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WHEN THE FAMILY OF FUNCTIONS VANISHING AT INFINITY IS AN IDEAL OF $C(X)$

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ABSTRACT. We prove that $C_\infty(X)$ is an ideal in $C(X)$ if and only if every open locally compact subset of X is bounded. In particular, if X is a locally compact Hausdorff space, $C_\infty(X)$ is an ideal of $C(X)$ if and only if X is a pseudocompact space. It is shown that the existence of some special functions in $C_\infty(X)$ causes $C_\infty(X)$ not to be an ideal of $C(X)$. Finally we will characterize the spaces X for which $C_\infty(X)$ and $C_K(X)$, or $C_\psi(X)$, coincide.

Introduction. Throughout this paper X stands for a completely regular Hausdorff space and $C(X)(C^*(X))$ for the ring of all (bounded) continuous real valued functions on X. In [**1**], Azarpanah considered essential ideals in $C(X)$ and characterized those X for which the ideal $C_K(X)$ of all functions in $C(X)$ with compact support is an essential ideal in $C(X)$. He considered also the subset $C_\infty(X)$ of all those functions in $C(X)$ which vanish at infinity. It gives an impression there that $C_\infty(X)$ might always be an ideal of $C(X)$. This, however, is not always true, e.g., $X = \mathbf{R}$.

We prove that $C_\infty(X)$ will be an ideal of $C(X)$ if and only if every open locally compact subset of X is bounded. In particular, for a locally compact Hausdorff space X, $C_\infty(X)$ is an ideal in $C(X)$ if and only if X is a pseudocompact space. We note that $Y \subseteq X$ is said to be bounded if for every $f \in C(X)$, $f(Y)$ is a bounded set in $\mathbf{R}$. We will show that the existence of a function $f \in C_\infty(X) \setminus C_K(X)$ whose zero-set $Z(f)$ is an open set, causes $C_\infty(X)$ not to be an ideal of $C(X)$. We also observe that the existence of a function h in $C_\infty(X)$ with $Z(h)$ a Lindelöf and bounded set causes $C_\infty(X)$ not to be an ideal of $C(X)$, unless X is a compact space.

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Topological spaces X for which $C_K(X)$, or $C_\infty(X)$, and the socle $C_F(X)$ of $C(X)$ coincide are characterized in [**1**]. In [**6**] it is shown that $C_F(X) = \{f \in C(X) : X \setminus Z(f) \text{ is finite}\}$. It is also well known [**4**, 7G.2], that if X is a locally compact noncompact Hausdorff space, then $C_\infty(X) = C_K(X)$ if and only if every σ-compact subset of X is contained in a compact subset of X. We will generalize this result for completely regular Hausdorff spaces. $C_K(X) = \{f \in C(X) : \operatorname{cl}(X \setminus Z(f)) \text{ is compact}\}$ is easily seen to be an ideal of $C(X)$, but $C_\infty(X) = \{f \in C(X) : \{x \in X : |f(x)| \geq (1/n)\} \text{ is compact, for all } n \in \mathbf{N}\}$ is a subring of $C(x)$, [**4**, 7G.2], and not always an ideal of $C(X)$. For example $C_\infty(\mathbf{R})$ is not an ideal of $C(\mathbf{R})$. To see this, we consider the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 1/(1+x^2)$ which is in $C_\infty(\mathbf{R})$. Now the function g defined by $g(x) = 1 + x^2$ is in $C(\mathbf{R})$ but $fg \notin C_\infty(\mathbf{R})$. $C_\infty(X)$ may sometimes be an ideal of $C(X)$, for example, $C_\infty(\mathbf{Q}) = (0)$ is an ideal in $C(\mathbf{Q})$, [**4**, 7F]. Whenever X is a locally compact Hausdorff space and every σ-compact subset of X is contained in a compact subset of X, then $C_\infty(X) = C_K(X)$ and hence is an ideal of $C(X)$. We also see in [**1**, Theorem 4.5] that $C_\infty(X) = C_F(X)$ if and only if X is a pseudo-discrete space (every compact subset has finite interior), with only a finite number of isolated points. Since $C_F(X)$ is an ideal of $C(X)$, then in this case $C_\infty(X)$ is an ideal of $C(X)$.

We note that X is a locally compact σ-compact space if and only if $X = \cup_{n=1}^{\infty} A_n$ such that A_n is compact and $A_n \subseteq \operatorname{int} A_{n+1}$ for all $n \in \mathbf{N}$ [**3**, p. 250]. The reader is referred to [**4**] for undefined terms and notations.

1. When is $C_\infty(X)$ an ideal in $C(X)$? To prove the main result of this section, we need the following lemma.

Lemma 1.1. *Let A be an open subset of X. Then $A = X \setminus Z(f)$ for some $f \in C_\infty(X)$ if and only if A is a σ-compact locally compact subset of X.*

Proof. Let $A = X \setminus Z(f)$ for some $f \in C_\infty(X)$. Then $A = \cup_{n=1}^{\infty} A_n$ where $A_n = \{x \in X : |f(x)| \geq (1/n)\}$. A_n is compact and hence A is σ-compact. If $x \in A$, there exists $n_0 \in \mathbf{N}$ such that $x \in \{y \in X :$

$|f(y)| > (1/n_0)\} \subseteq A_{n_0}$. Thus we get A is a locally compact subset of X. This proves the necessity. For sufficiency, let A be a σ-compact locally compact subset of X. Then $A = \cup_{n=1}^{\infty} A_n$ with A_n compact and $A_n \subseteq \operatorname{int} A_{n+1}$. Now for each $n \in \mathbf{N}$, there exists $f_n \in C(X)$ such that $f_n(X) \subseteq [0,1]$, $f_n(A_n) = \{1\}$, $f_n(X \setminus \operatorname{int} A_{n+1}) = \{0\}$. Then $f = \sum_{n=1}^{\infty} f_n/2^n$ is an element of $C(X)$ by the Weierstrass M-test. Clearly, $A = X \setminus Z(f)$. We claim that $f \in C_\infty(X)$. Let $x_0 \notin A_{n+1}$. Then $f_1(x_0) = \cdots = f_n(x_0) = 0$ and so $f(x_0) \leq (1/2^{n+1}) + \cdots \leq (1/2^n) < (1/n)$. So $x_0 \notin \{x \in X : |f(x)| \geq (1/n)\}$, and hence $\{x \in X : |f(x)| \geq (1/n)\} \subseteq A_{n+1}$ and so we get $f \in C_\infty(X)$. $\square$

The lemma above gives a new representation for $Z[C_\infty(X)]$. If $\mathbf{F}$ is the collection of all closed subsets of X, then

$$Z[C_\infty(X)] = \{H \in \mathbf{F} : X \setminus H \text{ is a locally compact } \sigma\text{-compact}\}.$$

Corollary 1.2. *$C_\infty(X)$ contains a unit of $C(X)$ if and only if X is a locally compact σ-compact space.*

Next we prove the main result of this section.

Theorem 1.3. *Let X be a completely regular Hausdorff space. The following conditions are equivalent*:

(a) *$C_\infty(X)$ is an ideal in $C(X)$.*

(b) *Every open locally compact subset of X is bounded.*

(c) *Every open locally compact σ-compact subset of X is bounded.*

Proof. (a) $\Rightarrow$ (b). Let Y be an open locally compact subset of X. Suppose that Y is not bounded. Then $g \in C(X)$, $g \geq 0$ and points $a_n \in Y$ exist such that $g(a_n) \geq 2^n$ for all $n \in \mathbf{N}$. We can also assume that $g(a_{n+1}) > g(a_n) + 1$, for every $n \in \mathbf{N}$. Since Y is locally compact and open in X, for each $n \in \mathbf{N}$, there exists an open set A_n in X such that $a_n \in A_n$, $\operatorname{cl} A_n$ is compact and $\operatorname{cl} A_n \subseteq Y$. Put $U_n = g^{-1}\{(g(a_n) - (1/4), g(a_n) + (1/4))\} \cap A_n$. Since $\operatorname{cl} U_n \subseteq \operatorname{cl} A_n$, therefore $\operatorname{cl} U_n$ is compact. If $m \neq n$, $\operatorname{cl} U_m \cap \operatorname{cl} U_n = \varnothing$. For every $n \in \mathbf{N}$, choose an open set V_n in X such that $a_n \in V_n \subseteq \operatorname{cl} V_n \subseteq$

U_n. Now for every $n \in \mathbf{N}$ we define $f_n \in C(X)$, $0 \leq f_n \leq 1$, $f_n(\operatorname{cl} V_n) = \{1\}$ and $f_n(X \setminus U_n) = \{0\}$. Let $f = \sum_{n=1}^{\infty} f_n/2^n$, then by the Weierstrass M-test, $f \in C(X)$. To show that $f \in C_\infty(X)$, let $n_0 \in \mathbf{N}$ and $K = \operatorname{cl} U_1 \cup \cdots \cup \operatorname{cl} U_{n_0}$. Clearly K is compact and if $x \in X \setminus K$, then $f(x) = \sum_{n>n_0} f_n(x)/2^n < (1/n_0)$. Thus $\{x \in X : |f(x)| \geq (1/n_0)\} \subseteq K$ and so is compact. Hence $f \in C_\infty(X)$. Now we claim that $fg \notin C_\infty(X)$. Let $C = \{x \in X : (fg)(x) \geq 1\}$, since $(fg)(a_n) = f(a_n)g(a_n) \geq 1$, for all $n \in \mathbf{N}$, then $\{a_n\} \subseteq C$. But $g \in C(X)$ is not bounded on $\{a_n\}$ implies that C cannot be compact, i.e., $fg \notin C_\infty(X)$. Thus $C_\infty(X)$ is not an ideal in $C(X)$ which is a contradiction. Hence Y is bounded and then (b) follows.

(b) $\Rightarrow$ (c). Easy.

(c) $\Rightarrow$ (a). Since $C_\infty(X)$ is a subring of $C(X)$, it is enough to prove that $fg \in C_\infty(X)$ for every $f \in C(X)$ and any $g \in C_\infty(X)$. By Lemma 1.1, $X \setminus Z(g) = Y$ is an open locally compact σ-compact subset of X and hence, by (c), $f(Y)$ is a bounded subset of $\mathbf{R}$. Now it is easy to see that $g^{1/3} \in C_\infty(X)$, since $g \in C_\infty(X)$. Moreover, $Z(g^{1/3}) = Z(g)$ implies that $(fg^{1/3})(X) = (fg^{1/3})(Y) \cup \{0\}$. Since $f(Y)$ is a bounded set in $\mathbf{R}$ and $g^{1/3} \in C_\infty(X)$ is a bounded function on X, we get $(fg^{1/3})(Y)$ is a bounded set in $\mathbf{R}$ implies that $fg^{1/3}$ is bounded on X and so belongs to $C^*(X)$. Since $C_\infty(X)$ is a ring, $g^{2/3} \in C_\infty(X)$. However, if $h \in C^*(X)$ and $k \in C_\infty(X)$, it is easy to check that $hk \in C_\infty(X)$. Therefore $fg = (fg^{1/3})g^{2/3} \in C_\infty(X)$, thus (a) holds. □

Corollary 1.4. *Let X be a locally compact Hausdorff space. Then $C_\infty(X)$ is an ideal in $C(X)$ if and only if X is a pseudocompact space.*

Proof. Suppose X is a pseudocompact space. If $g \in C_\infty(X)$ and $f \in C(X)$, then f is a bounded function and so $fg \in C_\infty(X)$ easily. Thus it follows that $C_\infty(X)$ is an ideal of $C(X)$. Conversely, if $C_\infty(X)$ is an ideal of $C(X)$, X itself is an open locally compact subset and so by Theorem 1.3, X is bounded, i.e., X is a pseudocompact space. □

We note by [**4**, Theorem 8.2] that a Lindelöf space is realcompact and by [**4**, 5H.2], a realcompact pseudocompact space is a compact space. Hence, if X is a Lindelöf pseudocompact space, then X is a compact space.

Corollary 1.5. *Suppose that there exists $h \in C_\infty(X)$ with $Z(h)$ Lindelöf and bounded. If $C_\infty(X)$ is an ideal in $C(X)$, then X is a compact space.*

Proof. By Lemma 1.1, $X \setminus Z(h)$ is an open locally compact σ-compact subset of X. Hence by Theorem 1.3, $X \setminus Z(h)$ is bounded. Now if $f \in C(X)$, then $f|_{X \setminus Z(h)}$ is bounded and also $f|_{Z(h)}$ is bounded. Thus f is a bounded function and hence we get X a pseudocompact space. $X = (X \setminus Z(h)) \cup Z(h)$ yields X is a Lindelöf space. Since X is now both Lindelöf and pseudocompact, we get that X is a compact space. $\square$

Remark 1.6. A compact set is both Lindelöf and bounded. In the Tychonoff plank T, the right edge is both Lindelöf and bounded (since T itself is pseudocompact) but is not compact.

To prove the last result of this section, we need the following:

Lemma 1.7. *Suppose $X = Y \oplus Z$, i.e., Y and Z are disjoint open subsets of X such that $X = Y \cup Z$. $C_\infty(X)$ is an ideal of $C(X)$ if and only if $C_\infty(Y)$ is an ideal of $C(Y)$ and $C_\infty(Z)$ is an ideal of $C(Z)$.*

Proposition 1.8. *Suppose there exists $f \in C_\infty(X) \setminus C_K(X)$ with $Z(f)$ an open set. Then $C_\infty(X)$ is not an ideal of $C(X)$.*

Proof. Since $Z(f)$ is open and already it is closed, $X = Y \oplus Z(f)$, where $Y = X \setminus Z(f)$. Suppose $C_\infty(X)$ is an ideal of $C(X)$. Then, by Lemma 1.7, $C_\infty(Y)$ is an ideal of $C(Y)$. Now $f|_Y \in C_\infty(Y)$ and f does not vanish on Y. Hence $1/f$ is defined on Y and belong to $C(Y)$. Now $1_Y = (f|_Y)1/f \in C_\infty(Y)$ implies that $Y = \{y \in Y : 1_Y(y) \geq (1/2)\}$ is compact. This yields that $f \in C_K(X)$ since $Y = X \setminus Z(f)$ and Y is closed so that Y is support of f. This is a contradiction and hence $C_\infty(X)$ is not an ideal of $C(X)$. $\square$

2. $C_\infty(X)$ **and related ideals in** $C(X)$**.** Topological spaces X for which $C_K(X) = C_F(X)$ and $C_\infty(X) = C_F(X)$ are characterized in [**1**]. Locally compact Hausdorff spaces X for which $C_\infty(X) = C_K(X)$ are also characterized in [**4**, 7G.2]. Another related ideal which is denoted by $C_\psi(X)$ in [**5**] is the set of all functions in $C(X)$ with pseudocompact support. Mandelker in [**7**, Theorem 2.1] has shown that in any topological space X, any bounded support is pseudocompact and this fact implies that $C_\psi(X)$ is an ideal in $C(X)$, see [**7**, Corollary 2]. In the case of a completely regular Hausdorff space, it is well known that if the closure of any open set is bounded, then it is pseudocompact, see [**2**, Theorem 4.1]. It is easy to see that $C_K(X) \subseteq C_\psi(X)$ and whenever $C_K(X) = C_\psi(X)$, then the space X is called ψ-compact, see [**5**] and [**7**] for more details.

In this section we will prove that for a completely regular Hausdorff space X, $C_\infty(X) = C_K(X)$ if and only if every open locally compact and σ-compact subset of X is contained in a compact subset of X. We also show that $C_\infty(X) \subseteq C_\psi(X)$ if and only if $C_\infty(X)$ is an ideal of $C(X)$ and, for a locally compact Hausdorff space X, $C_\infty(X) = C_\psi(X)$ implies that X is compact.

Proposition 2.1. *Let X be a completely regular Hausdorff space. $C_\infty(X) = C_K(X)$ if and only if every open locally compact σ-compact subset of X is contained in a compact set in X.*

Proof. Suppose the condition holds. It is enough to prove that $C_\infty(X) \subseteq C_K(X)$. Let $f \in C_\infty(X)$. Then by Lemma 1.1, $X \setminus Z(f)$ is an open locally compact and σ-compact subset of X. Hence, $X \setminus Z(f)$ is contained in a compact set C. Thus $\mathrm{Supp}\,(f) = \mathrm{cl}\,(X \setminus Z(f)) \subseteq C$ and hence $\mathrm{Supp}\,(f)$ is compact, i.e., $f \in C_K(X)$. Conversely, suppose $C_\infty(X) = C_K(X)$. Let A be an open locally compact and σ-compact subset of X. By Lemma 1.1, there exists $f \in C_\infty(X)$ such that $A = X \setminus Z(f)$. Now $f \in C_K(X)$ implies that $A = X \setminus Z(f) \subseteq \mathrm{cl}\,(X \setminus Z(f)) = \mathrm{Supp}\,(f)$. Since $\mathrm{Supp}\,(f)$ is compact, then A is contained in a compact set and hence the proposition holds. □

Remark 2.2. If X is a locally compact Hausdorff space, any σ-compact set is contained in an open locally compact σ-compact set and hence

Proposition 2.1 yields the characterization mentioned in [**4**, 7G.2].

Remark 2.3. If X is a space such that $C_\infty(X) = C_K(X)$, since $C_K(X)$ is always an ideal in $C(X)$, we get $C_\infty(X)$ is an ideal of $C(X)$ and hence X satisfies the conditions of Theorem 1.3. For another example of a space with conditions of Theorem 1.3, let $p \in \beta\mathbf{N} \setminus \mathbf{N}$, and let $X = \beta\mathbf{N} \setminus \{p\}$. Then X is a locally compact countably compact space and hence is pseudocompact also. Thus, by Corollary 1.4, $C_\infty(X)$ is an ideal of $C(X)$. Now $\mathbf{N}$ is an open locally compact and σ-compact subset of X and, since $\mathbf{N}$ is dense in $\beta\mathbf{N}$, it is not contained in any compact subset of X. Hence $C_\infty(X) \neq C_K(X)$.

By Theorem 1.3, the following proposition implies that $C_\infty(X) \subseteq C_\psi(X)$ if and only if $C_\infty(X)$ is an ideal of $C(X)$.

Proposition 2.4. *$C_\infty(X) \subseteq C_\psi(X)$ if and only if every open locally compact subset of X is bounded.*

Proof. Let $C_\infty(X) \subseteq C_\psi(X)$. By Theorem 1.3, it is enough to show that every open locally compact σ-compact subset A of X is bounded. By Lemma 1.1, $A = X \setminus Z(f)$ for some $f \in C_\infty(X)$. Now, by our hypothesis, $f \in C_\psi(X)$, i.e., $\mathrm{cl}\,(X \setminus Z(f))$ is pseudocompact and hence $A = X \setminus Z(f)$ is bounded. Conversely, let every open locally compact subset of X be bounded and $f \in C_\infty(X)$. Then $X \setminus Z(f)$ is an open locally compact set by Lemma 1.1, hence $X \setminus Z(f)$ is bounded. Therefore, $\mathrm{cl}\,(X \setminus Z(f))$ is bounded and by Theorem 4.1 in [**2**], $\mathrm{cl}\,(X \setminus Z(f))$ is pseudocompact, i.e., $f \in C_\psi(X)$.

We conclude the article with the following proposition which is clear by our Corollary 1.4 and Corollary 2 in [**7**].

Corollary 2.5. *Let X be a locally compact Hausdorff space. Then $C_\infty(X) = C_\psi(X)$ if and only if X is compact.*

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