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RECURSIVE SEQUENCES AND FAITHFULLY FLAT EXTENSIONS

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ABSTRACT. For any faithfully flat morphism $A \to B$ of Noetherian normal domains, a power series with coefficients in A which is rational over B, is already rational over A. The proof uses the fact that a sequence is recursive whenever it is recursive over some faithfully flat extension.

0. Introduction. It is well known that a necessary condition for a ring morphism $A \to B$ to be faithfully flat is that *any linear system of equations with coefficients from A which has a solution over B, must already have a solution over A.* In fact, if we strengthen this condition to *any solution over B comes from solutions over A by base change*, then this also becomes a sufficient condition for being faithfully flat. We could paraphrase the necessary condition as follows: *any linear system of equations over A which is solvable over a faithfully flat extension B of A, is already solvable over A.*

In this paper I present another necessary condition of the same flavor. The key definition is that of a (linear) recursive sequence $(x_n)_n$ over a ring A, as a sequence satisfying some fixed linear relation over A among t consecutive terms. I show that if $A \to B$ is faithfully flat and $(x_n)_n$ is a sequence of elements in A satisfying a linear recursion relation with coefficients in B, then it already satisfies such a recursion relation (of the same length) with coefficients in A. As there is a strong connection between recursive sequences and rational power series, I obtain the following corollary. Assume, moreover, that A and B are normal domains; then any power series over A which is rational (meaning that it can be written as a quotient of two polynomials) over B, is already rational over A. Any direct attempt, however, to prove this corollary just using faithfully flatness seems to fail, as far as I can tell.

1. Definition. Let A be a Noetherian ring and let $\mathbf{x} = (x_n)_{n<\omega}$ be a (countable) sequence of elements of A. We say that $\mathbf{x}$ is *recursive over A*

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(*of length* t), if there exists a linear form $L(X) = r_0X_0 + \cdots + r_{t-1}X_{t-1} \in A[X]$, where $t \in \mathbf{N}$ and $X = (X_i)_{i<t}$, such that for all $n \gg 0$ we have that

$$L(x_n, \dots, x_{n+t-1}) = x_{n+t}.$$

Here we have used the notation $n \gg 0$ as an abbreviation for *there exists* n_0, *such that for all* $n \geq n_0$.

2. Theorem. *Let* $A \to B$ *be a morphism of Noetherian rings and let* $\mathbf{x}$ *be a sequence in* A. *If* B *is faithfully flat over* A *and* $\mathbf{x}$ *is recursive over* B, *then* $\mathbf{x}$ *is already recursive over* A (*of the same length*).

Proof. Let $\mathcal{L}_t(\mathbf{x})$ be the collection of all linear forms $L(X) \in A[X]$, where $X = (X_i)_{i \leq t}$ and $t \in \mathbf{N}$, such that for all $n \gg 0$, we have that $L(x_n, \dots, x_{n+t}) = 0$. Evidently, $\mathcal{L}_t(\mathbf{x})$ carries the structure of an A-module. Let e be the $(t+1)$-tuple $(0, \dots, 0, -1)$, and let

$$\mathfrak{a}_t^A(\mathbf{x}) = \{L(e) \mid L \in \mathcal{L}_t(\mathbf{x})\}.$$

As the latter is the image of $\mathcal{L}_t(\mathbf{x})$ under the morphism $A[X] \to A$ defined by substituting e for X, it is an ideal of A.

We claim that $\mathbf{x}$ is recursive over A of length t, if and only if, $\mathfrak{a}_t^A(\mathbf{x}) = A$. Indeed, if $\mathbf{x}$ is recursive over A of length t, then there is a linear form $L(X) \in A[X]$, with $X = (X_i)_{i<t}$ such that for all $n \gg 0$ we have that $L(x_n, \dots, x_{n+t-1}) = x_{n+t}$. Let $L'(X, X_t) = L(X) - X_t$ so that $L' \in \mathcal{L}_t(\mathbf{x})$. Since $1 = L'(e)$, we proved one direction and the converse follows along the same lines.

Now, by assumption, $\mathbf{x}$ is recursive over B of length t so that by the criterion we just established, $\mathfrak{a}_t^B(\mathbf{x}) = B$. Hence let $L(X) = b_0X_0 + \cdots + b_tX_t \in B[X]$ be a witness to this, where $X = (X_i)_{i \leq t}$, i.e., such that for some $n_0 \in \mathbf{N}$, we have, for all $n \geq n_0$, that $L(x_n, \dots, x_{t+n}) = 0$ and $L(e) = 1$. Therefore $b_t = -1$. Put $b = (b_i)_{i<t} \in B^t$. For every $n \geq n_0$, let $M_n(Y) \in A[Y]$, where $Y = (Y_i)_{i \leq t}$, be defined as

$$M_n(Y) = x_nY_0 + \cdots + x_{n+t}Y_t.$$

By Noetherianity, there exists some $n_1 \geq n_0$ such that each M_n, for $n \geq n_0$, lies in the ideal of $A[Y]$ generated by $M_{n_0}, \dots, M_{n_1}$. In other words, there exists $p_{n,k}(Y) \in A[Y]$ such that, for all $n \geq n_0$ and all k with $n_0 \leq k \leq n_1$, we have that

$$M_n(Y) = \sum_{k=n_0}^{n_1} p_{n,k}(Y) M_k(Y). \tag{1}$$

By construction, $M_k(b, -1) = 0$ for $n_0 \leq k \leq n_1$. By flatness, we can find finitely many $a^{(j)} \in A^t$, $e^{(j)} \in A$ and $\beta^{(j)} \in B$ such that

$$M_k(a^{(j)}, e^{(j)}) = 0, \tag{2}$$

for all $j < s$ and $n_0 \leq k \leq n_1$, and

$$b = \sum_{j<s} \beta^{(j)} a^{(j)} \qquad -1 = \sum_{j<s} \beta^{(j)} e^{(j)}. \tag{3}$$

However, from (1) and (2), it then follows that $M_n(a^{(j)}, e^{(j)}) = 0$ for all $j < s$ and all $n \geq n_0$. This means that $L^{(j)}(X) = a_0^{(j)} X_0 + \cdots + a_{t-1}^{(j)} + e^{(j)} X_t$ lies in $\mathcal{L}_t(\mathbf{x})$, for each $j < s$, where $a^{(j)} = (a_i^{(j)})_{i<t}$. Hence $-e^{(j)} \in \mathfrak{a}_t^A(\mathbf{x})$. Together with (3), we therefore conclude that $\mathfrak{a}_t^A(\mathbf{x})B = B$. But faithfully flatness then implies that $\mathfrak{a}_t^A(\mathbf{x}) = A$, which by the above criterion means that $\mathbf{x}$ is recursive over A of length t. $\square$

3. Proposition. *Let A be a Noetherian ring and $\mathbf{x}$ a sequence in A. Let $\xi_{\mathbf{x}}(T) \in A[[T]]$ be the generating series of $\mathbf{x}$, i.e.,*

$$\xi_{\mathbf{x}}(T) = \sum_{n=0}^{\infty} x_n T^n.$$

Then $\mathbf{x}$ is recursive over A, if and only if, $\xi_{\mathbf{x}}(T)$ lies in $A(T)$, where the latter ring is the localization of $A[T]$ with respect to the multiplicative set $1 + (T)A[T]$.

Proof. Suppose that $\mathbf{x}$ is recursive and let $\xi(T) = \xi_{\mathbf{x}}(T)$. There is some n_0 and some $a_k \in A$ for $k < t$ such that

$$x_n = a_0 x_{n-t} + \cdots + a_{t-1} x_{n-1} \tag{4}$$

for all $n \geq n_0$. We want to show that

$$\xi(T) \equiv Q(T)\xi(T) \mod A[T] \tag{5}$$

for some polynomial $Q(T) \in (T)A[T]$. Indeed, if (5) holds, then

$$\xi(T) = Q(T)\xi(T) + P(T)$$

for some $P(T) \in A[T]$ and hence $\xi(T) = P(T)/(1 - Q(T))$ as required.

We work in the A-module $A[[T]]/A[T]$. Clearly

$$\xi(T) \equiv T^{n_0} \cdot \xi_{\mathbf{x}'}(T) \mod A[T],$$

where $\mathbf{x}'$ is the sequence obtained from $\mathbf{x}$ by deleting the first n_0 elements. Hence in order to prove (5) we may work with this new recursive sequence and hence assume from the start that $n_0 = 0$. We have, using (4), that

$$\begin{aligned}
\xi(T) &= \sum_{n<t} x_n T^n + \sum_{n \geq t} x_n T^n \\
&\equiv \sum_{n \geq t} (a_0 x_{n-t} + \cdots + a_{t-1} x_{n-1}) T^n \mod A[T] \\
&\equiv a_0 T^t \xi(T) + \cdots + a_{t-1} T \xi(T) \mod A[T].
\end{aligned}$$

This proves (5).

The converse is an easy exercise and is left to the reader. □

4. Corollary. *Let $\varphi : A \to B$ be a morphism of Noetherian rings. If φ is faithfully flat, then*

$$A(T) = A[[T]] \cap B(T), \tag{6}$$

for T a single variable.

Proof. The $\subset$-inclusion is immediate, hence take F in the righthand side of (6). Taking the coefficients of F as the members of a sequence $\mathbf{x}$ over A, we have that $F(T) = \xi_{\mathbf{x}}(T)$. Since $F(T) \in B(T)$, the sequence $\mathbf{x}$ is recursive over B by Proposition 3. Therefore, by faithfully flatness and Theorem 2, $\mathbf{x}$ is also recursive over A which by Proposition 3 again means that $F \in A(T)$. $\square$

5. Corollary. *Let $A \to B$ be a faithfully flat morphism of Noetherian normal domains. Then any power series with coefficients in A which is rational over B, is already rational over A.*

Proof. A Noetherian normal domain A has the Fatou property by Chabert [**1**, Section 3], meaning that $A(T)$ is equal to the intersection of the fraction field of $A[T]$ with $A[[T]]$, i.e., $A(T)$ is the ring of rational power series and we can apply Corollary 4.

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