \le C\|f - \mathbf{S}_{2^k-1} f\|_{H^p}.$$

(ii) $\Rightarrow$ (i). Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ satisfies (ii). We have $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$, so using the continuous form of Minkowski inequality, Lemma 3.4 of [**12**] and (ii), we see that, for $0 < r < 1$,

$$
\begin{aligned}
M_p(r, f') &= \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{n=1}^{\infty} n a_n r^{n-1} e^{i(n-1)\theta} \right|^p d\theta \right)^{1/p} \\
&\le \sum_{k=0}^{\infty} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{n=2^k}^{2^{k+1}-1} n a_n r^{n-1} e^{i(n-1)\theta} \right|^p d\theta \right)^{1/p} \\
&\le \sum_{k=0}^{\infty} 2^{k+1} r^{2^k-1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{n=2^k}^{2^{k+1}-1} a_n e^{in\theta} \right|^p d\theta \right)^{1/p} \\
&= 2 \sum_{k=0}^{\infty} 2^k r^{2^k-1} \|\Delta_k f\|_{H^p} \\
&\le C \sum_{k=0}^{\infty} 2^k r^{2^k-1} \omega(2^{-k}).
\end{aligned}
$$

Now notice that the last term in this chain of inequalities coincides with the right-hand side of (2.6) which was proved to be smaller than $C(\omega(1-r)/(1-r))$. Consequently, we have that

$$
M_p(r, f') \le \frac{\omega(1-r)}{1-r}, \quad 0 < r < 1.
$$

This shows that $f \in \Lambda(p, \omega)$. This finishes the proof. $\square$

Remark 2. In the case of the spaces Λ^p_α, this result was proved by Bourdon, Shapiro and Sledd in [**12**, Theorem 3.1].

Remark 3. Theorem 1 for $1 \le p < \infty$ can be deduced from Theorem 2. Indeed, suppose that f is as in (2.2) and that $1 \le p < \infty$ and $\omega \in \mathcal{AW}$. Using Theorem 8.20 in Chapter V of [**29**], we see that

$$
M_p(r, f') \approx M_2(r, f').
$$

Hence, using Theorem 2, we obtain

$$
f \in \Lambda(p, \omega) \Longleftrightarrow f \in \Lambda(2, \omega) \Longleftrightarrow \|\Delta_k f\|_{H^2} = O\left(\omega\left(\frac{1}{2^k} \right) \right). \tag{2.11}
$$

Now

$$\|\Delta_k f\|_{H^2}^2 = \sum_{n_j \in I(k)} |a_j|^2. \tag{2.12}$$

Since there are at most $C_\lambda = (2/\log\lambda) + 1$ of the n_j's in the set $I(k)$, using the facts that ω is increasing and that $\omega(\delta)/\delta$ is "essentially" decreasing, it is clear that (2.11) and (2.12) imply that

$$f \in \Lambda(p,\omega) \Longleftrightarrow a_k = O\left(\omega\left(\frac{1}{n_k}\right)\right).$$

Hence we have proved Theorem 1 in the case $1 \le p < \infty$.

3. Radial variation of mean Lipschitz functions. If f is a function which is analytic in Δ and $\theta \in [-\pi, \pi)$, we define

$$V(f,\theta) = \int_0^1 |f'(re^{i\theta})|\, dr. \tag{3.1}$$

Then $V(f,\theta)$ denotes the radial variation of f along the radius $[0, e^{i\theta})$, i.e., the length of the image of this radius under the mapping f. Also define the exceptional set $E(f)$ associated to f as

$$E(f) = \{e^{i\theta} \in \mathbf{T} : V(f,\theta) = \infty\}. \tag{3.2}$$

In this section we shall be mainly interested in studying the radial variation of functions in the mean Lipschitz spaces $\Lambda(p,\omega)$ with $1 \le p \le \infty$ and $\omega \in \mathcal{AW}$. We start with the following simple observation.

Proposition 1. *Let $1 \le p \le \infty$ and $\omega \in \mathcal{AW}$. If $f \in \Lambda(p,\omega)$, then the exceptional set $E(f)$ has linear measure zero.*

Proof. Using Fubini's theorem and bearing in mind that $M_1(r,f') \le M_p(r,f')$, we obtain

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} V(f,\theta)\, d\theta = \int_0^1 M_1(r,f')\, dr &\le \int_0^1 M_p(r,f')\, dr \\ &\le C\int_0^1 \frac{\omega(1-r)}{1-r}\, dr < \infty. \end{aligned}$$

Clearly this implies that $V(f,\theta) < \infty$ for almost every $\theta \in [-\pi, \pi)$. $\square$

Using the Fejér-Riesz inequality (see [**14**, p. 46]), we conclude that: If $f' \in H^1$ then $V(f,\theta) < \infty$ for every $\theta \in [-\pi, \pi)$, or equivalently,

$$f' \in H' \Longrightarrow E(f) = \varnothing. \tag{3.3}$$

Using results from [**19**] and [**9**], we can prove that (3.3) is sharp in a very strong sense. Indeed, we can prove the following result.

Theorem 3. *Let $\phi : [0,1) \to [0,\infty)$ be a function with $\phi(r) \to \infty$ as $r \to 1$. Then a function f exists, analytic in Δ, satisfying*

$$M_1(r, f') = O(\phi(r)), \quad \text{as } r \to 1, \tag{3.4}$$

for which $E(f) \neq \varnothing$.

Proof of Theorem 3. Using Theorem 3.2 of [**9**] (see also [**19**]) we see that a function f exists, analytic in Δ, which satisfies (3.4) and

$$|f(r)| \to \infty, \quad \text{as } r \to 1. \tag{3.5}$$

Now it is clear that (3.5) implies that $\int_0^1 |f'(r)|\, dr = \infty$ and then $E(f) \neq \varnothing$. $\square$

In Theorem 4, for any given $p \in (1, \infty)$, we obtain a characterization of those weights $\omega \in \mathcal{AW}$ for which the implication

$$f \in \Lambda(p, \omega) \Longrightarrow E(f) = \varnothing,$$

holds.

Theorem 4. *Let $1 < p < \infty$ and $\omega \in \mathcal{AW}$. Then the following two conditions are equivalent.*

(i) $\omega(t)/t^{1+(1/p)} \in L^1([0,1])$.

(ii) *$E(f)=\varnothing$ for every $f\in\Lambda(p,\omega)$.*

Proof of Theorem 4. To prove the implication (i) $\Rightarrow$ (ii) we shall use the following result.

Lemma 3. *If $1\le p<\infty$, $\omega\in\mathcal{AW}$ and $f\in\Lambda(p,\omega)$, then for $0<r<1$ and $p<q\le\infty$,*

$$M_q(r,f')\le C\frac{\omega(1-r)}{(1-r)^{1+(1/p)-(1/q)}}.$$

This lemma can be proved with the arguments used in the proof of Theorem 5.9 of [**14**].

Proof of (i) $\Rightarrow$ (ii). Let p and ω be as in Theorem 4 and take $f\in\Lambda(p,\omega)$. Using Lemma 3 with $q=\infty$, we obtain for every θ,

$$|f'(re^{i\theta})|\le M_\infty(r,f')\le C\frac{\omega(1-r)}{(1-r)^{1+(1/p)}},\quad 0<r<1,$$

which, with (i) gives

$$V(f,\theta)=\int_0^1|f'(re^{i\theta})|\,dr\le C\int_0^1\frac{\omega(1-r)}{(1-r)^{1+(1/p)}}\,dr<\infty,\quad\text{for every }\theta.$$

Hence $E(f)=\varnothing$. $\square$

Proof of (ii) $\Rightarrow$ (i). We shall argue by contradiction. Hence, let $1<p<\infty$ and $\omega\in\mathcal{AW}$, and suppose that

$$\frac{\omega(t)}{t^{1+(1/p)}}\notin L^1([0,1]). \tag{3.6}$$

Set

$$f(z)=\int_0^1\frac{\omega(t)}{t(1+t-z)^{1/p}}\,dt,\quad z\in\Delta. \tag{3.7}$$

It is clear that f is holomorphic in Δ. In fact, looking at the proof of Theorem 1.2 of [**9**], we see that

$$f \in \Lambda(p, \omega), \tag{3.8}$$

and

$$|f'(r)| = f'(r) \geq C \frac{\omega(1-r)}{(1-r)^{1+(1/p)}}, \quad \frac{1}{2} < r < 1. \tag{3.9}$$

Then (3.6) gives that $\int_0^1 |f'(r)|\, dr = \infty$ and hence $E(f) \neq \varnothing$. Consequently, the implication (ii) $\Rightarrow$ (i) in Theorem 4 holds.

Remark 4. If $\omega \in \mathcal{AW}$ and $f \in \Lambda(\infty, \omega)$ then, for $0 < r < 1$ and for every θ we have

$$|f'(re^{i\theta})| \leq \frac{\omega(1-r)}{1-r}$$

which, bearing in mind that $(\omega(t)/t) \in L^1([0,1])$, implies

$$V(f, \theta) \leq \int_0^1 \frac{\omega(1-r)}{1-r}\, dr < \infty, \quad \text{for every } \theta.$$

That is, we have proved: For every $\omega \in \mathcal{AW}$, we have that

$$f \in \Lambda(\infty, \omega) \Longrightarrow E(f) = \varnothing. \tag{3.10}$$

Let $\mathcal{D}$ be the family of those functions f, holomorphic in Δ, with finite Dirichlet integral, that is, those which satisfy

$$\mathcal{D}(f) = \frac{1}{\pi} \iint_\Delta |f'(z)|^2\, dx\, dy < \infty. \tag{3.11}$$

Geometrically, this is equivalent to saying that f maps Δ onto a Riemann surface of finite area. We recall that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$\mathcal{D}(f) = \sum_{n=0}^{\infty} n|a_n|^2. \tag{3.12}$$

The following well-known result is due to Beurling [**7**] (see also Chapter VIII of [**25**]).

Theorem C. *If $f \in \mathcal{D}$, then the set of points $e^{i\theta}$ for which $V(f,\theta) = \infty$ is a set of logarithmic capacity zero.*

We refer to [**17**], [**22**] and [**25**] for the definition and basic results about logarithmic capacity.

The conclusion of Theorem C is not true for $f \in \mathrm{BMOA}$. Actually it is not possible to state any result similar to Theorem C with BMOA in the place of $\mathcal{D}$ because a function $f \in \mathrm{BMOA}$ exists such that $V(f,\theta) = \infty$ for every θ. Indeed, Zygmund proved in [**28**] that if g is a function analytic in Δ which is given by a power series with Hadamard gaps,

$$g(z) = \sum_{k=0}^{\infty} a_k z^{n_k}, \quad \text{with } \frac{n_{k+1}}{n_k} > \lambda \quad \text{and} \quad \lambda > 1, \tag{3.13}$$

then

$$\sum_{k=1}^{\infty} |a_k| \le A_\lambda V(g,\theta), \quad \text{for every } \theta, \tag{3.14}$$

with A_λ being positive constant which depends only on λ.

Now if g is as in (3.13) and $\sum_{k=0}^{\infty} |a_k|^2 < \infty$, then $g \in \mathrm{BMOA}$ (see, e.g., [**3**, p. 25]). Using this fact and Zygmund's result, we easily see that if we take

$$f(z) = \sum_{k=1}^{\infty} \frac{z^{2^k}}{k}, \quad z \in \Delta,$$

then we have $V(f,\theta) = \infty$ for every θ and $f \in \mathrm{BMOA}$. This proves our assertions. See also [**24**].

On the other hand, extensions of Theorem C have been proved for certain Dirichlet-type spaces. Let us mention the following:

(i) For $0 < a < 1$, let $\mathcal{D}_a$ be the space of all functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$, analytic in Δ with

$$\sum_{n=1}^{\infty} n^{1-a} |a_n|^2 < \infty. \tag{3.15}$$

Zygmund proved (see [**22**, pp. 49–51]) the following result:

$$\text{If } f \in \mathcal{D}_a \text{ and } 0 < a < 1, \text{ then the exceptional set } E(f) \text{ has zero } a\text{-capacity.} \tag{3.16}$$

We refer to [**22**] and [**17**] for the definitions and results about capacities and Hausdorff measures.

(ii) If W is a positive, increasing and continuous function on $[0, \infty)$, we define $\mathcal{D}_W$ to be the space of those functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ that are holomorphic in Δ and for which

$$\sum_{n=1}^{\infty} W(n)|a_n|^2 < \infty.$$

The radial variation of functions in the spaces $\mathcal{D}_W$ has recently been studied by Twomey in [**26**]. The results obtained by Twomey for these spaces include the classical results for the classes $\mathcal{D}_a$ that we have just mentioned.

It is also worth mentioning that if f is analytic, univalent and bounded in Δ, then $f \in \mathcal{D}$, and hence $E(f)$, has logarithmic capacity zero. Using a standard method which relates any univalent function to a bounded univalent function (see, e.g., [**15**, p. 11]), this result can be extended to any univalent function. That is, we have:

Remark 5. If f is a function which is analytic and univalent in Δ, then the exceptional set $E(f)$ has logarithmic capacity zero.

Next we shall study the possibility of obtaining results similar to Theorem C for mean Lipschitz spaces. First let us consider the classical Λ^p_α-spaces. It is clear that $\mathcal{D} \subset \Lambda^2_{1/2}$. Then, using Theorem 5.9 of [**14**] or [**12**, Corollary 2.3] and Theorem A, we have that

$$\mathcal{D} \subset \Lambda^2_{1/2} \subset \Lambda^p_{1/p} \subset \mathrm{BMOA}, \quad \text{if } 2 \le p < \infty. \tag{3.17}$$

Let $p \ge 2$ and suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \Lambda^p_{1/p}$. We have

$$\begin{aligned}\int_0^1 (1-r)^a M_2(r, f')^2\, dr &\le \int_0^1 (1-r)^a M_p(r, f')^2\, dr \\ &\le C \int_0^1 (1-r)^{a-2+(2/p)}\, dr.\end{aligned}$$

Consequently,

$$\int_0^1 (1-r)^a M_2(r,f')^2\,dr < \infty \quad \text{for all } a > 1 - \frac{2}{p}. \tag{3.18}$$

But

$$\begin{aligned} \int_0^1 (1-r)^a M_2(r,f')^2\,dr &= \sum_{n=1}^{\infty} n^2 |a_n|^2 \int_0^1 (1-r)^a r^{2n-2}\,dr \\ &\approx \sum_{n=1}^{\infty} n^2|a_n|^2 B(a+1,n) \\ &\approx \sum_{n=1}^{\infty} n^{1-a}|a_n|^2. \end{aligned} \tag{3.19}$$

Here $B(\ ,\ )$ denotes the beta function. We have used the well-known estimate

$$B(a+1,n) \approx n^{-1-a}.$$

From (3.18) and (3.19) we obtain:

(3.20) If $2 \le p < \infty$ and $a > 1-(2/p)$, then $\Lambda^p_{1/p} \subset \mathcal{D}_a$.

Now we can prove the following result.

Theorem 5. (i) *Let $2 < p < \infty$ and $f \in \Lambda^p_{1/p}$. Then, for every $a > 1-(2/p)$, the exceptional set $E(f)$ has zero a-capacity.*

(ii) *If $1 < p \le 2$ and $f \in \Lambda^p_{1/p}$, then the exceptional set $E(f)$ has Hausdorff dimension equal to zero.*

Proof. Part (i) follows from (3.20) and (3.16).

If $1 < p \le 2$ and $f \in \Lambda^p_{1/p}$, then $f \in \Lambda^2_{1/2}$ ([**12**, p. 88]) and then (3.20) and (3.16) show that $E(f)$ has a-capacity zero for all $a > 0$ and it is well known (see, e.g., [**22**, p. 34]) that this implies that $E(f)$ has Hausdorff dimension zero. ◻

It would be interesting to know how sharp Theorem 5 (i) is and, especially, it is natural to ask whether or not the condition $f \in \Lambda^2_{1/2}$

implies that $E(f)$ has logarithmic capacity zero. We do not know the answer to this question, and then we shall look for some condition on an analytic function in Δ, stronger than the condition "$f \in \Lambda^2_{1/2}$" and weaker than the condition "$f \in \mathcal{D}$," but enough to insure that the set $E(f)$ has logarithmic capacity zero.

Bennett and Stoll proved in [**6**] that if f is analytic in Δ and

$$\sup_{0<r<1} \int_{-\pi}^{\pi} |\operatorname{Re} f'(re^{i\theta})| \, d\theta < \infty,$$

then $f \in$ BMOA. Actually, using Theorem 1.1 of [**14**], it is clear that this result is equivalent to the following:

Theorem D. *Let f be an analytic function in Δ such that f' is the Cauchy-Stieltjes integral of a finite complex measure μ on $\mathbf{T}$, i.e.,*

$$f'(z) = \int_{\mathbf{T}} \frac{d\mu(\xi)}{1 - \bar{\xi} z}, \quad z \in \Delta.$$

Then $f \in$ BMOA.

For simplicity, let us define $\mathcal{K}$ as the space of those functions f, analytic in Δ, which are the Cauchy-Stieltjes integral of a finite complex Borel measure on $\mathbf{T}$. Then Theorem D is equivalent to the following

$$f' \in \mathcal{K} \Longrightarrow f \in \text{BMOA}. \tag{3.21}$$

Next we prove that this result can be improved and that the conclusion of Theorem C remains true with the condition "$f' \in \mathcal{K}$" in the place of "$f \in \mathcal{D}$."

Theorem 6. *Let f be an analytic function in Δ such that $f' \in \mathcal{K}$. Then the following two assertions hold.*

(i) $f \in \Lambda^2_{1/2}$.

(ii) *The exceptional set $E(f)$ has logarithmic capacity zero.*

Proof. Let f be as in the theorem. Using Theorem 1.1 of [**14**], we see that f can be written as

$$f = f_1 - f_2 + i(f_3 - f_4), \tag{3.22}$$

with f_j $(j = 1, 2, 3, 4)$ analytic in Δ and

$$\operatorname{Re} f_j'(z) > 0, \quad z \in \Delta, \quad j = 1, 2, 3, 4. \tag{3.23}$$

Now it is well known that (3.23) implies that

$$M_2(r, f_j') \le C M_2\left(r, \frac{1+z}{1-z}\right) = O\left(\frac{1}{(1-r)^{1/2}}\right), \quad \text{as } r \to 1$$

(see, e.g., [**14**]). Then, using (3.22), we deduce that

$$M_2(r, f') = O\left(\frac{1}{(1-r)^{1/2}}\right), \quad \text{as } r \to 1,$$

that is, $f \in \Lambda^2_{1/2}$. This proves (i).

Also, (3.23) implies that the f_j's are univalent. Then, using Remark 5, we see that the exceptional sets $E(f_j)$ $(j = 1, 2, 3, 4)$ have logarithmic capacity zero. Then, using (3.22), it follows that the exceptional set $E(f)$ has logarithmic capacity zero. This proves (ii). $\square$

Let us remark that the first part of Theorem 6 can be improved. Baernstein and Brown introduced in [**4**] the space weak-H^1. A function f, analytic in Δ, is said to belong to weak-H^1 if f belongs to the Nevanlinna uniform class N^+ and a constant $b > 0$ exists such that

$$|\{t \in [-\pi, \pi] : |f(e^{i\theta})| > \lambda\}| \le b\lambda^{-1}, \quad \lambda > 0.$$

Here $|E|$ denotes the one-dimensional Lebesgue measure of E. It is well known that

$$H^1 \subset \mathcal{K} \subset \text{weak-}H^1. \tag{3.24}$$

Proposition 3 of [**4**] shows that if f is analytic in Δ, then

$$f' \in \text{weak-}H^1 \Longrightarrow f \in \Lambda^2_{1/2}, \tag{3.25}$$

a result which is stronger than Theorem 6 (i). In view of this, it is natural to ask whether or not the condition "$f' \in$ weak-H^1" implies that the exceptional set $E(f)$ has logarithmic capacity zero. We do not know the answer to this question.

Let us finish remarking that the inclusions in (3.24) are strict. Clearly, the function $f(z) = (1/(1-z))$ belongs to $\mathcal{K}\backslash H^1$. Hence, $H^1 \subsetneq \mathcal{K}$.

To show that the inclusion $\mathcal{K} \subset$ weak-H^1 is also strict, set

$$F(z) = \frac{z}{(1-z)^{1+i}}, \quad z \in \Delta. \tag{3.26}$$

We will show that $F \in$ weak-$H^1\backslash\mathcal{K}$. We have

$$|F(z)| = \frac{|z|}{|1-z|} e^{\mathrm{Arg}\,(1-z)} \leq \frac{e^{\pi/2}}{|1-z|}, \quad z \in \Delta, \tag{3.27}$$

which easily implies that $F \in$ weak-H^1. The fact that $F \notin \mathcal{K}$ was proved by Bass in his Ph.D. Thesis. Let us just say that it can be deduced from the fact that F is univalent (see Exercise 19 in [**15**, p. 72]) and the results of [**5**].

Acknowledgments. We wish to thank the referee for his/her helpful comments and remarks.

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