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HANKEL TRANSFORMATION AND HANKEL CONVOLUTION OF TEMPERED BEURLING DISTRIBUTIONS

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ABSTRACT. In this paper we complete the distributional theory of Hankel transformation developed in [**5**] and [**18**]. New Fréchet function spaces $\mathcal{H}_\mu(w)$ are introduced. The functions in $\mathcal{H}_\mu(w)$ have a growth in infinity restricted by the Beurling type function w. We study on $\mathcal{H}_\mu(w)$ and its dual the Hankel transformation and the Hankel convolution.

1. Introduction. The Hankel integral transformation is usually defined by

$$h_\mu(\phi)(x) = \int_0^\infty (xy)^{-\mu} J_\mu(xy)\phi(y)y^{2\mu+1}\, dy, \quad x \in (0,\infty),$$

where J_μ represents the Bessel function of the first kind and order μ. We will assume throughout this paper that $\mu > -1/2$. Note that if ϕ is a Lebesgue measurable function on $(0,\infty)$ and

$$\int_0^\infty x^{2\mu+1}|\phi(x)|\, dx < \infty,$$

then, since the function $z^{-\mu}J_\mu(z)$ is bounded on $(0,\infty)$, the Hankel transform $h_\mu(\phi)$ is a bounded function on $(0,\infty)$. Moreover, $h_\mu(\phi)$ is continuous on $(0,\infty)$ and, according to the Riemann-Lebesgue theorem for Hankel transforms ([**17**]), $\lim_{x\to\infty} h_\mu(\phi)(x) = 0$.

The study of the Hankel transformation in distribution spaces was started by Zemanian ([**18**], [**19**]). In [**18**] the Hankel transform of distribution of slow growth was defined. More recently, Betancor and

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Rodríguez-Mesa [**5**] have investigated the h_μ transform of generalized functions with exponential growth. Our objective in this paper that is motivated by the studies of Björck [**8**] is to define the Hankel transformation on new distribution spaces that are, in a certain sense, between the spaces considered in [**5**] and [**18**]. Thus we complete the investigations in [**5**] and [**18**].

Zemanian [**18**] introduced the space $\mathcal{H}_\mu$ that consists of all those complex valued and smooth functions ϕ defined on $(0,\infty)$ such that, for every $m,n \in \mathbf{N}$,

$$\gamma^\mu_{m,n}(\phi) = \sup_{x\in(0,\infty)} (1+x^2)^m \Big| \Big(\frac{1}{x}D\Big)^n (x^{-\mu-1/2}\phi(x)) \Big| < \infty.$$

On $\mathcal{H}_\mu$ he considers the topology generated by the family $\{\gamma^\mu_{m,n}\}_{m,n\in\mathbf{N}}$ of semi-norms. Then $\mathcal{H}_\mu$ is a Fréchet space and the Hankel transformation H_μ defined by

$$H_\mu(\phi)(x) = \int_0^\infty (xy)^{1/2} J_\mu(xy)\phi(y)\,dy, \quad x \in (0,\infty),$$

is an automorphism of $\mathcal{H}_\mu$ ([**18**, Lemma 8]). Note that the two forms h_μ and H_μ of Hankel transforms are related through

$$H_\mu(\phi)(x) = x^{\mu+1/2} h_\mu(y^{-\mu-1/2}\phi)(x), \quad x \in (0,\infty).$$

The Hankel transformation H_μ is defined on the dual $\mathcal{H}'_\mu$ of $\mathcal{H}_\mu$ by transposition.

Altenburg [**1**] developed for the h_μ transformation a theory similar to that of Zemanian. Note that the space $\mathcal{H}_{-1/2}$ coincides with the space $\mathcal{H}$ considered in [**1**].

In [**5**] the space χ_μ constituted by all the complex valued and smooth functions ϕ defined on $(0,\infty)$ satisfying that

$$\eta^\mu_{m,n}(\phi) = \sup_{x\in(0,\infty)} e^{mx} \Big| \Big(\frac{1}{x}D\Big)^n (x^{-\mu-1/2}\phi(x)) \Big| < \infty,$$

for each $m,n \in \mathbf{N}$ is considered. In [**5**, Theorem 2.1] a characterization of the image by H_μ of the space χ_μ as a certain space of entire functions with a restricted growth on horizontal strips is given. The

Hankel transform H_μ is defined on the corresponding dual spaces by transposition.

In this paper we analyze the behavior of Hankel transformations and Hankel convolutions in the intermediate, in a suitable sense, spaces between the spaces $\mathcal{H}_\mu$ of functions with growth at infinity restricted by polynomials in x and the spaces χ_μ of functions with growth at infinity restricted by polynomials in e^x. We introduce here the space $\mathcal{H}_\mu(w)$ constituted by functions whose growth is restricted by e^{nw}, $n \in \mathbf{N}$, where w is a function that we will define precisely later.

Hirschman [**13**], Haimo [**12**] and Cholewinski [**9**] investigated the Hankel convolution operation.

The convolution associated with the h_μ transformation is defined as follows. The Hankel convolution $f \#_\mu g$ of order μ of the measurable functions f and g is given through

$$(f \#_\mu g)(x) = \int_0^\infty f(y)({}_\mu\tau_x g)(y) \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\, dy,$$

where the Hankel translation operator ${}_\mu\tau_x g$, $x \in (0,\infty)$, of g is defined by

$$({}_\mu\tau_x g)(y) = \int_0^\infty g(z) D_\mu(x,y,z) \frac{z^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\, dz,$$

provided that the above integrals exist. Here D_μ is the following function

$$D_\mu(x,y,z) = (2^\mu \Gamma(\mu+1))^2 \int_0^\infty (xt)^{-\mu} J_\mu(xt)(yt)^\mu J_\mu(yt)(zt)^{-\mu} \cdot J_\mu(zt) t^{2\mu+1}\, dt, \quad x,y,z \in (0,\infty).$$

Moreover, we define ${}_\mu\tau_0 g = g$.

The study of the $\#_\mu$-convolution on L_p-spaces was developed in [**12**] and [**13**].

If we denote by $L_{1,\mu}$ the space of complex valued and measurable functions f on $(0,\infty)$ such that $\int_0^\infty |f(x)| x^{2\mu+1}\, dx < \infty$, the following interchange formula

$$h_\mu(f \#_\mu g) = h_\mu(f) h_\mu(g),$$

holds for every $f, g \in L_{1,\mu}$.

A straightforward manipulation in $\#_\mu$ allows to define a convolution operator for the transformation H_μ.

The investigation of the distributional Hankel convolution was started by de Sousa-Pinto [**15**] who considered only $\mu = 0$. Betancor and Marrero ([**3**], [**4**] and [**14**]) studied the Hankel convolution on the Zemanian spaces. In [**5**], Betancor and Rodríguez-Mesa analyzed the $\#_\mu$-convolution of distributions with exponential growth.

In the sequel, since we think any confusion is possible, to simplify we will write $\#$, τ_x, $x \in [0, \infty)$ and D instead of $\#_\mu$, ${}_\mu\tau_x$, $x \in [0, \infty)$ and D_μ, respectively.

As in [**8**] we consider continuous, increasing and nonnegative functions w defined on $[0, \infty)$ such that $w(0) = 0$, $w(1) > 0$, and it satisfies the following three properties

(α) $w(x + y) \leq w(x) + w(y)$, $x, y \in [0, \infty)$,

(β) $\int_1^\infty (w(x)/x^2)\, dx < \infty$, and

(γ) there exist $a \in \mathbf{R}$ and $b > 0$ such that $w(x) \geq a + b \log(1 + x)$, $x \in [0, \infty)$.

We say $w \in \mathcal{M}$ when w satisfies the above conditions. Note that if w is extended to $\mathbf{R}$ as an even function, then w satisfies the subadditivity property in (α) for every $x, y \in \mathbf{R}$.

Beurling [**7**] developed a general theory of distributions that extends the Schwartz theory. Some aspects of that theory were presented and completed by Björck [**8**]. Inspired by the works of Beurling [**7**] and Björck [**8**], we started in [**2**] the study of Beurling distributions for Hankel transforms. We now collect some definitions and properties presented in [**2**] and that will be useful in the sequel.

Let $w \in \mathcal{M}$. For every $a > 0$ the space $\mathcal{B}_\mu^a(w)$ is constituted by all those complex-valued and smooth functions ϕ on $(0, \infty)$ such that $\phi(x) = 0$, $x \geq a$, ϕ and $h_\mu(\phi) \in L_{1,\mu}$ and that

$$\delta_n^\mu(\phi) = \int_0^\infty |h_\mu(\phi)(x)| e^{nw(x)} x^{2\mu+1}\, dx < \infty,$$

for every $n \in \mathbf{N}$. $\mathcal{B}_\mu^a(w)$ is a Fréchet space when we consider on it the topology generated by the system $\{\delta_n^\mu\}_{n \in \mathbf{N}}$ of semi-norms. It is clear that $\mathcal{B}_\mu^a(w)$ is continuously contained in $\mathcal{B}_\mu^b(w)$ when $0 < a < b$. The union space $\mathcal{B}_\mu(w) = \cup_{a>0}\mathcal{B}_\mu^a(w)$ is endowed with the inductive topology.

For every $x \in (0,\infty)$, the Hankel translation τ_x defines a continuous linear mapping from $\mathcal{B}_\mu(w)$ into itself ([**2**, Proposition 2.13]). Then we can define the Hankel convolution $T\#\phi$ of $T \in \mathcal{B}_\mu(w)'$, the dual space of $\mathcal{B}_\mu(w)$ and $\phi \in \mathcal{B}_\mu(w)$ by

$$(T\#\phi)(x) = \langle T, \tau_x\phi\rangle, \quad x \in [0,\infty).$$

By $\mathcal{E}_\mu(w)$ we denote the space of pointwise multipliers of $\mathcal{B}_\mu(w)$. $\mathcal{E}_\mu(w)$ is endowed with the topology induced by the topology of pointwise convergence of the space $\mathcal{L}(\mathcal{B}_\mu(w))$ of continuous linear mapping from $\mathcal{B}_\mu(w)$ into itself. The space $\mathcal{E}_\mu(w)'$ dual of $\mathcal{E}_\mu(w)$ is characterized as the subspace of $\mathcal{B}_\mu(w)'$ defining Hankel convolution operators on $\mathcal{B}_\mu(w)$ ([**2**, Proposition 3.9]).

This paper is organized as follows. In Section 2 we introduce the space $\mathcal{H}_\mu(w)$ of functions and we study its main properties. The dual space $\mathcal{H}_\mu(w)'$ of $\mathcal{H}_\mu(w)$ is considered in Section 3. Also we analyze the Hankel transformation and the Hankel convolution on $\mathcal{H}_\mu(w)'$.

Throughout this paper we always denote by C a suitable positive constant that can change from one line to another one.

2. The space $\mathcal{H}_\mu(w)$. In the sequel w is a function in $\mathcal{M}$. We now introduce the function spaces $\mathcal{H}_\mu(w)$. A function $\phi \in L_{1,\mu}$ is in $\mathcal{H}_\mu(w)$ when ϕ and $h_\mu(\phi)$ are smooth functions and, for every $m, n \in \mathbf{N}$,

$$\alpha_{m,n}(\phi) = \sup_{x\in(0,\infty)} e^{mw(x)}\left|\left(\frac{1}{x}D\right)^n \phi(x)\right| < \infty,$$

and

$$\beta_{m,n}^\mu(\phi) = \sup_{x\in(0,\infty)} e^{mw(x)}\left|\left(\frac{1}{x}D\right)^n h_\mu(\phi)(x)\right| < \infty.$$

On $\mathcal{H}_\mu(w)$ we consider the topology generated by the family $\{\alpha_{m,n}, \beta_{m,n}^\mu\}_{m,n\in\mathbf{N}}$ of semi-norms.

In the following we establish some properties of $\mathcal{H}_\mu(w)$ that can be proved by invoking well-known properties of the Hankel transformation h_μ and the conditions imposed on the function w.

Proposition 2.1. (i) *The space* $\mathcal{H}_\mu(w)$ *is a Fréchet space and it is continuously contained in* $\mathcal{H}_{-1/2}$. *Moreover, if* $w(x) = \log(1+x)$, $x \in [0,\infty)$, *then* $\mathcal{H}_\mu(w) = \mathcal{H}_{-1/2}$, *where the equality is algebraical and topological.*

(ii) *The Hankel transformation* h_μ *is an automorphism of* $\mathcal{H}_\mu(w)$.

(iii) *The Bessel operator* $\Delta_\mu = x^{-2\mu-1}Dx^{2\mu+1}D$ *defines a continuous linear mapping from* $\mathcal{H}_\mu(w)$ *into itself.*

(iv) *If* P *is a polynomial, then the mapping* $\phi \to P(x^2)\phi$ *is linear and continuous from* $\mathcal{H}_\mu(w)$ *into itself.*

We now introduce a new family of semi-norms on $\mathcal{H}_\mu(w)$ that is equivalent to $\{\alpha_{m,n}, \beta^\mu_{m,n}\}_{m,n\in\mathbf{N}}$ and that will be very useful in the sequel.

Proposition 2.2. *For every* $m, n \in \mathbf{N}$, *we define*

$$A^\mu_{m,n}(\phi) = \sup_{x\in(0,\infty)} e^{mw(x)}|\Delta^n_\mu \phi(x)|, \quad \phi \in \mathcal{H}_\mu(w),$$

and

$$B^\mu_{m,n}(\phi) = \sup_{x\in(0,\infty)} e^{mw(x)}|\Delta^n_\mu h_\mu(\phi)(x)|, \quad \phi \in \mathcal{H}_\mu(w),$$

where Δ_μ *represents the Bessel operator* $x^{-2\mu-1}Dx^{2\mu+1}D$. *The family* $\{A^\mu_{m,n}, B^\mu_{m,n}\}_{m,n\in\mathbf{N}}$ *of semi-norms generates the topology of* $\mathcal{H}_\mu(w)$.

Proof. Proposition 2.1 (ii) and (iii) imply that the topology defined on $\mathcal{H}_\mu(w)$ by $\{\alpha_{m,n}, \beta^\mu_{m,n}\}_{m,n\in\mathbf{N}}$ is stronger than the one induced on it by $\{A^\mu_{m,n}, B^\mu_{m,n}\}_{m,n\in\mathbf{N}}$.

We now are going to see that $\{A^\mu_{m,n}, B^\mu_{m,n}\}_{m,n\in\mathbf{N}}$ generates on $\mathcal{H}_\mu(w)$ a topology finer than the one defined on it by $\{\alpha_{m,n}, \beta^\mu_{m,n}\}_{m,n\in\mathbf{N}}$.

For every $k \in \mathbf{N}$ and $\phi \in \mathcal{H}_\mu(w)$, we have that

$$\Big(\frac{1}{x}D\Big)^k \phi(x) = x^{-2\mu-2k}\int_0^x x_k \int_0^{x_k} x_{k-1} \cdots \int_0^{x_2} x_1^{2\mu+1}\Delta^k_\mu \phi(x_1)\,dx_1 \dots dx_k, \quad x \in (0,\infty), \tag{2.1}$$

and

$$\Big(\frac{1}{x}D\Big)^k \phi(x) = (-1)^k x^{-2\mu-2k} \int_x^\infty x_k \int_{x_k}^\infty x_{k-1} \cdots \int_{x_2}^\infty x_1^{2\mu+1} \Delta_\mu^k \phi(x_1)\, dx_1 \dots dx_k, \quad x \in (0,\infty). \tag{2.2}$$

To prove (2.1) and (2.2), we must proceed inductively. We are going to show (2.1). To see (2.2), we can argue in a similar way.

Formula (2.1) holds when $k = 1$. Indeed, according to Proposition 2.1 (i) and by [**1**, Lemma 8 b)], it has, for every $\phi \in \mathcal{H}_\mu(w)$

$$h_{\mu+1}\Big(\Big(\frac{1}{x}D\Big)\phi\Big) = -h_\mu(\phi). \tag{2.3}$$

Moreover, by partial integration and by [**20** (7), Chapter 5], since the function $z^{1/2}J_\mu(z)$ is bounded on $(0,\infty)$, it has, for every $y \in (0,\infty)$ and $\phi \in \mathcal{H}_\mu(w)$,

$$\begin{aligned} h_{\mu+1}\Big(x^{-2\mu-2}\int_0^x x_1^{2\mu+1}\Delta_\mu \phi(x_1)\,dx_1\Big)(y) &= -y^{-2}\int_0^\infty \frac{d}{dx}((xy)^{-\mu}J_\mu(xy))\int_0^x x_1^{2\mu+1}\Delta_\mu\phi(x_1)\,dx_1\,dx \\ &= y^{-2}h_\mu(\Delta_\mu\phi)(y) \\ &= -h_\mu(\phi)(y). \end{aligned} \tag{2.4}$$

From (2.3) and (2.4) we deduce that (2.1) is true for every $\phi \in \mathcal{H}_\mu(w)$ when $k = 1$.

We now suppose that $l \in \mathbf{N}$ and that, for every $\phi \in \mathcal{H}_\mu(w)$, we have

$$\Big(\frac{1}{x}D\Big)^l \phi(x) = x^{-2\mu-2l}\int_0^x x_l \int_0^{x_l} x_{l-1} \cdots \int_0^{x_2} x_1^{2\mu+1}\Delta_\mu^l \phi(x_1) dx_l \dots dx_1, \quad x \in (0,\infty). \tag{2.5}$$

We have to see that (2.5) holds when l is replaced by $l+1$ for every $\phi \in \mathcal{H}_\mu(w)$. Let $\phi \in \mathcal{H}_\mu(w)$. According to [**1**, Lemma 8], we can write

$$\Big(\frac{1}{x}D\Big)^{l+1}\phi = (-1)^{l+1}h_{\mu+l+1}(h_\mu\phi).$$

On the other hand, it is easy to see that from the induction hypothesis (2.5) it deduces that, since $\Delta_\mu\phi \in \mathcal{H}_\mu(w)$, Proposition 2.1,

$$\text{(2.6)}\quad x^{-2\mu-2(l+1)} \int_0^x x_{l+1} \int_0^{x_{l+1}} x_l \cdots \int_0^{x_2} x_1^{2\mu+1} \Delta_\mu^{l+1}\phi(x_1)\, dx_1 \dots dx_{l+1} = \Lambda_{\mu+l}\Big(\Big(\frac{1}{x}D\Big)^l \Delta_\mu\phi\Big)(x), \quad x \in (0,\infty),$$

where Λ_μ denotes the operator defined by

$$(\Lambda_\mu\psi)(x) = x^{-2\mu-2} \int_0^x t^{2\mu+1}\psi(t)\, dt, \quad x \in (0,\infty),$$

for every $\psi \in \mathcal{H}_\mu(w)$.

Moreover, from (2.3), it follows that

$$\Big(\frac{1}{x}D\Big)^l \Delta_\mu\phi = \Delta_{\mu+l}\Big(\frac{1}{x}D\Big)^l \phi. \tag{2.7}$$

On the other hand, by partial integration and by [**1**, Lemma 8 b)], we obtain that, for every $\psi \in \mathcal{H}_{-1/2}$,

$$\begin{aligned} h_{\mu+l+1}&(\Lambda_{\mu+l}\Delta_{\mu+l}\psi)(y) \\ &= -y^{-2} \int_0^\infty \frac{d}{dx}((xy)^{-\mu-l}J_{\mu+l}(xy)) \int_0^x t^{2\mu+2l+1}\Delta_{\mu+l}\psi(t)\, dt\, dx \\ &= -h_{\mu+l}(\psi)(y), \quad y \in (0,\infty). \end{aligned}$$

Hence,

$$\Lambda_{\mu+l}\Delta_{\mu+l}\psi = \Big(\frac{1}{x}D\Big)\psi, \quad \psi \in \mathcal{H}_{-1/2}. \tag{2.8}$$

From (2.6), (2.7) and (2.8), according to Proposition 2.1 (i), it implies that

$$\Big(\frac{1}{x}D\Big)^{l+1}\phi(x) = x^{-2\mu-2(l+1)} \int_0^x x_{l+1} \int_0^{x_{l+1}} x \cdots \int_0^{x_2} x_1^{2\mu+1}\Delta_\mu^{l+1}\phi(x_1)\, dx_1 \dots dx_{l+1}, \quad x \in (0,\infty).$$

Thus (2.1) is proved.

Now let $m, n \in \mathbf{N}$. Assume that $\phi \in \mathcal{H}_\mu(w)$. From (2.1) it follows that

$$\begin{aligned} &e^{mw(x)}\left|\left(\frac{1}{x}D\right)^n \phi(x)\right| \\ &\leq C \sup_{z\in(0,\infty)} |\Delta_\mu^n \phi(z)| x^{-2\mu-2n} \int_0^x x_n \int_0^{x_n} x_{n-1} \cdots \int_0^{x_2} x_1^{2\mu+1}\, dx_1 \ldots dx_n \\ &\leq C \sup_{z\in(0,\infty)} |\Delta_\mu^n \phi(z)|, \quad x \in (0,1). \end{aligned}$$

Also, by using (2.2), since w is increasing and it satisfies the (γ)-property, we obtain for $l \in \mathbf{N}$ large enough,

$$\begin{aligned} e^{mw(x)}\left|\left(\frac{1}{x}D\right)^n \phi(x)\right| &\leq x^{-2\mu-2n} \int_x^\infty x_n \int_{x_n}^\infty x_{n-1} \\ &\qquad \ldots \int_{x_2}^\infty x_1^{2\mu+1} e^{mw(x_1)} |\Delta_\mu^n \phi(x_1)|\, dx_1 \ldots dx_n \\ &\leq C \sup_{z\in(0,\infty)} e^{(m+l)w(z)} |\Delta_\mu^n \phi(z)|, \quad x \geq 1. \end{aligned}$$

Hence, it concludes that, for a certain $l \in \mathbf{N}$,

$$\alpha_{m,n}(\phi) \leq C A^\mu_{m+l,n}(\phi).$$

According to Proposition 2.1 (ii) $h_\mu(\phi)$ is also in $\mathcal{H}_\mu(w)$ and then the following inequality also holds

$$\beta^\mu_{m,n}(\phi) \leq C B^\mu_{m+l,n}(\phi).$$

Thus we prove that the topology generated by $\{A^\mu_{m,n}, B^\mu_{m,n}\}_{m,n\in\mathbf{N}}$ on $\mathcal{H}_\mu(w)$ is finer than the one induced on it by $\{\alpha_{m,n}, \beta^\mu_{m,n}\}_{m,n\in\mathbf{N}}$ and the proof is completed. □

Through the proof of Proposition 2.2 we also show the following characterizations of the space $\mathcal{H}_\mu(w)$.

Proposition 2.3. *A function $\phi \in \mathcal{H}_\mu(w)$ if and only if $\phi \in \mathcal{H}_{-1/2}$ and ϕ satisfies one of the three following conditions*:

(i) *For every* $m, n \in \mathbf{N}$, $A^{\mu}_{m,n}(\phi) < \infty$ *and* $B^{\mu}_{m,n}(\phi) < \infty$,

(ii) *For every* $m, n \in \mathbf{N}$, $A^{\mu}_{m,n}(\phi) < \infty$ *and* $\beta^{\mu}_{m,n}(\phi) < \infty$,

(iii) *For every* $m, n \in \mathbf{N}$, $\alpha_{m,n}(\phi) < \infty$ *and* $B^{\mu}_{m,n}(\phi) < \infty$.

Moreover, the families of semi-norms $\{A^{\mu}_{m,n}, B^{\mu}_{m,n}\}_{m,n\in\mathbf{N}}$, $\{A^{\mu}_{m,n}, \beta^{\mu}_{m,n}\}_{m,n\in\mathbf{N}}$ *and* $\{\alpha_{m,n}, B^{\mu}_{m,n}\}_{m,n\in\mathbf{N}}$ *generates the topology of* $\mathcal{H}_{\mu}(w)$. □

We now analyze the behavior of Hankel translation operator on $\mathcal{H}_{\mu}(w)$.

Proposition 2.4. (i) *Let* $x \in (0, \infty)$. *The Hankel translation operator* τ_x *defines a continuous linear mapping from* $\mathcal{H}_{\mu}(w)$ *into itself.*

(ii) *Let* $\phi \in \mathcal{H}_{\mu}(w)$. *The (nonlinear) mapping* F_{ϕ} *defined by* $F_{\phi}(x) = \tau_x\phi$, $x \in [0, \infty)$, *is continuous from* $[0, \infty)$ *into* $\mathcal{H}_{\mu}(w)$.

Proof. (i) Let $\phi \in \mathcal{H}_{\mu}(w)$ and $m, n \in \mathbf{N}$. Since $\Delta_{\mu}\tau_x\phi = \tau_x\Delta_{\mu}\phi$ ([**14**, Proposition 2.1]) and since w is increasing and it satisfies the (α)-property, we can write

$$\begin{aligned}
e^{mw(y)}&|\Delta^n_{\mu}(\tau_x\phi)(y)| \\
&\le e^{mw(y)}\tau_x(|\Delta^n_{\mu}\phi|)(y) \\
&\le e^{m(w(y)-w(|x-y|))}\int_{|x-y|}^{x+y} D(x,y,z)e^{mw(z)}|\Delta^n_{\mu}\phi(z)|\frac{z^{2\mu+1}}{2^{\mu}\Gamma(\mu+1)}\,dz \\
&\le e^{mw(x)}\sup_{z\in(0,\infty)} e^{mw(z)}|\Delta^n_{\mu}\phi(z)|\int_0^{\infty} D(x,y,z)\frac{z^{2\mu+1}}{2^{\mu}\Gamma(\mu+1)}\,dz,
\end{aligned}$$

for each $y \in (0, \infty)$.

Hence, by [**13**, (2)], it concludes

$$A^{\mu}_{m,n}(\tau_x\phi) \le e^{mw(x)}A^{\mu}_{m,n}(\phi). \tag{2.9}$$

On the other hand, by [**3**, (3.1)] and [**20** (7), Chapter 5], since the function $z^{-\mu}J_\mu(z)$ is bounded on $(0,\infty)$, it follows

$$\begin{aligned} e^{mw(y)}\Big|\Big(\frac{1}{y}D\Big)^n h_\mu(\tau_x\phi)(y)\Big| &= e^{mw(y)}\Big|\Big(\frac{1}{y}D\Big)^n(2^\mu\Gamma(\mu+1)(xy)^{-\mu}J_\mu(xy)h_\mu(\phi)(y))\Big| \\ &\leq C\sum_{j=0}^n e^{mw(y)}\Big|\Big(\frac{1}{y}D\Big)^{n-j}h_\mu(\phi)(y)\Big|x^{2j}, \quad y\in(0,\infty). \end{aligned}$$

Then

$$\beta^\mu_{m,n}(\tau_x\phi)\leq C(1+x^{2n})\sum_{j=0}^n \beta^\mu_{m,j}(\phi). \tag{2.10}$$

From (2.9) and (2.10) we deduce that τ_x is continuous from $\mathcal{H}_\mu(w)$ into itself.

(ii) Let $\phi\in\mathcal{H}_\mu(w)$. Assume that $x_0\in(0,\infty)$ and $m,n\in\mathbf{N}$. We can write for every $x\in[(x_0/2),(3x_0/2)]$ and $y\geq 2x_0$,

$$\begin{aligned} e^{mw(y)}&|\Delta^n_\mu((\tau_x\phi)-(\tau_{x_0}\phi))(y)| \\ &\leq e^{(m+1)[w(y)-w(y-(3x_0/2))]-w(y)}\sup_{z\in(0,\infty)}e^{(m+1)w(z)}|\Delta^n_\mu\phi(z)| \\ &\quad\cdot\int_{y-(3x_0/2)}^{y+(3x_0/2)}|D(x,y,z)-D(x_0,y,z)|\,\frac{z^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dz \\ &\leq 2e^{(m+1)w(3x_0/2)-w(y)}\sup_{z\in(0,\infty)}e^{(m+1)w(z)}|\Delta^n_\mu\phi(z)|. \end{aligned}$$

Hence, if $\varepsilon>0$, then there exists $y_1\geq 2x_0$ such that, for every $x\in[(x_0/2),(3x_0/2)]$ and $y\geq y_1$,

$$e^{mw(y)}|\Delta^n_\mu((\tau_x\phi)-(\tau_{x_0}\phi))(y)|<\varepsilon.$$

On the other hand, since w is increasing on $[0,\infty)$, it has

$$\begin{aligned} \sup_{y\in(0,y_1)}&e^{mw(y)}|\Delta^n_\mu((\tau_x\phi)-(\tau_{x_0}\phi))(y)| \\ &\leq e^{mw(y_1)}\sup_{y\in(0,y_1)}|\Delta^n_\mu((\tau_x\phi)-(\tau_{x_0}\phi))(y)|. \end{aligned}$$

Therefore, according to [**14**, p. 359], since Δ_μ is a continuous operator from $\mathcal{H}_{-1/2}$ into itself, we deduce that if $\varepsilon > 0$, then

$$\sup_{y\in(0,y_1)} e^{mw(y)}|\Delta_\mu^n((\tau_x\phi) - (\tau_{x_0}\phi))(y)| < \varepsilon,$$

provided that $x \in (0,\infty)$ and $|x - x_0| < \delta$, for some $\delta > 0$.

Thus we conclude that, for every $\varepsilon > 0$, there exists $\delta > 0$ for which

$$A^\mu_{m,n}(\tau_x\phi - \tau_{x_0}\phi) < \varepsilon,$$

when $x \in (0,\infty)$ and $|x - x_0| < \delta$.

Moreover, the Leibniz rule and again [**3**, (3.1)] and [**20** (7), Chapter 5] lead to

$$\begin{aligned}\Big(\frac{1}{y}\frac{d}{dy}\Big)^n&(h_\mu(\tau_x\phi - \tau_{x_0}\phi)(y))\\ &= 2^\mu\Gamma(\mu+1)\sum_{j=0}^n \binom{n}{j}(-1)^j\Big(\frac{1}{y}\frac{d}{dy}\Big)^{n-j} h_\mu(\phi)(y)\\ &\quad\cdot(x^{2j}(xy)^{-\mu-j}J_{\mu+j}(xy) - x_0^{2j}(x_0y)^{-\mu-j}J_{\mu+j}(x_0y)),\\ &\qquad x,y\in(0,\infty).\end{aligned}$$

Hence, the boundedness of the function $z^{-\mu}J_\mu(z)$, $z \in (0,\infty)$, implies that if $\varepsilon > 0$,

$$\begin{aligned}e^{mw(y)}\Big|\Big(\frac{1}{y}\frac{d}{dy}\Big)^n&(h_\mu(\tau_x\phi - \tau_{x_0}\phi)(y))\Big|\\ &\le Ce^{-w(y)}\sum_{j=0}^n (x^{2j} + x_0^{2j})\beta^\mu_{m+1,n-j}(\phi)\\ &< \varepsilon,\end{aligned}$$

for each $x \in (0, 2x_0)$ and $y \ge y_1$, where y_1 is a large enough positive number.

On the other hand, since the function $f_j(x,y) = x^{2j}(xy)^{-\mu-j}J_{\mu+j}(xy)$, $x, y \in [0,\infty)$, is continuous (and hence uniformly continuous in each compact subset of $[0,\infty)\times[0,\infty)$), for every $j \in \mathbf{N}$, if $\varepsilon > 0$ we can find $\delta > 0$ such that $|f_j(x,y) - f_j(x_0,y)| < \varepsilon$, for every $y \in [0,y_1]$, $x \in [0,\infty)$, $|x-x_0| < \delta$ and $j = 0,\dots,n$. Then

$$\sup_{y\in(0,y_1)} e^{mw(y)}\Big|\Big(\frac{1}{y}\frac{d}{dy}\Big)^n (h_\mu(\tau_x\phi - \tau_{x_0}\phi)(y))\Big| \le C\varepsilon \sum_{j=0}^{n} \alpha^{\mu}_{m,j}(\phi),$$

for every $x \in (0,\infty)$ and $|x-x_0| < \delta$.

Thus, it is concluded that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\alpha^{\mu}_{m,n}(\tau_x\phi - \tau_{x_0}\phi) < \varepsilon,$$

provided that $x \in (0,\infty)$ and $|x-x_0| < \delta$.

Hence F_ϕ is a continuous function on x_0.

To see that F_ϕ is continuous in $x = 0$, we can proceed in a similar way. □

Next we study the pointwise multiplication and the Hankel convolution on $\mathcal{H}_\mu(w)$.

Proposition 2.5. *The bilinear mappings defined by*

$$(\phi,\psi) \longrightarrow \phi\psi$$

and

$$(\phi,\psi) \longrightarrow \phi\#\psi$$

are continuous from $\mathcal{H}_\mu(w) \times \mathcal{H}_\mu(w)$ *into* $\mathcal{H}_\mu(w)$.

Proof. By virtue of the interchange formula [**14**, Theorem 2.d]

$$h_\mu(\phi\#\psi) = h_\mu(\phi)h_\mu(\psi), \quad \phi,\psi \in \mathcal{H}_\mu(w),$$

the continuity of the pointwise multiplication mapping is equivalent to the one of the Hankel convolution mapping.

Let $m,n \in \mathbf{N}$. Assume that $\phi,\psi \in \mathcal{H}_\mu(w)$. We can write, from the Leibniz rule, that

$$\alpha_{m,n}(\phi\psi) \le C \sum_{j=0}^{n} \alpha_{m,n-j}(\phi)\alpha_{0,j}(\psi).$$

On the other hand, since $\Delta_\mu(\phi\#\psi) = (\Delta_\mu\phi)\#\psi$ [**14**, Proposition 2.2] and since w is increasing on $[0,\infty)$ and it satisfies the (α)-property, it has

$$\begin{aligned}
&e^{mw(x)}|\Delta_\mu^n h_\mu(\phi\psi)(x)| \\
&\quad = e^{mw(x)}|((\Delta_\mu^n h_\mu(\phi))\#h_\mu(\psi))(x)| \\
&\quad \le e^{mw(x)}\int_0^\infty |\Delta_\mu^n(h_\mu\phi)(y)|e^{-mw(|x-y|)} \\
&\qquad \cdot \int_{|x-y|}^{x+y} D(x,y,z)|h_\mu(\psi)(z)|e^{mw(z)}\frac{z^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dz\,\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dy \\
&\quad \le \int_0^\infty |\Delta_\mu^n(h_\mu\phi)(y)|e^{mw(y)}\int_{|x-y|}^{x+y} D(x,y,z)|h_\mu(\psi)(z)|e^{mw(z)} \\
&\qquad \cdot \frac{z^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dz\,\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dy, \quad x\in(0,\infty).
\end{aligned}$$

Hence, since w verifies the (γ)-property and by taking into account [**13**], we can conclude

$$B_{m,n}^\mu(\phi\psi) \le CB_{m+l,n}^\mu(\phi)B_{m,0}^\mu(\psi),$$

for some $l \in \mathbf{N}$.

By virtue of Proposition 2.3, we have proved that the pointwise multiplication defines a continuous mapping from $\mathcal{H}_\mu(w)\times\mathcal{H}_\mu(w)$ into $\mathcal{H}_\mu(w)$.

Thus the proof of this proposition is complete. $\square$

Remark 1. The last proposition shows that each function in $\mathcal{H}_\mu(w)$ defines a multiplier in $\mathcal{H}_\mu(w)$. Also, in the proof of Proposition 2.4, it was established that for every $x \in (0,\infty)$ the function f_x defined by

$$f_x(y) = (xy)^{-\mu}J_\mu(xy), \quad y\in(0,\infty),$$

is a multiplier of $\mathcal{H}_\mu(w)$. It is an open problem to give a complete description of the space of multipliers of $\mathcal{H}_\mu(w)$.

In [**2**] we introduced the space $\mathcal{B}_\mu(w)$ (see Section 1 for definitions). $\mathcal{B}_\mu(w)$ can be considered a Beurling type function space for the Hankel

h_μ transformation. In the following we establish that $\mathcal{B}_\mu(w)$ is a dense subset of $\mathcal{H}_\mu(w)$.

Proposition 2.6. *The space $\mathcal{B}_\mu(w)$ is continuously contained in $\mathcal{H}_\mu(w)$. Moreover, $\mathcal{B}_\mu(w)$ is a dense subspace of $\mathcal{H}_\mu(w)$.*

Proof. Let $\phi \in \mathcal{B}^a_\mu(w)$, where $a > 0$. Since ϕ and $h_\mu(\phi) \in L_{\mu,1}$, according to [**13**, Corollary 2], it has

$$\phi(x) = \int_0^\infty (xy)^{-\mu} J_\mu(xy) h_\mu(\phi)(y) y^{2\mu+1}\, dy, \quad x \in (0,\infty).$$

Hence, by invoking [**20** (7), Chapter 5], since $z^{-\mu}J_\mu(z)$ is a bounded function on $(0,\infty)$ and w satisfies the (γ)-property for every $m, n \in \mathbf{N}$, we can find $l \in \mathbf{N}$ for which

$$\alpha_{m,n}(\phi) \le C \sup_{x\in(0,a)} e^{mw(x)} \int_0^\infty y^{2n+2\mu+1} |h_\mu(\phi)(y)|\, dy \le C\delta^\mu_l(\phi). \tag{2.11}$$

Here C is a positive constant that is not dependent on ϕ.

By virtue of the Paley-Wiener type theorem for the Hankel transform on $\mathcal{B}^a_\mu(w)$ ([**2**, Proposition 2.6]), $h_\mu(\phi)$ is an even entire function and, for every $m \in \mathbf{N}$, there exists $C_m > 0$ for which

$$|h_\mu(\phi)(x+iy)| \le C_m e^{-mw(x)+(a+1)|y|}, \quad x, y \in \mathbf{R}. \tag{2.12}$$

According to the well-known Cauchy integral formula, we can write

$$\frac{d^l}{dx^l} h_\mu(\phi)(x) = \frac{l!}{2\pi i} \int_{\mathcal{C}_x} \frac{h_\mu(\phi)(z)}{(z-x)^{l+1}}\, dz, \quad l \in \mathbf{N} \text{ and } x \in \mathbf{R}, \tag{2.13}$$

where $\mathcal{C}_x$ represents the circled path having by parametric representation $z = x + e^{i\theta}$, $\theta \in [0, 2\pi)$.

Let $m, n \in \mathbf{N}$. From (2.12) and (2.13), it follows, since w satisfies the (α)-property, that

$$\Big|\frac{d^n}{dx^n} h_\mu(\phi)(x)\Big| \le C \int_0^{2\pi} e^{-mw(x+\cos\theta)+(a+1)|\sin\theta|}\, d\theta \le C e^{-mw(x)}, \quad x \ge 1.$$

Hence it follows

$$\Big|\Big(\frac{1}{x}\frac{d}{dx}\Big)^n h_\mu(\phi)(x)\Big| \le Ce^{-mw(x)}, \quad x \ge 1.$$

Moreover, by using again the above-mentioned properties of the Bessel functions, we have

$$\Big|\Big(\frac{1}{x}\frac{d}{dx}\Big)^n h_\mu(\phi)(x)\Big| \le C\int_0^a y^{2n+2\mu+1}|\phi(y)|\,dy \le C\alpha_{0,0}(\phi), \quad x\in(0,1).$$

Thus we conclude that $\beta^\mu_{m,n}(\phi) < \infty$.

We have proved that $\mathcal{B}^a_\mu(w)$ is contained in $\mathcal{H}_\mu(w)$.

To see that $\mathcal{B}^a_\mu(w)$ is continuously contained in $\mathcal{H}_\mu(w)$ we will use the closed graph theorem. Assume that $\{\phi_\nu\}_{\nu\in\mathbf{N}}$ is a sequence in $\mathcal{B}^a_\mu(w)$ such that $\phi_\nu \to \phi$ as $\nu \to \infty$, in $\mathcal{B}^a_\mu(w)$ and $\phi_\nu \to \psi$ as $\nu \to \infty$ in $\mathcal{H}_\mu(w)$. It is clear that $\phi_\nu(x) \to \psi(x)$ as $\nu \to \infty$ for every $x \in (0,\infty)$. Moreover, from (2.11) we deduce that $\phi_\nu(x) \to \phi(x)$ as $\nu \to \infty$ for each $x \in (0,\infty)$. Hence $\phi = \psi$. Thus we show that $\mathcal{B}^a_\mu(w)$ is continuously contained in $\mathcal{H}_\mu(w)$ for every $a > 0$. Then the inclusion $\mathcal{B}_\mu(w) \subset \mathcal{H}_\mu(w)$ is continuous.

We now see that $\mathcal{B}_\mu(w)$ is a dense subset of $\mathcal{H}_\mu(w)$. According to [**2**, Proposition 2.18] we choose $\psi \in \mathcal{B}^2_\mu(w)$ such that $0 \le \psi \le 1$ and $\psi(x) = 1$, $x \in (0,1)$. Assume that $\phi \in \mathcal{H}_\mu(w)$. We define for every $l \in \mathbf{N}\setminus\{0\}$, $\psi_l(x) = \psi(x/l)$, $x \in (0,\infty)$ and $\phi_l = \psi_l\phi$.

Let $m,n \in \mathbf{N}$. The Leibniz rule leads to, for every $l \in \mathbf{N}\setminus\{0\}$,

$$e^{mw(x)}\Big|\Big(\frac{1}{x}D\Big)^n(\phi_l(x) - \phi(x))\Big| \le S^1_l(x) + S^2_l(x), \quad x \in (0,\infty),$$

where

$$S^1_l(x) = \sum_{j=0}^{n-1}\binom{n}{j}e^{mw(x)}\Big|\Big(\frac{1}{x}D\Big)^j\phi(x)\Big|\Big|\Big(\frac{1}{x}D\Big)^{n-j}\psi\Big(\frac{x}{l}\Big)\Big|, \quad x\in(0,\infty),$$

and

$$S^2_l(x) = e^{mw(x)}\Big|\Big(\frac{1}{x}D\Big)^l\phi(x)\Big|\Big|\psi\Big(\frac{x}{l}\Big) - 1\Big|, \quad x \in (0,\infty).$$

Standard arguments allow us now to conclude that

$$\alpha_{m,n}(\phi_l - \phi) \longrightarrow 0, \quad \text{as } l \to \infty.$$

On the other hand, by [**13**, Theorem 2d], since $\psi_l(0) = 1$, $l \in \mathbf{N} \setminus \{0\}$, we can write

$$\begin{aligned}
&\Delta_\mu^n h_\mu(\phi_l - \phi)(x) \\
&\quad = (h_\mu(\psi_l) \# \Delta_\mu^n h_\mu(\phi))(x) - \Delta_\mu^n h_\mu(\phi)(x) \\
&\quad = \int_0^\infty h_\mu(\psi_l)(y)(\tau_x(\Delta_\mu^n h_\mu(\phi))(y) - \Delta_\mu^n h_\mu(\phi)(x)) \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\, dy,
\end{aligned}$$

for each $x \in (0, \infty)$ and $l \in \mathbf{N} \setminus \{0\}$.

Fix $l \in \mathbf{N} \setminus \{0\}$. To simplify we denote by $\Phi = \Delta_\mu^n h_\mu(\phi)$. It is not hard to see that $h_\mu(\psi_l)(y) = l^{2(\mu+1)} h_\mu(\psi)(yl)$, $y \in (0, \infty)$. Then

$$\begin{aligned}
&\Delta_\mu^n h_\mu(\phi_l - \phi)(x) \\
&\quad = \int_0^\infty h_\mu(\psi)(y)\Big(\tau_x(\Phi)\Big(\frac{y}{l}\Big) - \Phi(x)\Big) \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\, dy, \quad x \in (0, \infty).
\end{aligned}$$

We now consider $\alpha \in (0, 1)$ that will be specified later. We divide the last integral into two parts.

According to [**13**, (2)], since w is an increasing function on $[0, \infty)$, we have that

$$\begin{aligned}
&\Big| \int_{x+l^\alpha}^\infty h_\mu(\psi)(y) \int_{|x-y/l|}^{x+y/l} D\Big(x, \frac{y}{l}, z\Big) \\
&\qquad\qquad \cdot (\Phi(z) - \Phi(x)) \frac{z^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\, dz \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\, dy\Big| \\
&\quad \leq C \sup_{z \in (0,\infty)} |\Phi(z)| \int_{x+l^\alpha}^\infty |h_\mu(\psi)(y)| y^{2\mu+1}\, dy \\
&\quad \leq C \int_{x+l^\alpha}^\infty e^{-(m+k)w(y)} y^{2\mu+1}\, dy \\
&\qquad\qquad \cdot \sup_{z \in (0,\infty)} |\Phi(z)| \sup_{z \in (0,\infty)} |h_\mu(\psi)(z)| e^{(m+k)w(z)} \\
&\quad \leq C e^{-mw(x)} \int_{l^\alpha}^\infty e^{-kw(y)} y^{2\mu+1}\, dy \\
&\qquad\qquad \cdot \sup_{z \in (0,\infty)} |\Phi(z)| \sup_{z \in (0,\infty)} |h_\mu(\psi)(z)| e^{(m+k)w(z)},
\end{aligned}$$

for every $x \in (0,\infty)$ and $k \in \mathbf{N}$.

Hence, since w satisfies the (γ)-property, by choosing $k \in \mathbf{N}$ large enough it follows that

$$\begin{aligned}\sup_{x\in(0,\infty)} \Big| e^{mw(x)} \int_{x+l^\alpha}^{\infty} h_\mu(\psi)(y) \int_{|x-y/l|}^{x+y/l} D\Big(x, \frac{y}{l}, z\Big)(\Phi(z)-\Phi(x)) \\ \cdot \frac{z^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\, dz\, \frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\, dy\Big| \\ \leq C \int_{l^\alpha}^\infty e^{-kw(y)} y^{2\mu+1}\, dy \sup_{z\in(0,\infty)} |\Phi(z)| \sup_{z\in(0,\infty)} |h_\mu(\psi)(z)| e^{(m+k)w(z)} \\ \to 0, \quad \text{as } l\to\infty.\end{aligned}$$

On the other hand, by again using [**13**, (2)], one obtains, for every $x \in (0,\infty)$,

$$\begin{aligned}\Big| e^{mw(x)} \int_0^{x+l^\alpha} h_\mu(\psi)(y) \int_{|x-y/l|}^{x+y/l} D\Big(x, \frac{y}{l}, z\Big)(\Phi(z)-\Phi(x)) \\ \cdot \frac{z^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\, dz\, \frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\, dy\Big| \\ \leq C \sup_{z\in(0,\infty)} |h_\mu(\phi)(z)| e^{mw(x)} (x+l^\alpha)^{2\mu+2} \sup_{\substack{|x-y/l|\leq z\leq x+y/l \\ 0<y<x+l^\alpha}} |\Phi(z)-\Phi(x)|.\end{aligned}$$

Moreover, we have that, for each $\eta \in (0, x+l^\alpha)$ and $x\in(0,\infty)$,

$$\begin{aligned}\Big|\Phi\Big(x+\frac{\eta}{l}\Big)-\Phi(x)\Big| &\leq \int_x^{x+(\eta/l)} \Big|\frac{d}{dt}\Phi(t)\Big|\, dt \\ &\leq \frac{1}{l}(x+l^\alpha) \sup_{-x-l^\alpha\leq\xi\leq x+l^\alpha} \Big|\Big(\frac{d}{dt}\Phi\Big)\Big(x+\frac{\xi}{l}\Big)\Big|.\end{aligned}$$

Also, we can write

$$\Big|\Phi\Big(x+\frac{\eta}{l}\Big)-\Phi(x)\Big| \leq \frac{1}{l}(x+l^\alpha) \sup_{-x-l^\alpha\leq\xi\leq x+l^\alpha} \Big|\Big(\frac{d}{dt}\Phi\Big)\Big(x+\frac{\xi}{l}\Big)\Big|,$$

for each $x\in(0,\infty)$ and $\eta \in (-x-l^\alpha, 0)$.

If it is necessary above we consider the even and smooth extension of Φ to $\mathbf{R}$. Hence, it has

$$\Big|e^{mw(x)}\int_0^{x+l^\alpha} h_\mu(\psi)(y)\int_{|x-y/l|}^{x+y/l} D\Big(x,\frac{y}{l},z\Big)(\Phi(z)-\Phi(x)) \cdot \frac{z^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dz\,\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dy\Big|$$
$$\leq C\sup_{z\in(0,\infty)}|h_\mu(\psi)(z)|e^{mw(x)}\frac{1}{l}(x+l^\alpha)^{2\mu+4} \cdot \sup_{-x-l^\alpha\leq\xi\leq x+l^\alpha}\Big|\Big(\frac{1}{t}\frac{d}{dt}\Phi\Big)\Big(x+\frac{\xi}{l}\Big)\Big|$$
$$\leq C\sup_{z\in(0,\infty)}|h_\mu(\psi)(z)|e^{mw(x)-kw(x-(x/l)-l^{\alpha-1})}\frac{1}{l}(x+l^\alpha)^{2\mu+4} \cdot \sup_{z\in(0,\infty)}\Big|\frac{1}{z}\frac{d}{dz}\Phi(z)\Big|e^{kw(z)},$$

provided that $x\geq 2$, $k,l\in\mathbf{N}$ and $l\geq 2$. Note that if $x,l\geq 2$, $x\geq(l^\alpha/(l-1))$. Then

$$\Big|e^{mw(x)}\int_0^{x+l^\alpha} h_\mu(\psi)(y)\int_{|x-y/l|}^{x+y/l} D\Big(x,\frac{y}{l},z\Big)(\Phi(z)-\Phi(x)) \cdot \frac{z^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dz\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dy\Big|$$
$$\leq Cl^{\alpha(2\mu+4)-1}(x+1)^{2\mu+4}e^{mw(x)-kw[x-(x/l)-l^{\alpha-1}]},$$

when $x\geq 2$, $l,k\in\mathbf{N}$ and $l\geq 2$.

Since w is increasing on $[0,\infty)$ and w verifies the (α)-property, we have that

$$w\Big(x-\frac{x}{l}-l^{\alpha-1}\Big)\geq\frac{1}{2}w(x)-w(1),\quad x\geq 2,\quad l,k\in\mathbf{N}\text{ and }l\geq 2.$$

hence, by choosing k large enough, since w satisfies the (γ)-property, it follows

$$\Big|e^{mw(x)}\int_0^{x+l^\alpha} h_\mu(\psi)(y)\int_{|x-y/l|}^{x+y/l} D\Big(x,\frac{y}{l},z\Big)(\Phi(z)-\Phi(x)) \cdot \frac{z^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dz\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dy\Big|$$
$$\leq Cl^{\alpha(2\mu+4)-1},\quad x\geq 2,\quad l,k\in\mathbf{N}\text{ and }l\geq 2.$$

Assume now that $0 < \alpha < 1/(2\mu + 4)$. Then we conclude that

$$\sup_{x\geq 2}\Big|e^{mw(x)}\int_0^{x+l^\alpha} h_\mu(\psi)(y)\int_{|x-y/l|}^{x+y/l} D\Big(x,\frac{y}{l},z\Big)(\Phi(z)-\Phi(x)) \cdot \frac{z^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dz\,\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dy\Big| \to 0,$$

as $l \to \infty$.

By proceeding in a similar way we obtain that

$$\sup_{0\leq x\leq 2}\Big|e^{mw(x)}\int_0^{x+l^\alpha} h_\mu(\psi)(y)\int_{|x-y/l|}^{x+y/l} D\Big(x,\frac{y}{l},z\Big)(\Phi(z)-\Phi(x)) \cdot \frac{z^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dz\,\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dy\Big|$$

$$\leq C\sup_{z\in(0,\infty)}|h_\mu(\psi)(z)|\frac{1}{l}(2+l^\alpha)^{2\mu+4}\sup_{z\in(0,\infty)}\Big|\frac{1}{z}\frac{d}{dz}\Phi(z)\Big| \to 0, \quad \text{as } l\to\infty,$$

provided that $0 < \alpha < 1/(2^\mu + 4)$.

Thus, we deduce that

$$B^\mu_{m,n}(\phi_l - \phi) \longrightarrow 0, \quad \text{as } l \to \infty.$$

By taking into account Proposition 2.3, the proof is now complete. □

Remark 2. According to [**2**, Corollary 2.8], the (β)-property (for w) is essential to establish the nontriviality of the space $\mathcal{B}_\mu(w)$. However the space $\mathcal{H}_\mu(w)$ is nontrivial although w does not verify (β). Indeed, the function $\phi(x) = e^{-x^2/2}$, $x \in [0,\infty)$, is in $\mathcal{H}_\mu(w)$ (see [**10**, (10)]) provided that $w(x) \leq Cx^l$, when x is large for some $l < 2$.

Next we establish a result concerning approximated identity in $\mathcal{H}_\mu(w)$ involving Hankel convolution. This property, whose proof will be omitted, can be proved following a procedure similar to the one employed to prove [**3**, Proposition 3.5] and [**6**, Proposition 2.3].

Proposition 2.7. *Assume that $\psi \in \mathcal{B}_\mu(w)$ and that $\int_0^\infty \psi(x)x^{2\mu+1}\,dx = 2^\mu\Gamma(\mu+1)$. Then, for every $\phi \in \mathcal{H}_\mu(w)$, $\phi\#\psi_m \to \phi$, as $m \to \infty$, in $\mathcal{H}_\mu(w)$ where, for each $m \in \mathbf{N}$, $\psi_m(x) = m^{2\mu+2}\psi(mx)$, $x \in (0,\infty)$.*

3. Hankel transformation and Hankel convolution on the space $\mathcal{H}_\mu(w)'$ dual of $\mathcal{H}_\mu(w)$. In this section we study the Hankel transformation and the Hankel convolution on $\mathcal{H}_\mu(w)'$, the dual space of $\mathcal{H}_\mu(w)$. Our results can be seen as an extension of the ones presented in [**5**] and [**14**].

Suppose that f is a measurable function on $(0,\infty)$ such that, for some $k \in \mathbf{N}$,

$$\int_0^\infty e^{-kw(x)}|f(x)|x^{2\mu+1}\,dx < \infty,$$

then f defines an element $T_f \in \mathcal{H}_\mu(w)'$ by

$$\langle T_f, \phi\rangle = \int_0^\infty f(x)\phi(x)\frac{x^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dx, \quad \phi \in \mathcal{H}_\mu(w).$$

Indeed, for every $\phi \in \mathcal{H}_\mu(w)$, it has

$$|\langle T_f, \phi\rangle| \le C\int_0^\infty e^{-kw(x)}|f(x)|x^{2\mu+1}\,dx\alpha_{k,0}(\phi).$$

In particular the space $\mathcal{H}_\mu(w)$ can be identified with a subspace of $\mathcal{H}_\mu(w)'$.

On the other hand, if $\phi \in \mathcal{H}_\mu(w)$ then $\phi \in \mathcal{E}_\mu(w)$, the space of pointwise multipliers of $\mathcal{B}_\mu(w)$. Indeed, let $\phi \in \mathcal{H}_\mu(w)$. Assume that $\psi \in \mathcal{B}_\mu^a(w)$ with $a > 0$. Then $\phi(x)\psi(x) = 0$, $x \ge a$. Moreover, for every $n \in \mathbf{N}$,

$$\delta_n^\mu(\phi\psi) = \int_0^\infty e^{nw(x)}|h_\mu(\phi\psi)(x)|x^{2\mu+1}\,dx \le C\delta_n^\mu(\psi)\beta_{l,0}^\mu(\phi),$$

where $l \in \mathbf{N}$ is chosen large enough and it is not depending on ϕ.

Note that we also have proved that $\mathcal{H}_\mu(w)$ is continuously contained in $\mathcal{E}_\mu(w)$. Hence, the dual space $\mathcal{E}_\mu(w)'$ of $\mathcal{E}_\mu(w)$ is contained in $\mathcal{H}_\mu(w)'$.

We define the Hankel transformation on $\mathcal{H}_\mu(w)'$ by transposition. That is, if $T \in \mathcal{H}_\mu(w)'$, the Hankel transform $h_\mu' T$ of T is the element of $\mathcal{H}_\mu(w)'$ given through

$$\langle h_\mu' T, \phi\rangle = \langle T, h_\mu\phi\rangle, \quad \phi \in \mathcal{H}_\mu(w).$$

The generalized Hankel transformation h'_μ can be seen as an extension of the Hankel transformation h_μ. Let $\psi \in \mathcal{H}_\mu(w)$. Since $h_\mu(\psi) \in \mathcal{H}_\mu(w)$, $h_\mu(\psi)$ defines an element $T_{h_\mu(\phi)}$ of $\mathcal{H}_\mu(w)'$ by

$$\langle T_{h_\mu(\psi)}, \phi\rangle = \int_0^\infty h_\mu(\psi)(x)\phi(x)\frac{x^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dx, \quad \phi \in \mathcal{H}_\mu(w).$$

Moreover, Parseval equality for Hankel transformations leads to

$$\begin{aligned}\langle T_{h_\mu}(\psi), \phi\rangle &= \int_0^\infty \psi(x)h_\mu(\phi)(x)\frac{x^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dx \\ &= \langle T_\psi, h_\mu(\phi)\rangle, \quad \phi \in \mathcal{H}_\mu(w).\end{aligned}$$

Thus we have shown that $T_{h_\mu(\psi)} = h'_\mu(T_\psi)$.

We now determine the Hankel transform of the distributions in $\mathcal{E}_\mu(w)'$.

Proposition 3.1. *If $T \in \mathcal{E}_\mu(w)'$, the Hankel transform $h'_\mu T$ coincides with the functional defined by the function*

$$F(x) = 2^\mu\Gamma(\mu+1)\langle T(y), (xy)^{-\mu}J_\mu(xy)\rangle, \quad x \in (0,\infty).$$

Then $h'_\mu T$ is a continuous function on $[0,\infty)$ and there exist $C > 0$ and $r \in \mathbf{N}$ for which

$$|h'_\mu(T)(x)| \le Ce^{rw(x)}, \quad x \in (0,\infty).$$

Proof. Let $T = \mathcal{E}_\mu(w)'$. We have to see that

(3.1)
$$\langle h'_\mu(T), \phi\rangle = \langle T, h_\mu(\phi)\rangle = \int_0^\infty \langle T(y), (xy)^{-\mu}J_\mu(xy)\rangle\phi(x)x^{2\mu+1}\,dx,$$

for every $\phi \in \mathcal{H}_\mu(w)$.

In [**2**, Proposition 3.4] we proved that, for every $x \in (0,\infty)$, the function f_x defined by $f_x(y) = (xy)^{-\mu}J_\mu(xy)$, $y \in (0,\infty)$ is in $\mathcal{E}_\mu(w)$. Hence, we can define the function

$$F(x) = \langle T(y), (xy)^{-\mu}J_\mu(xy)\rangle, \quad x \in [0,\infty).$$

Thus F is a continuous function on $[0,\infty)$. Indeed, let $x_0 \in [0,\infty)$. To see that F is continuous in x_0, it is sufficient to show that, for every $n \in \mathbf{N}$ and $\phi \in \mathcal{B}_\mu(w)$,

$$\delta_n^\mu(\phi(y)((xy)^{-\mu}J_\mu(xy) - (x_0y)^{-\mu}J_\mu(x_0y))) \longrightarrow 0, \quad \text{as } x \to x_0.$$

Assume that $n \in \mathbf{N}$ and $\phi \in \mathcal{B}_\mu(w)$. By virtue of [**3**, (3.4)], it follows for every $x, z \in [0,\infty)$,

$$\begin{aligned} h_\mu(\phi(y)((xy)^{-\mu}J_\mu(xy) - (x_0y)^{-\mu}J_\mu(x_0y)))(z) \\ = \frac{1}{2^\mu\Gamma(\mu+1)}(\tau_x(h_\mu\phi)(z) - \tau_{x_0}(h_\mu\phi)(z)). \end{aligned}$$

According to Proposition 2.4 (ii) and Proposition 2.6, the mapping G defined by

$$G(x) = \tau_x(h_\mu\phi), \quad x \in [0,\infty),$$

is continuous from $[0,\infty)$ into $\mathcal{H}_\mu(w)$. Moreover, since w satisfies the (γ)-property, there exists $l \in \mathbf{N}$ such that

$$\begin{aligned} &\delta_n^\mu(\phi((x.)^{-\mu}J_\mu(x.) - (x_0.)^{-\mu}J_\mu(x_0.))) \\ &\quad = \frac{1}{2^\mu\Gamma(\mu+1)}\int_0^\infty e^{nw(z)}|\tau_x(h_\mu\phi)(z) - \tau_{x_0}(h_\mu\phi)(z)|z^{2\mu+1}\,dz \\ &\quad \le C\alpha_{n+l,0}(\tau_x(h_\mu\phi) - \tau_{x_0}(h_\mu\phi)), \quad x \in [0,\infty). \end{aligned}$$

Hence,

$$\delta_n^\mu(\phi(y)((xy)^{-\mu}J_\mu(xy) - (x_0y)^{-\mu}J_\mu(x_0y))) \longrightarrow 0, \quad \text{as } x \to x_0.$$

Moreover, since $T \in \mathcal{E}_\mu(w)'$, there exist $C > 0$, $r \in \mathbf{N}$ and $\phi_1, \dots, \phi_r \in \mathcal{B}_\mu(w)$,

$$|\langle T, \Phi\rangle| \le C \max_{j=1,\dots,r} \delta_r^\mu(\phi_j\Phi), \quad \Phi \in \mathcal{E}_\mu(w).$$

In particular, since w has the (γ)-property for every $x \in (0,\infty)$,

$$\begin{aligned} |\langle T(y), (xy)^{-\mu}J_\mu(xy)\rangle| &\le C \max_{j=1,\dots,r}\int_0^\infty e^{rw(x)}|\tau_x(h_\mu\phi_j)(y)|y^{2\mu+1}\,dy \\ &\le C \max_{j=1,\dots,r} \alpha_{r+l,0}(\tau_x(h_\mu\phi_j)), \end{aligned}$$

for some $l \in \mathbf{N}$. Then by (2.9), it follows that

$$(3.2)\quad |\langle T(y), (xy)^{-\mu} J_\mu(xy)\rangle| \leq C e^{(r+l)w(x)} \max_{j=1,\dots,r} \beta^\mu_{r+l,0}(\phi_j), \quad x \in [0,\infty).$$

From (3.2) we infer that the integral in (3.1) is absolutely convergent for every $\phi \in \mathcal{H}_\mu(w)$.

Assume that $\phi \in \mathcal{H}_\mu(w)$. It is clear that

$$\lim_{b\to\infty} \int_b^\infty \langle T(y), (xy)^{-\mu} J_\mu(xy)\rangle \phi(x) x^{2\mu+1}\, dx = 0.$$

Let $b > 0$. We can write

$$(3.3)\quad \int_0^b \langle T(y), (xy)^{-\mu} J_\mu(xy)\rangle \phi(x) x^{2\mu+1}\, dx = \lim_{n\to\infty} \Big\langle T(y), \frac{b}{n} \sum_{j=1}^n \Big(\frac{jb}{n} y\Big)^{-\mu} J_\mu\Big(\frac{jb}{n} y\Big) \phi\Big(\frac{jb}{n}\Big) \Big(\frac{jb}{n}\Big)^{2\mu+1} \Big\rangle.$$

We are going to see that

$$(3.4)\quad \int_0^b (xy)^{-\mu} J_\mu(xy) \phi(x) x^{2\mu+1}\, dx = \lim_{n\to\infty} \frac{b}{n} \sum_{j=1}^n \Big(\frac{jb}{n} y\Big)^{-\mu} J_\mu\Big(\frac{jb}{n} y\Big) \phi\Big(\frac{jb}{n}\Big) \Big(\frac{jb}{n}\Big)^{2\mu+1},$$

in the sense of convergence of $\mathcal{E}_\mu(w)$.

Indeed, let $\psi \in \mathcal{B}_\mu(w)$ and $m \in \mathbf{N}$. It has, for some $l \in \mathbf{N}$,

$$\begin{aligned}
\delta_m^\mu\Big(\psi(y)\Big(\int_0^b (xy)^{-\mu} J_\mu(xy)\phi(x)x^{2\mu+1}\,dx \\
&- \frac{b}{n}\sum_{j=1}^n \Big(\frac{jb}{n}y\Big)^{-\mu} J_\mu\Big(\frac{jb}{n}y\Big)\phi\Big(\frac{jb}{n}\Big)\Big(\frac{jb}{n}\Big)^{2\mu+1}\Big)\Big) \\
&\le C\alpha_{l,0}\Big(h_\mu\Big(\psi(y)\Big(\int_0^b (xy)^{-\mu} J_\mu(xy)\phi(x)x^{2\mu+1}\,dx \\
&- \frac{b}{n}\sum_{j=1}^n \Big(\frac{jb}{n}y\Big)^{-\mu} J_\mu\Big(\frac{jb}{n}y\Big)\phi\Big(\frac{jb}{n}\Big)\Big(\frac{jb}{n}\Big)^{2\mu+1}\Big)\Big)\Big) \\
&\le C\alpha_{l,0}\Big(\int_0^b \phi(x)x^{2\mu+1}\tau_x(h_\mu\psi)(z)\,dx \\
&- \frac{b}{n}\sum_{j=1}^n \phi\Big(\frac{jb}{n}\Big)\Big(\frac{jb}{n}\Big)^{2\mu+1}\tau_{jb/n}(h_\mu\psi)(z)\Big).
\end{aligned}$$

Note that from (2.9), it follows that

$$\begin{aligned}
e^{lw(z)}\Big|\int_0^b \phi(x)x^{2\mu+1}\tau_x(h_\mu\psi)(z)\,dx \\
&- \frac{b}{n}\sum_{j=1}^n \phi\Big(\frac{jb}{n}\Big)\Big(\frac{jb}{n}\Big)^{2\mu+1}\tau_{jb/n}(h_\mu\psi)(z)\Big| \\
&\le Ce^{-w(z)}\Big(\int_0^b |\phi(x)|x^{2\mu+1}e^{(l+1)w(x)}\,dx \\
&+ \frac{b}{n}\sum_{j=1}^n \Big|\phi\Big(\frac{jb}{n}\Big)\Big|\Big(\frac{jb}{n}\Big)^{2\mu+1}e^{(l+1)w(jb/n)}\Big) \\
&\le Ce^{-w(z)}, \quad z \in (0,\infty).
\end{aligned}$$

Hence, if $\varepsilon > 0$ then there exists $z_0 \in (0,\infty)$ such that

$$\begin{aligned}
\sup_{z\ge z_0} e^{lw(z)}\Big|\int_0^b \phi(x)x^{2\mu+1}\tau_x(h_\mu\psi)(z)\,dx \\
&- \frac{b}{n}\sum_{j=1}^n \phi\Big(\frac{jb}{n}\Big)\Big(\frac{jb}{n}\Big)^{2\mu+1}\tau_{jb/n}(h_\mu\psi)(z)\Big| < \varepsilon.
\end{aligned}$$

On the other hand, since the function H defined by

$$H(x,z) = \phi(x)x^{2\mu+1}\tau_x(h_\mu\psi)(z), \quad x,z \in [0,\infty),$$

is uniformly continuous in $(x,z) \in [0,b] \times [0,z_0]$, it has

$$\lim_{n\to\infty} \frac{b}{n} \sum_{j=1}^{n} \phi\Big(\frac{jb}{n}\Big)\Big(\frac{jb}{n}\Big)^{2\mu+1} \tau_z(h_\mu\psi)\Big(\frac{jb}{n}\Big) = \int_0^b \phi(x)x^{2\mu+1}\tau_z(h_\mu\psi)(x)\,dx,$$

uniformly in $[0,x_0]$.

From the above arguments we conclude (3.4) in the sense of convergence in $\mathcal{E}_\mu(w)$. Hence it has that

$$\int_0^b \langle T(y),(xy)^{-\mu}J_\mu(xy)\rangle\phi(x)x^{2\mu+1}\,dx = \Big\langle T(y), \int_0^b (xy)^{-\mu}J_\mu(xy)\phi(x)x^{2\mu+1}\,dx\Big\rangle.$$

Also,

$$\lim_{b\to\infty}\int_b^\infty (xy)^{-\mu}J_\mu(xy)\phi(x)x^{2\mu+1}\,dx = 0$$

in the sense of convergence in $\mathcal{E}_\mu(w)$.

Indeed, assume that $b>0$, $\psi \in \mathcal{B}_\mu(w)$ and $m \in \mathbf{N}$. For a certain $l \in \mathbf{N}$ we have that

$$\begin{aligned}\delta_m^\mu\Big(\psi(y)\int_b^\infty (xy)^{-\mu}J_\mu(xy)\phi(x)x^{2\mu+1}\,dx\Big) &\le C\alpha_{l,0}\Big(h_\mu\Big(\psi(y)\int_b^\infty (xy)^{-\mu}J_\mu(xy)\phi(x)x^{2\mu+1}\,dx\Big)\Big)\\ &\le C\sup_{z\in(0,\infty)} e^{lw(z)}\Big|\int_b^\infty \phi(x)\tau_z(h_\mu\psi)(x)x^{2\mu+1}\,dx\Big|\\ &\le C\int_b^\infty |\phi(x)|e^{lw(x)}x^{2\mu+1}\,dx\beta_{l,0}^\mu(\psi).\end{aligned}$$

Hence,

$$\lim_{b\to\infty} \delta_m^\mu\Big(\psi(y)\int_b^\infty (xy)^{-\mu}J_\mu(xy)\phi(x)x^{2\mu+1}\,dx\Big) = 0.$$

Standard arguments allow us now to show that (3.1) holds. □

Proposition 2.4 (i) allows us to define the Hankel convolution $T\#\phi$ of $T\in\mathcal{H}_\mu(w)'$ and $\phi\in\mathcal{H}_\mu(w)$ as follows

$$(T\#\phi)(x) = \langle T, \tau_x\phi\rangle, \quad x\in[0,\infty).$$

Note that the last definition extends the Hankel convolution from $\mathcal{H}_\mu(w)\times\mathcal{H}_\mu(w)$ to $\mathcal{H}_\mu(w)'\times H_\mu(w)$. Indeed, let $\phi,\psi\in\mathcal{H}_\mu(w)$. We can write

$$\begin{aligned}(T_\phi\#\psi)(x) = \langle T_\phi, \tau_x\psi\rangle &= \int_0^\infty \phi(y)(\tau_x\psi)(y)\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dy\\ &= (\phi\#\psi)(x), \quad x\in[0,\infty).\end{aligned}$$

We now prove that $T\#\phi\in\mathcal{H}_\mu(w)'$ for every $T\in\mathcal{H}_\mu(w)'$ and $\phi\in\mathcal{H}_\mu(w)$.

Proposition 3.2. *Let $T\in\mathcal{H}_\mu(w)'$ and $\phi\in\mathcal{H}_\mu(w)$. Then $T\#\phi$ is a continuous function on $[0,\infty)$. Moreover, there exist $C>0$ and $r\in\mathbf{N}$ such that*

$$|(T\#\phi)(x)| \le Ce^{rw(x)}, \quad x\in[0,\infty).$$

Hence, $T\#\phi$ defines an element of $\mathcal{H}_\mu(w)'$.

Proof. According to Proposition 2.4 (ii), $T\#\phi$ is a continuous function on $[0,\infty)$. Moreover, since $T\in\mathcal{H}_\mu(w)'$, from Proposition 2.3 it implies that there exist $C>0$ and $r\in\mathbf{N}$ such that

$$|\langle T,\psi\rangle| \le C\max_{0\le n\le r}\{A^\mu_{r,n}(\psi), \beta^\mu_{r,n}(\psi)\}, \quad \psi\in\mathcal{H}_\mu(w).$$

In particular, we have that

$$|(T\#\phi)(x)| \le C\max_{0\le n\le r}\{A^\mu_{r,n}(\tau_x\phi), \beta^\mu_{r,n}(\tau_x\phi)\}, \quad x\in[0,\infty).$$

From (2.9), it is deduced that

$$A^{\mu}_{r,n}(\tau_x \phi) \leq e^{rw(x)} A^{\mu}_{r,n}(\phi), \quad x \in [0,\infty) \text{ and } n \in \mathbf{N}.$$

Also (2.10) implies, since w satisfies the (γ)-property, that

$$\begin{aligned}\beta^{\mu}_{r,n}(\tau_x \phi) &\leq C(1+x^{2n}) \sum_{j=0}^{n} \beta^{\mu}_{r,j}(\phi) \\ &\leq C e^{lw(x)} \sum_{j=0}^{n} \beta^{\mu}_{r,j}(\phi), \quad x \in [0,\infty) \quad \text{and } n \in \mathbf{N},\end{aligned}$$

for some $l \in \mathbf{N}$.

Hence, for a certain $m \in \mathbf{N}$,

$$|(T\#\phi)(x)| \leq C e^{mw(x)}, \quad x \in [0,\infty). \qquad \square$$

We now introduce, for every $m \in \mathbf{N}$, the space $\mathcal{A}_m(w)$ constituted by all those functions f defined on $(0,\infty)$ such that

$$\sup_{x \in (0,\infty)} e^{-mw(x)} |f(x)| < \infty.$$

A careful reading of the proof of Proposition 3.2 allows us to deduce that if $T \in \mathcal{H}_\mu(w)'$, there exists $r \in \mathbf{N}$ such that $T\#\phi \in \mathcal{A}_r(w)$ for every $\phi \in \mathcal{H}_\mu(w)$.

Next we establish an associative property for the distributional convolution.

Proposition 3.3. *Let $T \in \mathcal{H}_\mu(w)'$ and $\phi, \psi \in \mathcal{H}_\mu(w)$. Then*

$$(T\#\phi)\#\psi = T\#(\phi\#\psi). \tag{3.5}$$

Proof. As it was shown in Proposition 3.2, $T\#\phi$ defines an element of $\mathcal{H}_\mu(w)'$ and we have

$$\begin{aligned}((T\#\phi)\#\psi)(x) &= \int_0^\infty (T\#\phi)(y)(\tau_x \psi)(y) \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\, dy \\ &= \int_0^\infty \langle T, \tau_y \phi \rangle (\tau_x \psi)(y) \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\, dy, \quad x \in (0,\infty).\end{aligned}$$

Equality (3.5) will be proved when we see that, for every $x \in (0, \infty)$,

$$(3.6)\qquad \int_0^\infty \langle T, \tau_y \phi\rangle (\tau_x \psi)(y) \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\, dy = \Big\langle T(z), \int_0^\infty (\tau_y \phi)(z)(\tau_x \psi)(y) \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\, dy \Big\rangle.$$

Indeed, we have

$$\int_0^\infty (\tau_y \phi)(z)(\tau_x \psi)(y) \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\, dy = \int_0^\infty (\tau_z \phi)(y)(\tau_x \psi)(y) \frac{y^{2\mu+1}}{2\mu \Gamma(\mu+1)}\, dy = (\tau_z \phi \# \psi)(x) = \tau_x(\phi \# \psi)(z), \quad x, z \in [0, \infty).$$

Our objective is to prove (3.6). We will use a procedure similar to the one employed in the proof of Proposition 3.1.

Let $x \in [0, \infty)$. By virtue of Proposition 3.2, it follows that

$$(3.7)\qquad \lim_{b\to\infty} \int_b^\infty \langle T, \tau_y \phi\rangle (\tau_x \psi)(y) \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\, dy = 0.$$

Assume that $m, n \in \mathbf{N}$. According to (2.9), we can write

$$A_{m,n}^\mu\Big(\int_b^\infty (\tau_z \phi)(y)(\tau_x \psi)(y) \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\, dy\Big) \le \int_b^\infty e^{mw(y)} |(\tau_x \psi)(y)| \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\, dy A_{m,n}^\mu(\phi), \quad b > 0.$$

Hence, from Proposition 2.4 (i) it is inferred that

$$\lim_{b\to\infty} A_{m,n}^\mu\Big(\int_b^\infty (\tau_z \phi)(y)(\tau_x \psi)(y) \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\, dy\Big) = 0.$$

On the other hand, for every $b > 0$,

$$\Big(\frac{1}{t} D\Big)^n h_\mu\Big(\int_b^\infty (\tau_z \phi)(y)(\tau_x \psi)(y) \frac{y^{2\mu+1}}{2\mu \Gamma(\mu+1)}\, dy\Big)(t) = \sum_{j=0}^n (-1)^j \binom{n}{j} \int_b^\infty (\tau_x \psi)(y) y^{2j} (yt)^{-\mu-j} J_{\mu+j}(yt) y^{2\mu+1}\, dy \cdot \Big(\frac{1}{t} D\Big)^{n-j} h_\mu(\phi)(t), \quad t \in (0, \infty).$$

Therefore, by Proposition 2.4 (i) and taking into account the boundedness of the function $z^{-\mu}J_\mu(z)$ on $(0,\infty)$, we have

$$\beta^\mu_{m,n}\Big(\int_b^\infty (\tau_z\phi)(y)(\tau_x\psi)(y)\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dy\Big)$$
$$\le C\sum_{j=0}^n \beta^\mu_{m,n-j}(\phi)\int_b^\infty |(\tau_x\psi)(y)|y^{2j+2\mu+1}\,dy \longrightarrow 0, \quad \text{as } b\to\infty.$$

Thus we see that

$$\int_b^\infty (\tau_z\phi)(y)(\tau_x\psi)(y)\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dy \longrightarrow 0, \quad \text{as } b\to\infty, \tag{3.8}$$

in the sense of convergence in $\mathcal{H}_\mu(w)$.

Now let $b>0$. By using, as in the proof of Proposition 3.1, Riemann sums, we can prove that

$$\int_0^b \langle T,\tau_y\phi\rangle(\tau_x\psi)(y)y^{2\mu+1}\,dy = \Big\langle T(z),\int_0^b (\tau_y\phi)(z)(\tau_x\psi)(y)y^{2\mu+1}\,dy\Big\rangle. \tag{3.9}$$

By combining (3.7), (3.8) and (3.9) we deduce (3.6) and thus the proof of (3.5) is complete. □

A useful special case of Proposition 3.3 follows.

Corollary 3.4. *Let $T\in\mathcal{H}_\mu(w)'$ and $\phi,\psi\in\mathcal{H}_\mu(w)$. Then*

$$\langle T\#\phi,\psi\rangle = \langle T,\phi\#\psi\rangle. \tag{3.10}$$

Proof. To see (3.10), it is sufficient to take $x=0$ in (3.5). □

Remark 3. Note that the property in Corollary 3.4 is equivalent to the one in Proposition 3.3. Indeed, let $T\in\mathcal{H}_\mu(w)'$ and $\phi,\psi\in\mathcal{H}_\mu(w)$.

If $x \in [0,\infty)$, $\tau_x\psi \in \mathcal{H}_\mu(w)$ (Proposition 2.4 (i)). Then from Corollary 3.4 we deduce

$$\begin{aligned}((T\#\phi)\#\psi)(x) &= \langle T, \phi\#(\tau_x\psi)\rangle \\ &= \langle T, \tau_x(\phi\#\psi)\rangle \\ &= (T\#(\phi\#\psi))(x), \quad x \in [0,\infty).\end{aligned}$$

Thus Proposition 3.3 is established.

We now obtain a distributional version of the interchange formula.

Proposition 3.5. *Let $T \in \mathcal{H}_\mu(w)'$ and $\phi \in \mathcal{H}_\mu(w)$. Then*

$$h'_\mu(T\#\phi) = h'_\mu(T)h_\mu(\phi).$$

Proof. Assume that $\psi \in \mathcal{H}_\mu(w)$. According to Corollary 3.4, we can write

$$\begin{aligned}\langle h'_\mu(T\#\phi), \psi\rangle &= \langle T\#\phi, h_\mu(\psi)\rangle = \langle T, \phi\#h_\mu(\psi)\rangle \\ &= \langle T, h_\mu(h_\mu(\phi)\psi)\rangle = \langle h'_\mu(T)h_\mu(\phi), \psi\rangle. \qquad \square\end{aligned}$$

Another consequence of Corollary 3.4 is the following.

Proposition 3.6. *The space $\mathcal{A}(w) = \cup_{m\in\mathbf{N}}\mathcal{A}_m(w)$ is a weak $*$ dense subspace of $\mathcal{H}_\mu(w)'$.*

Proof. To see this property it is sufficient to take into account the remark after Proposition 3.2 and to use Proposition 2.7 and Corollary 3.4.

We now introduce the space $\mathcal{F}_\mu(w)$ that consists of all those $T \in \mathcal{B}_\mu(w)'$ for which there exists a function G_T belonging to $\mathcal{A}_m(w)$ for some $m \in \mathbf{N}$, such that

$$\langle T, \phi\rangle = \int_0^\infty G_T(y)h_\mu(\phi)(y)\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dy, \quad \phi \in \mathcal{B}_\mu(w). \tag{3.11}$$

Note that the righthand side of (3.11) defines a continuous functional on $\mathcal{H}_\mu(w)$. Hence, T can be extended to $\mathcal{H}_\mu(w)$ as an element of $\mathcal{H}_\mu(w)'$. We continue denoting by T that extension to $\mathcal{H}_\mu(w)$. Moreover, for every $\phi \in \mathcal{H}_\mu(w)$, it has

$$\begin{aligned}\langle h'_\mu T, \phi\rangle &= \langle T, h_\mu(\phi)\rangle \\ &= \int_0^\infty G_T(y) h_\mu(h_\mu(\phi))(y) \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\, dy \\ &= \int_0^\infty G_T(y)\phi(y) \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\, dy.\end{aligned}$$

Hence, $h'_\mu T$ coincides with the functional generated by G_T on $\mathcal{H}_\mu(w)'$.

We also can prove that if $T \in \mathcal{F}_\mu(w)$ and $\phi \in \mathcal{H}_\mu(w)$, then $T\#\phi$ and $T.\phi$ are in $\mathcal{F}_\mu(w)$.

Remark 4. In a forthcoming paper we will continue the study of the tempered Beurling-type distributions and the Hankel transformation following the ideas of von Grudzinski [**11**].

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