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# ANALYTIC FOURIER-FEYNMAN TRANSFORM AND CONVOLUTION OF FUNCTIONALS ON ABSTRACT WIENER SPACE

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ABSTRACT. Huffman, Park and Skoug obtained various results for the $L_p$ analytic Fourier-Feynman transform and the convolution of functionals in some Banach algebra $\mathcal{S}$ on classical Wiener space. Recently, Ahn studied $L_1$ analytic Fourier-Feynman transform theory for functionals in the Fresnel class $\mathcal{F}(B)$ of abstract Wiener space $(B,\nu)$.

In this paper we first define an $L_p$ analytic Fourier-Feynman transform and a convolution of functionals on a product abstract Wiener space and establish various relationships between the Fourier-Feynman transform and convolution for functionals in the generalized Fresnel class $\mathcal{F}_{A_1,A_2}$ containing $\mathcal{F}(B)$. Also we obtain Parseval's relation for those functionals. Results of Huffman, Park, Skoug and Ahn are corollaries of our results.

**1. Introduction.** The concept of an $L_1$ analytic Fourier-Feynman transform for functionals on classical Wiener space was introduced by Brue in [**3**]. In [**4**], Cameron and Storvick introduced an $L_2$ analytic Fourier-Feynman transform on classical Wiener space. In [**11**], Johnson and Skoug developed an $L_p$ analytic Fourier-Feynman transform theory for $1 \le p \le 2$ that extended the results in [**3**], [**4**] and gave various relationships between the $L_1$ and $L_2$ theories. In [**8**] Huffman, Park and Skoug defined a convolution product for functionals on classical Wiener space, and they obtained various results for the Fourier-Feynman transform and the convolution product [**8**], [**9**], [**10**]. Moreover, Chang, Kim and Yoo [**6**] introduced the integral transform which is an extension of Fourier-Wiener transform on the abstract Wiener space, and they established the relationship between the integral transform of functionals in some classes and the integral transform of their convolution.

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Recently, Ahn [**1**] introduced an $L_1$ analytic Fourier-Feynman transform and a convolution on the Fresnel class $\mathcal{F}(B)$ of abstract Wiener space, and he obtained similar results as in [**9**].

On the other hand, for a successful treatment of certain physical problems by means of a Feynman integral (e.g., the anharmonic oscillator of [**2**], Section 5) Kallianpur and Bromley introduced a larger class $\mathcal{F}_{A_1,A_2}$ than the Fresnel class $\mathcal{F}(B)$ and showed the existence of the analytic Feynman integral of functionals in $\mathcal{F}_{A_1,A_2}$ [**12**].

In this paper we define an $L_p$ analytic Fourier-Feynman transform and a convolution of functionals defined on a product abstract Wiener space and establish various relationships between the Fourier-Feynman transforms of functionals in $\mathcal{F}_{A_1,A_2}$ and the Fourier-Feynman transform of their convolution. In addition, we establish a Parseval's relation for functionals in $\mathcal{F}_{A_1,A_2}$ from this relationship. Results in [**1**], [**9**] are corollaries of our results.

**2. Definitions and preliminaries.** Let $(H, B, \nu)$ be an abstract Wiener space, and let $\{e_j\}$ be a complete orthonormal system in $H$ such that the $e_j$'s are in $B^*$, the dual of $B$. For each $h \in H$ and $x \in B$, we define a stochastic inner product $(h,x)^\sim$ as follows:

$$(2.1)\qquad (h,x)^\sim = \begin{cases} \lim_{n\to\infty} \sum_{j=1}^{n} \langle h, e_j\rangle (x, e_j) & \text{if the limit exists} \\ 0 & \text{otherwise,} \end{cases}$$

where $(\cdot,\cdot)$ denotes the natural dual pairing between $B$ and $B^*$. It is well known [**12**], [**13**] that, for each $h(\neq 0)$ in $H$, $(h,\cdot)^\sim$ is a Gaussian random variable on $B$ with mean zero and variance $|h|^2$, that is,

$$(2.2)\qquad \int_B \exp\{i(h,x)^\sim\}\, d\nu(x) = \exp\left\{ -\frac{1}{2}|h|^2 \right\}.$$

A subset $E$ of a product abstract Wiener space $B^2$ is said to be scale-invariant measurable provided $\{(\alpha x_1, \beta x_2) : (x_1, x_2) \in E\}$ is abstract Wiener measurable for every $\alpha > 0$ and $\beta > 0$, and a scale-invariant measurable set $N$ is said to be scale-invariant null provided $(\nu\times\nu)(\{(\alpha x_1, \beta x_2) : (x_1,x_2) \in N\}) = 0$ for every $\alpha > 0$ and $\beta > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere ($s$ almost everywhere). A function $F$

is said to be scale-invariant measurable provided $F$ is defined on a scale-invariant measurable set and $F((\alpha\cdot,\beta\cdot))$ is abstract Wiener measurable for every $\alpha>0$ and $\beta>0$. Given two complex-valued functions $F$ and $G$ on $B^2$, we say that $F=G$ $s$ almost everywhere, and write $F\approx G$, if $F(\alpha x_1,\beta x_2)=G(\alpha x_1,\beta x_2)$ for $\nu\times\nu$ almost every $(x_1,x_2)\in B^2$ for all $\alpha>0$ and $\beta>0$. For a functional $F$ on $B^2$, we will denote by $[F]$ the equivalence class of functionals which are equal to $F$ $s$ almost everywhere.

Let $\mathbf{C}$ denote the complex numbers, and let

$$\Omega=\{\vec{\lambda}=(\lambda_1,\lambda_2)\in\mathbf{C}^2:\operatorname{Re}\lambda_k>0 \text{ for } k=1,2\} \tag{2.3}$$

and

$$\tilde{\Omega}=\{\vec{\lambda}=(\lambda_1,\lambda_2)\in\mathbf{C}^2:\lambda_k\neq 0,\ \operatorname{Re}\lambda_k\geq 0 \text{ for } k=1,2\}. \tag{2.4}$$

Let $F$ be a complex-valued function on $B^2$ such that the integral

$$J_F(\lambda_1,\lambda_2)=\int_{B^2}F(\lambda_1^{-1/2}x_1,\lambda_2^{-1/2}x_2)d(\nu\times\nu)(x_1,x_2) \tag{2.5}$$

exists as a finite number for all real numbers $\lambda_1>0$ and $\lambda_2>0$. If there exists a function $J_F^*(\lambda_1,\lambda_2)$ analytic on $\Omega$ such that $J_F^*(\lambda_1,\lambda_2)=J_F(\lambda_1,\lambda_2)$ for all $\lambda_1>0$ and $\lambda_2>0$, then $J_F^*(\lambda_1,\lambda_2)$ is defined to be the analytic Wiener integral of $F$ over $B^2$ with parameter $\vec{\lambda}=(\lambda_1,\lambda_2)$, and for $\vec{\lambda}\in\Omega$ we write

$$\int_{B^2}^{\mathrm{anw}_{\vec{\lambda}}}F(x_1,x_2)d(\nu\times\nu)(x_1,x_2)=J_F^*(\lambda_1,\lambda_2). \tag{2.6}$$

Let $q_1$ and $q_2$ be nonzero real numbers and $F$ a functional on $B^2$ such that $\int_{B^2}^{\mathrm{anw}_{\vec{\lambda}}}F(x_1,x_2)d(\nu\times\nu)(x_1,x_2)$ exists for all $\vec{\lambda}\in\Omega$. If the following limit exists, then we call it the analytic Feynman integral of $F$ over $B^2$ with parameter $\vec{q}=(q_1,q_2)$, and we write

$$\begin{aligned}\int_{B^2}^{\mathrm{anf}_{\vec{q}}}&F(x_1,x_2)d(\nu\times\nu)(x_1,x_2)\\&=\lim_{\vec{\lambda}\to(-iq_1,-iq_2)}\int_{B^2}^{\mathrm{anw}_{\vec{\lambda}}}F(x_1,x_2)d(\nu\times\nu)(x_1,x_2),\end{aligned} \tag{2.7}$$

where $\vec{\lambda} = (\lambda_1, \lambda_2)$ approaches $(-iq_1, -iq_2)$ through $\Omega$.

**Notation 2.1.** (i) For $\vec{\lambda} = (\lambda_1, \lambda_2) \in \Omega$ and $(y_1, y_2) \in B^2$, let

$$(T_{\vec{\lambda}}(F))(y_1, y_2) = \int_{B^2}^{\text{anw}_{\vec{\lambda}}} F(x_1 + y_1, x_2 + y_2) d(\nu \times \nu)(x_1, x_2). \tag{2.8}$$

(ii) Let $1 < p \le 2$, and let $\{G_n\}$ and $G$ be scale-invariant measurable functionals such that, for each $\alpha > 0$ and $\beta > 0$,

$$\lim_{n\to\infty} \int_{B^2} |G_n(\alpha x_1, \beta x_2) - G(\alpha x_1, \beta x_2)|^{p'} d(\nu \times \nu)(x_1, x_2) = 0, \tag{2.9}$$

where $p$ and $p'$ are related by $(1/p) + (1/p') = 1$. Then we write

$$\lim_{n\to\infty} (w_s^{p'})(G_n) \approx G \tag{2.10}$$

and call $G$ the scale-invariant limit in the mean of order $p'$. A similar definition is understood when $n$ is replaced by the continuously varying parameter $\vec{\lambda}$.

**Definition 2.2.** Let $q_1$ and $q_2$ be nonzero real numbers. For $1 < p \le 2$, we define the $L_p$ analytic Fourier-Feynman transform $T_{\vec{q}}^{(p)}(F)$ of $F$ on $B^2$ by the formula ($\vec{\lambda} \in \Omega$)

$$(T_{\vec{q}}^{(p)}(F))(y_1, y_2) = \lim_{\vec{\lambda}\to(-iq_1,-iq_2)} (w_s^{p'})(T_{\vec{\lambda}}(F))(y_1, y_2), \tag{2.11}$$

whenever this limit exists. We define the $L_1$ analytic Fourier-Feynman transform $T_{\vec{q}}^{(1)}(F)$ of $F$ by ($\vec{\lambda} \in \Omega$)

$$(T_{\vec{q}}^{(1)}(F))(y_1, y_2) = \lim_{\vec{\lambda}\to(-iq_1,-iq_2)} (T_{\vec{\lambda}}(F))(y_1, y_2), \tag{2.12}$$

for $s$ almost everywhere, $(y_1, y_2) \in B^2$.

We note that, for $1 \le p \le 2$, $T_{\vec{q}}^{(p)}(F)$ is defined only $s$ almost everywhere. We also note that if $T_{\vec{q}}^{(p)}(F_1)$ exists and if $F_1 \approx F_2$, then $T_{\vec{q}}^{(p)}(F_2)$ exists and $T_{\vec{q}}^{(p)}(F_1) \approx T_{\vec{q}}^{(p)}(F_2)$.

**Definition 2.3.** Let $F$ and $G$ be functionals on $B^2$. For $\vec{\lambda} = (\lambda_1, \lambda_2) \in \Omega$, we define their convolution product, if it exists, by

(2.13)
$$(F * G)_{\vec{\lambda}}(y_1, y_2) = \int_{B^2}^{\mathrm{anw}_{\vec{\lambda}}} F\left(\frac{y_1+x_1}{\sqrt{2}}, \frac{y_2+x_2}{\sqrt{2}}\right) G\left(\frac{y_1-x_1}{\sqrt{2}}, \frac{y_2-x_2}{\sqrt{2}}\right) d(\nu\times\nu)(x_1, x_2).$$

For $\vec{q} = (q_1, q_2)$ with nonzero real numbers $q_1$ and $q_2$, we define their convolution product, if it exists, by

(2.14)
$$(F * G)_{\vec{q}}(y_1, y_2) = \int_{B^2}^{\mathrm{anf}_{\vec{q}}} F\left(\frac{y_1+x_1}{\sqrt{2}}, \frac{y_2+x_2}{\sqrt{2}}\right) G\left(\frac{y_1-x_1}{\sqrt{2}}, \frac{y_2-x_2}{\sqrt{2}}\right) d(\nu\times\nu)(x_1, x_2).$$

**Definition 2.4.** Let $A_1$ and $A_2$ be bounded, nonnegative self-adjoint operators on $H$. Let $\mathcal{F}_{A_1,A_2}$ be the space of all $s$-equivalence classes of functionals $F$ on $B^2$ which have the form

$$F(x_1, x_2) = \int_H \exp\left\{ i[(A_1^{1/2}h, x_1)^\sim + (A_2^{1/2}h, x_2)^\sim] \right\} d\sigma(h) \tag{2.15}$$

for some complex-valued countably additive Borel measure $\sigma$ on $H$.

As is customary, we will identify a functional with its $s$-equivalence class and think of $\mathcal{F}_{A_1,A_2}$ as a collection of functionals on $B^2$ rather than as a collection of equivalence classes.

Let $M(H)$ denote the space of complex-valued countably additive Borel measures on $H$. Under the total variation norm $\|\cdot\|$ and with convolution as multiplication, $M(H)$ is a commutative Banach algebra with identity [**2**]. In addition the map $\sigma \mapsto [F]$ defined by (2.15) sets up an algebra isomoprhism between $M(H)$ and $\mathcal{F}_{A_1,A_2}$ if the range of $A_1 + A_2$ is dense in $H$. In this case $\mathcal{F}_{A_1,A_2}$ becomes a Banach algebra under the norm $\|F\| = \|\sigma\|$ [**12**].

*Remark* 2.5. Let $\mathcal{F}(B)$ denote the class of all functions $F$ on $B$ of the form

$$F(x) = \int_H \exp\{i(h, x)^\sim\} \, d\sigma(h)$$

for some $\sigma \in M(H)$. Then we know that if $A_1$ is the identity operator on $H$ and $A_2 = 0$, then $\mathcal{F}_{A_1,A_2}$ is essentially the Fresnel class $\mathcal{F}(B)$.

**3. Transform and convolution of functionals in $\mathcal{F}_{A_1,A_2}$.** In this section we establish several results involving the concepts of '$L_p$ analytic Fourier-Feynman transform' and 'convolution' for functionals in the class $\mathcal{F}_{A_1,A_2}$. In addition, we establish some interesting formulas for functionals in $\mathcal{F}_{A_1,A_2}$.

We begin with the existence theorem of the $L_p$ analytic Fourier-Feynman transform for functionals in $\mathcal{F}_{A_1,A_2}$.

**Theorem 3.1.** *Let $F \in \mathcal{F}_{A_1,A_2}$ be given by*

$$F(x_1, x_2) = \int_H \exp\Big\{i[(A_1^{1/2}/h, x_1)^\sim + (A_2^{1/2}h, x_2)^\sim]\Big\}\, d\sigma(h), \tag{3.1}$$

*for s almost everywhere, $(x_1, x_2) \in B^2$, where $\sigma$ is an element of $M(H)$. Then, for all $p$ with $1 \le p \le 2$, the $L_p$ analytic Fourier-Feynman transform $T_{\vec{q}}^{(p)}(F)$, $\vec{q} = (q_1, q_2)$, exists for all nonzero real numbers $q_1$ and $q_2$, and belongs to $\mathcal{F}_{A_1,A_2}$. Moreover, $T_{\vec{q}}^{(p)}(F)$ is given by the formula*

(3.2)
$$(T_{\vec{q}}^{(p)}(F))(y_1, y_2) = \int_H \exp\Big\{i[(A_1^{1/2}h, y_1)^\sim + (A_2^{1/2}h, y_2)^\sim] - \frac{i}{2q_1}|A_1^{1/2}h|^2 - \frac{i}{2q_2}|A_2^{1/2}h|^2\Big\}\, d\sigma(h),$$

*for s almost everywhere, $(y_1, y_2) \in B^2$.*

*Proof.* For all $\lambda_1 > 0$, $\lambda_2 > 0$ and $s$ almost everywhere, $(y_1, y_2) \in B^2$, using the Fubini theorem and (2.2) we obtain

(3.3)

$$
\begin{aligned}
&(T_{\vec{\lambda}}(F))(y_1,y_2)\\
&= \int_{B^2} F(\lambda_1^{-1/2}x_1+y_1, \lambda_2^{-1/2}x_2+y_2)\,d(\nu\times\nu)(x_1,x_2)\\
&= \int_{B^2}\int_H \exp\Big\{i[(A_1^{1/2}h, \lambda_1^{-1/2}x_1+y_1)^\sim\\
&\qquad + (A_2^{1/2}h, \lambda_2^{-1/2}x_2+y_2)^\sim]\Big\}d\sigma(h)\,d(\nu\times\nu)(x_1,x_2)\\
&= \int_H \exp\Big\{i[(A_1^{1/2}h, y_1)^\sim + (A_2^{1/2}h, y_2)^\sim]\\
&\qquad - \frac{1}{2\lambda_1}|A_1^{1/2}h|^2 - \frac{1}{2\lambda_2}|A_2^{1/2}h|^2\Big\}\,d\sigma(h).
\end{aligned}
$$

Let $\vec{\lambda} = (\lambda_1,\lambda_2) \in \tilde{\Omega}$, and let $\{(\lambda_{1,n},\lambda_{2,n})\}$ be a sequence in $\tilde{\Omega}$ which converges to $\vec{\lambda}$. Then

$$
\Big|\exp\Big\{i[(A_1^{1/2}h, y_1)^\sim + (A_2^{1/2}h, y_2)^\sim] - \frac{1}{2\lambda_{1,n}}|A_1^{1/2}h|^2 - \frac{1}{2\lambda_{2,n}}|A_2^{1/2}h|^2\Big\}\Big| \le 1
$$

for all $n = 1,2,\dots$, and so, by the dominated convergence theorem, the last expression in (3.3) is a bounded continuous function of $\vec{\lambda} \in \tilde{\Omega}$. Also, by the Morera theorem, we can show that it is an analytic function of $\vec{\lambda} \in \Omega$. Hence for $\vec{\lambda} \in \Omega$ and $s$ almost everywhere, $(y_1,y_2) \in B^2$,

$$
(T_{\vec{\lambda}}(F))(y_1,y_2) = \int_H \exp\Big\{i[(A_1^{1/2}h, y_1)^\sim + (A_2^{1/2}h, y_2)^\sim] - \frac{1}{2\lambda_1}|A_1^{1/2}h|^2 - \frac{1}{2\lambda_2}|A_2^{1/2}h|^2\Big\}\,d\sigma(h).
$$

In case $p = 1$, by the dominated convergence theorem,

$$
\lim_{\vec{\lambda}\to(-iq_1,-iq_2)} (T_{\vec{\lambda}}(F))(y_1,y_2) = \int_H \exp\Big\{i[(A_1^{1/2}h, y_1)^\sim + (A_2^{1/2}h, y_2)^\sim] - \frac{i}{2q_1}|A_1^{1/2}h|^2 - \frac{i}{2q_2}|A_2^{1/2}h|^2\Big\}\,d\sigma(h),
$$

for $s$ almost everywhere, $(y_1,y_2)\in B^2$, where $\vec{\lambda}\to(-iq_1,-iq_2)$ through $\Omega$. If $1<p\le 2$, again by the dominated convergence theorem,

$$\lim_{\vec{\lambda}\to(-iq_1,-iq_2)}\int_{B^2}\bigg|\int_H \exp\Big\{i[(A_1^{1/2}h,y_1)^\sim+(A_2^{1/2}h,y_2)^\sim] - \frac{i}{2q_1}|A_1^{1/2}h|^2-\frac{i}{2q_2}|A_2^{1/2}h|^2\Big\}\,d\sigma(h) - (T_{\vec{\lambda}}(F))(y_1,y_2)\bigg|^{p'}\,d(\nu\times\nu)(y_1,y_2)=0$$

for $s$ almost everywhere, $(y_1,y_2)\in B^2$, where $\vec{\lambda}\to(-iq_1,-iq_2)$ through $\Omega$. Hence $(T_{\vec{q}}^{(p)}(F))(y_1,y_2)$ exists and is given by (3.2) for all desired values of $p$ and $\vec{q}$.

Finally, let $\sigma'$ be a set function on $\mathcal{B}(H)$, the Borel class of $H$, defined by

$$\sigma'(E)=\int_E \exp\Big\{-\frac{i}{2q_1}|A_1^{1/2}h|^2-\frac{i}{2q_2}|A_2^{1/2}h|^2\Big\}\,d\sigma(h),\quad E\in\mathcal{B}(H).$$

Then $\sigma'\in M(H)$ and

$$(T_{\vec{q}}^{(p)}(F))(y_1,y_2)=\int_H \exp\Big\{i[(A_1^{1/2}h,y_1)^\sim+(A_2^{1/2}h,y_2)^\sim]\Big\}\,d\sigma'(h), \tag{3.4}$$

for $s$ almost everywhere, $(y_1,y_2)\in B^2$. Thus, $T_{\vec{q}}^{(p)}(F)$ belongs to $\mathcal{F}_{A_1,A_2}$. $\square$

Next we obtain an inverse transform theorem for $F\in\mathcal{F}_{A_1,A_2}$.

**Theorem 3.2.** *Let $F\in\mathcal{F}_{A_1,A_2}$ be given by* (3.1). *Then, for all nonzero real numbers $q_1$ and $q_2$, and for $1\le p\le 2$,*

$$T_{-\vec{q}}^{(p)}(T_{\vec{q}}^{(p)}(F))\approx F, \tag{3.5}$$

*where $\vec{q}=(q_1,q_2)$ and $-\vec{q}=(-q_1,-q_2)$.*

*Proof.* Proceeding as in the proof of Theorem 3.1, for all $\lambda_1,\lambda_2>0$ and $s$ almost everywhere, $(y_1,y_2)\in B^2$, using (3.2), Fubini's theorem and (2.2), we have

$$
\begin{aligned}
&(T_{\vec\lambda}(T_{\vec q}^{(p)}(F)))(y_1,y_2)\\
&= \int_{B^2}(T_{\vec q}^{(p)}(F))(\lambda_1^{-1/2}x_1+y_1,\lambda_2^{-1/2}x_2+y_2)\,d(\nu\times\nu)(x_1,x_2)\\
&= \int_{B^2}\int_H \exp\Big\{i[(A_1^{1/2}h,\lambda_1^{-1/2}x_1+y_1)^\sim+(A_2^{1/2}h,\lambda_2^{-1/2}x_2+y_2)^\sim]\\
&\qquad\qquad -\frac{i}{2q_1}|A_1^{1/2}h|^2-\frac{i}{2q_2}|A_2^{1/2}h|^2\Big\}\,d\sigma(h)\,d(\nu\times\nu)(x_1,x_2)\\
&= \int_H \exp\Big\{i[(A_1^{1/2}h,y_1)^\sim+(A_2^{1/2}h,y_2)^\sim]-\frac{i}{2q_1}|A_1^{1/2}h|^2-\frac{i}{2q_2}|A_2^{1/2}h|^2\\
&\qquad\qquad -\frac{1}{2\lambda_1}|A_1^{1/2}h|^2-\frac{1}{2\lambda_2}|A_2^{1/2}h|^2\Big\}\,d\sigma(h).
\end{aligned}
$$

By the same method as in the proof of Theorem 3.1, we can show that the last expression is an analytic function of $\vec\lambda$ throughout $\Omega$, and it is a bounded continuous function of $\vec\lambda$ on $\tilde\Omega$ for all $(y_1,y_2)\in B^2$. Hence if we let $\vec\lambda\to(iq_1,iq_2)$ through values in $\Omega$, then we obtain $T_{-\vec q}^{(p)}(T_{\vec q}^{(p)}(F))\approx F$ as desired. □

**Theorem 3.3.** *Let $F$ and $G$ be elements of $\mathcal{F}_{A_1,A_2}$ with corresponding finite Borel measures $\sigma$ and $\rho$ in $M(H)$, respectively. Then their convolution product $(F*G)_{\vec q}$ exists for all nonzero real numbers $q_1,q_2$ and belongs to $\mathcal{F}_{A_1,A_2}$. Moreover, $(F*G)_{\vec q}$ is given by the formula*

$$
\begin{aligned}
&(F*G)_{\vec q}(y_1,y_2)\\
&\quad= \int_{H^2}\exp\Big\{\frac{i}{\sqrt2}[(A_1^{1/2}(h+k),y_1)^\sim+(A_2^{1/2}(h+k),y_2)^\sim]\\
&\qquad\qquad -\frac{i}{4q_1}|A_1^{1/2}(h-k)|^2-\frac{i}{4q_2}|A_2^{1/2}(h-k)|^2\Big\}\,d\sigma(h)\,d\rho(k),
\end{aligned}
$$

*for $s$ almost everywhere, $(y_1,y_2)\in B^2$.*

*Proof.* Proceeding as in the proof of Theorem 3.1, for all $\lambda_1,\lambda_2>0$ and $s$ almost everywhere, $(y_1,y_2)\in B^2$, using Fubini's theorem and

(2.2) we have

$$
\begin{aligned}
(F*G)_{\vec{\lambda}}(y_1,y_2)
&= \int_{B^2} F\bigg(\frac{y_1+\lambda_1^{-1/2}x_1}{\sqrt{2}}, \frac{y_2+\lambda_2^{-1/2}x_2}{\sqrt{2}}\bigg) \\
&\qquad \cdot G\bigg(\frac{y_1-\lambda_1^{-1/2}x_1}{\sqrt{2}}, \frac{y_2-\lambda_2^{-1/2}x_2}{\sqrt{2}}\bigg)\, d(\nu\times\nu)(x_1,x_2) \\
&= \int_{B_2}\int_H \exp\bigg\{\frac{i}{\sqrt{2}}[(A_1^{1/2}h, y_1+\lambda_1^{-1/2}x_1)^{\sim} \\
&\qquad + (A_2^{1/2}h, y_2+\lambda_2^{-1/2}x_2)^{\sim}]\bigg\}\, d\sigma(h) \\
&\quad \cdot \int_H \exp\bigg\{\frac{i}{\sqrt{2}}[(A_1^{1/2}k, y_1-\lambda_1^{-1/2}x_1)^{\sim} \\
&\qquad + (A_2^{1/2}k, y_2-\lambda_2^{-1/2}x_2)^{\sim}]\bigg\}\, d\rho(k)\, d(\nu\times\nu)(x_1,x_2) \\
&= \int_{H^2} \exp\bigg\{\frac{i}{\sqrt{2}}[(A_1^{1/2}(h+k), y_1)^{\sim} + (A_2^{1/2}(h+k), y_2)^{\sim}] \\
&\qquad - \frac{1}{4\lambda_1}|A_1^{1/2}(h-k)|^2 - \frac{1}{4\lambda_2}|A_2^{1/2}(h-k)|^2\bigg\}\, d\sigma(h)\, d\rho(k).
\end{aligned}
$$

The last expression is an analytic function of $\vec{\lambda}$ throughout $\Omega$ and it is a bounded continuous function of $\vec{\lambda}$ on $\tilde{\Omega}$ for all $(y_1,y_2)\in B^2$. Hence, if we let $\vec{\lambda}\to(-iq_1,-iq_2)$ through $\Omega$, then by the dominated convergence theorem, $(F*G)_{\vec{q}}$ exists and is given by (3.6) for all nonzero real numbers $q_1$ and $q_2$.

Finally, let $\mu$ be a set function on $\mathcal{B}(H^2)$, the Borel class of $H^2$, defined by

$$
\mu(E) = \int_E \exp\bigg\{-\frac{i}{4q_1}|A_1^{1/2}(h-k)|^2 - \frac{i}{4q^2}|A_2^{1/2}(h-k)|^2\bigg\}\, d\sigma(h)\, d\rho(k),
$$

for $E\in\mathcal{B}(H^2)$. Then $\mu$ is a complex Borel measure on $H^2$ and

$$
\begin{aligned}
(F*G)_{\vec{q}}(y_1,y_2) = \int_{H^2} \exp\bigg\{\frac{i}{\sqrt{2}}[(A_1^{1/2}(h+k), y_1)^{\sim} \\
+ (A_2^{1/2}(h+k), y_2)^{\sim}]\bigg\}\, d\mu(h,k),
\end{aligned}
$$

for $s$ almost everywhere, $(y_1, y_2) \in B^2$. Now define a function $\phi : H^2 \to H$ by $\phi(h,k) = (h+k)/\sqrt{2}$. Then $\phi$ is a Borel measurable function and so $\eta \equiv \mu \circ \phi^{-1}$ is in $M(H)$. Using the change of variable theorem, we have

$$(F * G)_{\vec{q}}(y_1, y_2) = \int_H \exp\left\{ i[(A_1^{1/2}h, y_1)^\sim + (A_2^{1/2}h, y_2)^\sim] \right\} d\eta(h),$$

for $s$ almost everywhere, $(y_1, y_2) \in B^2$ and so $(F * G)_{\vec{q}}$ belongs to $\mathcal{F}_{A_1,A_2}$. □

**Theorem 3.4.** *Let $F, G, \sigma$ and $\rho$ be given as in Theorem* 3.3. *Then, for all nonzero real numbers $q_1$ and $q_2$, and for $s$ almost everywhere, $(z_1, z_2) \in B^2$, $(T_{\vec{q}}^{(p)}(F * G)_{\vec{q}})(z_1, z_2)$ exists and*

$$\begin{aligned}(3.7)\qquad & (T_{\vec{q}}^{(p)}(F * G)_{\vec{q}})(z_1, z_2) \\ &= (T_{\vec{q}}^{(p)}(F))\left(\frac{z_1}{\sqrt{2}}, \frac{z_2}{\sqrt{2}}\right)(T_{\vec{q}}^{(p)}(G))\left(\frac{z_1}{\sqrt{2}}, \frac{z_2}{\sqrt{2}}\right)\end{aligned}$$

*for $1 \le p \le 2$. Moreover, both sides of* (3.7) *are given by the expression*

$$\begin{aligned}(3.8)\qquad & \int_{H^2} \exp\left\{ \frac{i}{\sqrt{2}}[(A_1^{1/2}(h+k), z_1)^\sim + (A_2^{1/2}(h+k), z_2)^\sim] \right. \\ & \left. - \frac{i}{2q_1}(|A_1^{1/2}h|^2 + |A_1^{1/2}k|^2) - \frac{i}{2q_2}(|A_2^{1/2}h|^2 + |A_2^{1/2}k|^2) \right\} \\ & d\sigma(h)\, d\rho(k).\end{aligned}$$

*Proof.* For $\lambda_1, \lambda_2 > 0$ and $s$ almost everywhere, $(z_1, z_2) \in B^2$, using (3.6), Fubini's theorem and (2.2), we see that

$$\begin{aligned}& (T_{\vec{\lambda}}(F * G)_{\vec{q}})(z_1, z_2) \\ &= \int_{B^2} (F * G)_{\vec{q}}(\lambda_1^{-1/2}y_1 + z_1, \lambda_2^{-1/2}y_2 + z_2)\, d(\nu \times \nu)(y_1, y_2)\end{aligned}$$

$$
\begin{aligned}
&= \int_{B^2}\int_{H^2} \exp\bigg\{\frac{i}{\sqrt{2}}[(A_1^{1/2}(h+k), \lambda_1^{-1/2}y_1+z_1)^\sim \\
&\qquad\qquad + (A_2^{1/2}(h+k), \lambda_2^{-1/2}y_2+z_2)^\sim] - \frac{i}{4q_1}|A_1^{1/2}(h-k)|^2 \\
&\qquad\qquad - \frac{i}{4q_2}|A_2^{1/2}(h-k)|^2\bigg\}\, d\sigma(h)\, d\rho(k)\, d(\nu\times\nu)(y_1,y_2) \\
&= \int_{H^2} \exp\bigg\{\frac{i}{\sqrt{2}}[(A_1^{1/2}(h+k), z_1)^\sim + (A_2^{1/2}(h+k), z_2)^\sim] \\
&\qquad - \frac{i}{4q_1}|A_1^{1/2}(h-k)|^2 - \frac{i}{4q_2}|A_2^{1/2}(h-k)|^2 \\
&\qquad - \frac{1}{4\lambda_1}|A_1^{1/2}(h+k)|^2 - \frac{1}{4\lambda_2}|A_2^{1/2}(h+k)|^2\bigg\}\, d\sigma(h)\, d\rho(k).
\end{aligned}
$$

The last expression is an analytic function of $\vec{\lambda}$ throughout $\Omega$ and it is a bounded continuous function of $\vec{\lambda}$ on $\tilde{\Omega}$ for all $(z_1, z_2) \in B^2$. Hence, if we let $\vec{\lambda} \to (-iq_1, -iq_2)$ through $\Omega$, $T_{\vec{q}}^{(p)}(F*G)_q$ exists and is given by (3.8). Moreover, by (3.2), the right-hand side of (3.7) has the expression (3.8) and so the result follows. □

**Theorem 3.5.** *Let $F$ and $G$ be given as in Theorem* 3.3. *Then, for all nonzero real numbers $q_1$ and $q_2$, and for s almost everywhere, $(y_1, y_2) \in B^2$,*

$$
\begin{aligned}
&(T_{\vec{q}}^{(p)}(F) * T_{\vec{q}}^{(p)}(G))_{-\vec{q}}(y_1, y_2) \\
&\qquad = T_{\vec{q}}^{(p)}\bigg(F\Big(\frac{\cdot}{\sqrt{2}}, \frac{\cdot}{\sqrt{2}}\Big)G\Big(\frac{\cdot}{\sqrt{2}}, \frac{\cdot}{\sqrt{2}}\Big)\bigg)(y_1, y_2)
\end{aligned}
\tag{3.9}
$$

*for $1 \le p \le 2$.*

*Proof.* We proved in Theorem 3.1 that $T_{\vec{q}}^{(p)}(F)$, $T_{\vec{q}}^{(p)}(G) \in \mathcal{F}_{A_1,A_2}$ and they are given by the expressions

$$
(T_{\vec{q}}^{(p)}(F))(y_1, y_2) = \int_H \exp\Big\{i[(A_1^{1/2}h, y_1)^\sim + (A_2^{1/2}h, y_2)^\sim]\Big\}\, d\sigma'(h),
$$

$$
(T_{\vec{q}}^{(p)}(G))(y_1, y_2) = \int_H \exp\Big\{i[(A_1^{1/2}k, y_1)^\sim + (A_2^{1/2}k, y_2)^\sim]\Big\}\, d\rho'(k),
$$

where

$$\sigma'(E) = \int_E \exp\left\{-\frac{i}{2q_1}|A_1^{1/2}h|^2 - \frac{i}{2q_2}|A_2^{1/2}h|^2\right\} d\sigma(h),$$
$$\rho'(E) = \int_E \exp\left\{-\frac{i}{2q_1}|A_1^{1/2}k|^2 - \frac{i}{2q_2}|A_2^{1/2}k|^2\right\} d\rho(k),$$

for $E \in \mathcal{B}(H)$. Hence (3.6) and a direct calculation show that, for fixed $p$ and $\vec{q}$ and for $s$ almost everywhere, $(y_1, y_2) \in B^2$, we obtain

$$\begin{aligned}
&(T_{\vec{q}}^{(p)}(F) * T_{\vec{q}}^{(p)}(G))_{-\vec{q}}(y_1, y_2) \\
&= \int_{H^2} \exp\bigg\{\frac{i}{\sqrt{2}}[(A_1^{1/2}(h+k), y_1)^\sim + (A_2^{1/2}(h+k), y_2)^\sim] \\
&\qquad + \frac{i}{4q_1}|A_1^{1/2}(h-k)|^2 + \frac{i}{4q_2}|A_2^{1/2}(h-k)|^2\bigg\} d\sigma'(h)\rho'(k) \\
&= \int_{H^2} \exp\bigg\{\frac{i}{\sqrt{2}}[(A_1^{1/2}(h+k), y_1)^\sim + (A_2^{1/2}(h+k), y_2)^\sim] \\
&\qquad - \frac{i}{4q_1}|A_1^{1/2}(h+k)|^2 - \frac{i}{4q_2}|A_2^{1/2}(h+k)|^2\bigg\} d\sigma(h)\rho(k).
\end{aligned}$$

On the other hand, for $\lambda_1, \lambda_2 > 0$ and $s$ almost everywhere, $(y_1, y_2) \in B^2$, using Fubini's theorem and (2.2),

$$\begin{aligned}
&T_{\vec{\lambda}}\bigg(F\Big(\frac{\cdot}{\sqrt{2}}, \frac{\cdot}{\sqrt{2}}\Big)G\Big(\frac{\cdot}{\sqrt{2}}, \frac{\cdot}{\sqrt{2}}\Big)\bigg)(y_1, y_2) \\
&= \int_{B^2} F\bigg(\frac{\lambda_1^{-1/2}x_1 + y_1}{\sqrt{2}}, \frac{\lambda_2^{-1/2}x_2 + y_2}{\sqrt{2}}\bigg) \\
&\qquad \cdot G\bigg(\frac{\lambda_1^{-1/2}x_1 + y_1}{\sqrt{2}}, \frac{\lambda_2^{-1/2}x_2 + y_2}{\sqrt{2}}\bigg) d(\nu \times \nu)(x_1, x_2) \\
&= \int_{B^2}\int_H \exp\bigg\{\frac{i}{\sqrt{2}}[(A_1^{1/2}h, \lambda_1^{-1/2}x_1 + y_1)^\sim \\
&\qquad + (A_2^{1/2}h, \lambda_2^{-1/2}x_2 + y_2)^\sim]\bigg\} d\sigma(h)
\end{aligned}$$

$$\cdot \int_H \exp\bigg\{\frac{i}{\sqrt{2}}[(A_1^{1/2}k, \lambda_1^{-1/2}x_1 + y_1)^{\sim} + (A_2^{1/2}k, \lambda_2^{-1/2}x_2 + y_2)^{\sim}]\bigg\}\, d\rho(k)\, d(\nu\times\nu)(x_1,x_2)$$

$$= \int_{H^2} \exp\bigg\{\frac{i}{\sqrt{2}}[(A_1^{1/2}(h+k), y_1)^{\sim} + (A_2^{1/2}(h+k), y_2)^{\sim}] - \frac{1}{4\lambda_1}|A_1^{1/2}(h+k)|^2 - \frac{1}{4\lambda_2}|A_2^{1/2}(h+k)|^2\bigg\}\, d\sigma(h)\, d\rho(k).$$

The last expression is an analytic function of $\vec{\lambda} \in \Omega$, and it is a bounded continuous function of $\vec{\lambda} \in \tilde{\Omega}$ for all $(y_1, y_2) \in B^2$. So, letting $\vec{\lambda} \to (-iq_1, -iq_2)$ through $\Omega$, we obtain the desired result. □

In our next theorem we establish an interesting Parseval's relation for functionals $F$ and $G$ in the class $\mathcal{F}_{A_1,A_2}$.

**Theorem 3.6.** *Let $F$ and $G$ be given as in Theorem* 3.3. *Then, for all nonzero real numbers $q_1$ and $q_2$, the Parseval's relation*

$$\int_{B^2}^{\mathrm{anf}_{-\vec{q}}} (T_{\vec{q}}^{(p)}(F))\Big(\frac{z_1}{\sqrt{2}}, \frac{z_2}{\sqrt{2}}\Big)(T_{\vec{q}}^{(p)}(G))\Big(\frac{z_1}{\sqrt{2}}, \frac{z_2}{\sqrt{2}}\Big)\, d(\nu\times\nu)(z_1,z_2) = \int_{B^2}^{\mathrm{anf}_{\vec{q}}} F\Big(\frac{z_1}{\sqrt{2}}, \frac{z_2}{\sqrt{2}}\Big) G\Big(-\frac{z_1}{\sqrt{2}}, -\frac{z_1}{\sqrt{2}}\Big)\, d(\nu\times\nu)(z_1,z_2),$$

*holds for* $1 \le p \le 2$.

*Proof.* Fix $p$ and $\vec{q} = (q_1, q_2)$. Then for $\lambda_1, \lambda_2 > 0$, using (3.8), Fubini's theorem and (2.2), we have

$$\int_{B^2} (T_{\vec{q}}^{(p)}(F*G)_{\vec{q}})(\lambda_1^{-1/2}z_1, \lambda_2^{-1/2}z_2)\, d(\nu\times\nu)(z_1,z_2)$$

$$= \int_{B^2}\int_{H^2} \exp\bigg\{\frac{i}{\sqrt{2}}[(A_1^{1/2}(h+k), \lambda_1^{-1/2}z_1)^{\sim} + (A_2^{1/2}(h+k), \lambda_2^{-1/2}z_2)^{\sim}] - \frac{i}{2q_1}(|A_1^{1/2}h|^2 + |A_1^{1/2}k|^2) - \frac{i}{2q_2}(|A_2^{1/2}h|^2 + |A_2^{1/2}k|^2)\bigg\}$$
$$d\sigma(h)\, d\rho(k)\, d(\nu\times\nu)(z_1,z_2)$$

$$= \int_{H^2} \exp\left\{ -\frac{1}{4\lambda_1}|A_1^{1/2}(h+k)|^2 - \frac{1}{4\lambda_2}|A_2^{1/2}(h+k)|^2 \right.$$
$$\left. - \frac{i}{2q_1}(|A_1^{1/2}h|^2 + |A_1^{1/2}k|^2) - \frac{i}{2q_2}(|A_2^{1/2}h|^2 + |A_2^{1/2}k|^2) \right\}$$
$$d\sigma(h)\, d\rho(k).$$

The last expression is an analytic function of $\vec{\lambda}$ throughout $\Omega$ and it is a continuous function of $\vec{\lambda}$ on $\tilde{\Omega}$. So, letting $\vec{\lambda} \to (iq_1, iq_2)$ through $\Omega$ and using (3.7) we obtain

$$\int_{B^2}^{\mathrm{anf}_{-\vec{q}}} (T_{\vec{q}}^{(p)}(F))\left(\frac{z_1}{\sqrt{2}}, \frac{z^2}{\sqrt{2}}\right)(T_{\vec{q}}^{(p)}(G))\left(\frac{z_1}{\sqrt{2}}, \frac{z_2}{\sqrt{2}}\right) d(\nu\times\nu)(z_1, z_2)$$
$$= \int_{B^2}^{\mathrm{anf}_{-\vec{q}}} (T_{\vec{q}}^{(p)}(F * G)_{\vec{q}})(z_1, z_2)\, d(\nu\times\nu)(z_1, z_2)$$
$$= \int_{H^2} \exp\left\{ -\frac{i}{4q_1}|A_1^{1/2}(h-k)|^2 - \frac{i}{4q_2}|A_2^{1/2}(h-k)|^2 \right\} d\sigma(h)\, d\rho(k).$$

On the other hand, for $\lambda_1, \lambda_2 > 0$,

$$\int_{B^2} F\left(\frac{\lambda_1^{-1/2}z_1}{\sqrt{2}}, \frac{\lambda_2^{-1/2}z_2}{\sqrt{2}}\right) G\left(-\frac{\lambda_1^{-1/2}z_1}{\sqrt{2}}, -\frac{\lambda_2^{-1/2}z_2}{\sqrt{2}}\right) d(\nu\times\nu)(z_1, z_2)$$
$$= \int_{B^2}\int_{H^2} \exp\left\{ \frac{i}{\sqrt{2}}[(A_1^{1/2}(h-k), \lambda_1^{-1/2}z_1)^{\sim} \right.$$
$$\left. + (A_2^{1/2}(h-k), \lambda_2^{-1/2}z_2)^{\sim}] \right\} d\sigma(h)\, d\rho(k)\, d(\nu\times\nu)(z_1, z_2)$$
$$= \int_{H^2} \exp\left\{ -\frac{1}{4\lambda_1}|A_1^{1/2}(h-k)|^2 - \frac{1}{4\lambda_2}|A_2^{1/2}(h-k)|^2 \right\} d\sigma(h)\, d\rho(k),$$

and the last expression is an analytic function of $\vec{\lambda}$ throughout $\Omega$ and it is a continuous function of $\vec{\lambda}$ on $\tilde{\Omega}$. So, letting $\vec{\lambda} \to (-iq_1, -iq_2)$ through $\Omega$ we obtain the desired result. □

The following corollary follows immediately from equation (3.10) by choosing $G \equiv F$ for (i) and $G \equiv 1$ for (ii) below.

**Corollary 3.7.** *Let $F, p$ and $\vec{q}$ be given as in Theorem* 3.6*. Then,*

(i)

$$\int_{B^2}^{\mathrm{anf}_{-\vec{q}}}\bigg[(T_{\vec{q}}^{(p)}(F))\bigg(\frac{z_1}{\sqrt{2}},\frac{z_2}{\sqrt{2}}\bigg)\bigg]^2\,d(\nu\times\nu)(z_1,z_2)$$
$$=\int_{B^2}^{\mathrm{anf}_{\vec{q}}}F\bigg(\frac{z_1}{\sqrt{2}},\frac{z_2}{\sqrt{2}}\bigg)F\bigg(-\frac{z_1}{\sqrt{2}},-\frac{z_2}{\sqrt{2}}\bigg)\,d(\nu\times\nu)(z_1,z_2).$$

(ii)

$$\int_{B^2}^{\mathrm{anf}_{-\vec{q}}}(T_{\vec{q}}^{(p)}(F))\bigg(\frac{z_1}{\sqrt{2}},\frac{z_2}{\sqrt{2}}\bigg)\,d(\nu\times\nu)(z_1,z_2)$$
$$=\int_{B^2}^{\mathrm{anf}_{\vec{q}}}F\bigg(\frac{z_1}{\sqrt{2}},\frac{z_2}{\sqrt{2}}\bigg)\,d(\nu\times\nu)(z_1,z_2).$$

From the proof of Theorem 3.6, we can easily obtain the following interesting alternative form of Parseval's relation.

**Corollary 3.8.** *Let $F, G, p$ and $\vec{q}$ be given as in Theorem* 3.6. *Then*

$$\int_{B^2}^{\mathrm{anf}_{-\vec{q}}}(T_{\vec{q}/2}^{(p)}(F))(z_1,z_2)(T_{\vec{q}/2}^{(p)}(G))(z_1,z_2)\,d(\nu\times\nu)(z_1,z_2)$$
$$=\int_{B^2}^{\mathrm{anf}_{\vec{q}}}F(z_1,z_2)G(-z_1,-z_2)\,d(\nu\times\nu)(z_1,z_2),$$

*where $\vec{q}/2=(q_1/2,q_2/2)$.*

From Theorem 3.2 and Theorem 3.6, we have the following multiplication formula.

**Corollary 3.9.** *Let $F, G, p$ and $\vec{q}$ be given as in Theorem* 3.6. *Then*

$$\int_{B^2}^{\mathrm{anf}_{-\vec{q}}}(T_{\vec{q}}^{(p)}(F))\bigg(\frac{z_1}{\sqrt{2}},\frac{z_2}{\sqrt{2}}\bigg)G\bigg(\frac{z_1}{\sqrt{2}},\frac{z_2}{\sqrt{2}}\bigg)\,d(\nu\times\nu)(z_1,z_2)$$
$$=\int_{B^2}^{\mathrm{anf}_{\vec{q}}}F\bigg(\frac{z_1}{\sqrt{2}},\frac{z_2}{\sqrt{2}}\bigg)(T_{-\vec{q}}^{(p)}(G))\bigg(-\frac{z_1}{\sqrt{2}},-\frac{z_2}{\sqrt{2}}\bigg)\,d(\nu\times\nu)(z_1,z_2).$$

**4. Corollaries.** In this section we give various corollaries which show that our results in Section 3 are indeed very general theorems. Below we list results of two types.

(i) *Abstract Wiener space.* If $A_1$ is the identity operator on $H$ and $A_2 = 0$, then $\mathcal{F}_{A_1,A_2}$ is essentially the Fresnel class $\mathcal{F}(B)$ and

$$(T^{(p)}_{(q_1,q_2)}(F))(y_1,y_2) = (T^{(p)}_{q_1}(F_0))(y_1),$$

where $F_0(y_1) = F(y_1,y_2)$ for all $(y_1,y_2)\in B^2$ and $(T^{(p)}_{q_1}(F_0))(y_1)$ means the $L_p$ analytic Fourier-Feynman transform on $B$.

**Theorem 4.1.** *Let $F$ and $G$ be in $\mathcal{F}(B)$. Then, for all nonzero real $q$ and for s almost everywhere, $z$ in $B$, $(T^{(p)}_q(F*G)_q)(z)$ exists and*

$$(T^{(p)}_q(F*G)_q)(z) = (T^{(p)}_q(F))\Big(\frac{z}{\sqrt{2}}\Big)(T^{(p)}_q(G))\Big(\frac{z}{\sqrt{2}}\Big)$$

*for* $1\le p\le 2$.

**Theorem 4.2.** *Let $F$ and $G$ be given as in Theorem* 4.1. *Then for all nonzero real numbers $q$ and for s almost everywhere, $z\in B$,*

$$(T^{(p)}_q(F) * T^{(p)}_q(G))_{-q}(z) = T^{(p)}_q\bigg(F\Big(\frac{\cdot}{\sqrt{2}}\Big)G\Big(\frac{\cdot}{\sqrt{2}}\Big)\bigg)(z)$$

*for* $1\le p\le 2$.

**Theorem 4.3.** *Let $F, G$ and $q$ be given as in Theorem* 4.1. *Then Parseval's relation*

$$\int_B^{\mathrm{anf}_{-q}} (T^{(p)}_q(F*G)_q)(z)\,d\nu(z) = \int_B^{\mathrm{anf}_q} F\Big(\frac{z}{\sqrt{2}}\Big)G\Big(-\frac{z}{\sqrt{2}}\Big)\,d\nu(z)$$

*holds for* $1\le p\le 2$.

**Corollary 4.4.** *Let $F, G, p$ and $q$ be given as in Theorem* 4.3. *Then,*

$$\int_B^{\mathrm{anf}_{-q}} (T^{(p)}_{q/2}(F))(z)(T^{(p)}_{q/2}(G))(z)\,d\nu(z) = \int_B^{\mathrm{anf}_q} F(z)G(-z)\,d\nu(z).$$

**Corollary 4.5.** *Let $F, G, p$ and $\vec{q}$ be given as in Theorem 4.3. Then*

$$\int_B^{\mathrm{anf}_{-q}} (T_q^{(p)}(F))\Big(\frac{z}{\sqrt{2}}\Big) G\Big(\frac{z}{\sqrt{2}}\Big)\, d\nu(z) = \int_B^{\mathrm{anf}_q} F\Big(\frac{z}{\sqrt{2}}\Big)(T_{-q}^{(p)}(G))\Big(-\frac{z}{\sqrt{2}}\Big)\, d\nu(z).$$

(ii) *Classical Wiener space.* Fix $T > 0$ and let $H_0 = H_0[0,T]$ be the space of real-valued functions $f$ on $[0,T]$ which are absolutely continuous and whose derivative $Df$ is in $L_2[0,T]$. The inner product on $H_0$ is given by

$$\langle f, g\rangle = \int_0^T (Df)(s)(Dg)(s)\, ds.$$

Then $H_0$ is a real separable infinite dimensional Hilbert space. Let $B_0 = C_0[0,T]$ be the space of all continuous functions $x$ on $[0,T]$ with $x(0) = 0$, and equip $B_0$ with the sup norm. Let $\nu_0$ be the classical Wiener measure. Then $(H_0, B_0, \nu_0)$ is an example of an abstract Wiener space. Note that if $\{e_n\}$ is a complete orthonormal set in $H_0$, then $\{De_n\}$ is also a complete orthonormal set in $L_2[0,T]$ and $(e_n, x)^\sim$ equals the Paley-Wiener-Zygmund stochastic integral $\int_0^T (De_n)(s)\tilde{d}x(s)$ for $s$ almost everywhere, $x \in B_0$. Moreover, we know that $F \in \mathcal{F}(B_0)$ if and only if $F \in \mathcal{S}$ where $\mathcal{S}$ is the Banach algebra introduced by Cameron and Storvick [**5**].

**Theorem 4.6** [**9**, Theorem 3.3]. *Let $F$ and $G$ be in $\mathcal{S}$. Then, for all nonzero real numbers $q$ and for $s$ almost everywhere, $z \in B_0$,*

$$(T_q^{(p)}(F * G)_q)(z) = (T_q^{(p)}(F))\Big(\frac{z}{\sqrt{2}}\Big)(T_q^{(p)}(G))\Big(\frac{z}{\sqrt{2}}\Big)$$

*for $1 \le p \le 2$.*

**Theorem 4.7** [**9**, Theorem 3.4]. *Let $F$ and $G$ be given as in Theorem 4.6. Then, for all nonzero real numbers $q$, the Parseval's* *identity*

$$\int_{B_0}^{\mathrm{anf}_{-q}} (T_q^{(p)}(F*G)_q)(z)\,d\nu_0(z) = \int_{B_0}^{\mathrm{anf}_q} F\Big(\frac{z}{\sqrt{2}}\Big)G\Big(-\frac{z}{\sqrt{2}}\Big)\,d\nu_0(z)$$

*holds for* $1 \le p \le 2$.

**Corollary 4.8** [**9**, Corollary 3.1 and its remark]. *Let* $F, G, p$ *and* $q$ *be given as in Theorem* 4.7. *Then*

(i)

$$\int_{B_0}^{\mathrm{anf}_{-q}} \Big[(T_q^{(p)}(F))\Big(\frac{z}{\sqrt{2}}\Big)\Big]^2\,d\nu_0(z) = \int_{B_0}^{\mathrm{anf}_q} F\Big(\frac{z}{\sqrt{2}}\Big)F\Big(-\frac{z}{\sqrt{2}}\Big)\,d\nu_0(z).$$

(ii)

$$\int_{B_0}^{\mathrm{anf}_{-q}} (T_q^{(p)}(F))\Big(\frac{z}{\sqrt{2}}\Big)\,d\nu_0(z) = \int_{B_0}^{\mathrm{anf}_q} F\Big(\frac{z}{\sqrt{2}}\Big)\,d\nu_0(z),$$

*and*

(iii)

$$\int_{B_0}^{\mathrm{anf}_{-q}} (T_{q/2}^{(p)}(F))(z)(T_{q/2}^{(p)}(G))(z)\,d\nu_0(z) = \int_{B_0}^{\mathrm{anf}_q} F(z)G(-z)\,d\nu_0(z).$$

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