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IRREDUCIBLE CONTINUA OF TYPE λ WITH ALMOST UNIQUE HYPERSPACE

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ABSTRACT. For an irreducible continuum X of type λ we study the family of all continua Y for which hyperspaces of subcontinua $C(X)$ and $C(Y)$ are homeomorphic. The family is determined if each layer of X is a layer of cohesion and the set of degenerate layers is dense in X.

1. Introduction. Given a (metric) continuum X, denote by $C(X)$ the hyperspace of subcontinua of X (i.e., the family of all subcontinua of X) metrized by the Hausdorff metric. A class Λ of continua is said to be *C-determined* (see [**16**, p. 33]), provided that for every $X, Y \in \Lambda$ if the hyperspaces $C(X)$ and $C(Y)$ are homeomorphic, then so are the continua X and Y. For various results on this subject, see e.g., [**16**, pp. 32–33], [**10**, pp. 437–438], [**8**], [**9**], [**14**] and [**15**]. The following concept is closely related to the above.

For a given continuum X, consider a family $\Im(X)$ of continua Y such that:

(1.1) no two distinct members of $\Im(X)$ are homeomorphic,

(1.2) $C(Y)$ is homeomorphic to $C(X)$ for each member Y of $\Im(X)$,

(1.3) $\Im(X)$ is the maximal family satisfying conditions (1.1) and (1.2), i.e., if Z is a continuum such that $C(Z)$ is homeomorphic to $C(X)$, then Z is homeomorphic to Y for some $Y \in \Im(X)$.

A continuum X is said to have *unique hyperspace* provided that the family $\Im(X)$ consists of one element only, viz. of X, [**1**, Definition 1]; *almost unique hyperspace* provided that the family $\Im(X)$ is finite and consists of more than one element, [**2**, Definition 1.1].

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Information about continua having unique or almost unique hyperspace can be found in the first-named author's doctoral dissertation, see [**1**], [**2**], [**3**] and [**4**].

The main purpose of this paper is to study the structure of members of the family $\Im(X)$ for a given irreducible continuum X of type λ. The principal result says that if X is a continuum of type λ, each layer of which is a layer of cohesion and the set of degenerate layers is dense in X, then:

(a) if both end layers of X are nondegenerate, then X has unique hyperspace;

(b) if exactly one end layer is degenerate, then X has either unique hyperspace or almost unique hyperspace;

(c) if both end layers of X are degenerate, then a complete description of the family $\Im(X)$ is presented.

As a consequence we generalize the results of [**8**], and we answer Question 5 of [**8**, p. 215] showing that there are arc-like continua X for which the family $\Im(X)$ consists of more than four elements which are arc-like, too. In fact, we find an arc-like continuum X for which there are infinitely many arc-like elements in the family $\Im(X)$. Recall that if X is an arc, then by [**1**, Lemma 11], the only element of $\Im(X) \setminus \{X\}$ is the circle; and in [**8**, Example 3] and [**5**, Example 5] arc-like continua X are constructed such that $\Im(X)$ has at least four elements.

The paper consists of five sections. After the introduction, preliminaries and auxiliary results are collected in Section 2. Section 3 is devoted to arc-like continua. It contains the results discussed above. Section 4, which concerns irreducible continua X of type λ, comprises the above-mentioned principal theorem that describes the family $\Im(X)$ depending on the end layers of the continuum, as summarized in cases (a), (b) and (c). Finally, closing remarks and questions are gathered in Section 5.

2. Preliminaries and auxiliary results. All spaces considered in the paper are assumed to be metric. Given a space X, we denote by $B_X(p, \varepsilon)$ the (open) ball in X centered at a point $p \in X$ and having the radius ε. For a subset $A \subset X$, we use the symbols $\operatorname{card} A$, $\operatorname{cl}_X(A)$, $\operatorname{int}_X(A)$ and $\operatorname{bd}_X(A)$ to denote the cardinality, closure, interior and boundary of A in X, respectively.

The symbols $\mathbf{N}$ and $\mathbf{R}$ stand for the sets of all positive integers and of all reals, correspondingly. The closed unit interval $[0,1]$ of reals is denoted by $\mathbf{I}$. If an arc is denoted by ab, we assume that a and b are the end points of the arc. By a *ray* we mean a homeomorphic copy of the real half-line $[0,\infty)$. The image of 0 is named the *end point* of the ray. To simplify notation we often omit the homeomorphism and refer to a ray as to the half-line itself.

A *continuum* means a nonempty compact connected space. The *hyperspace of subcontinua* $C(X)$ of a given space X is the family of all subcontinua of X metrized by the Hausdorff metric H (see [**10**, pp. 6, 11]). For $A \in C(X)$, we put $C(A,X) = \{B \in C(X) : A \subset B\}$. The hyperspace of singletons of X is denoted by $F_1(X)$. If two spaces, A and B, are homeomorphic, we write $A \approx B$. Thus, $X \approx F_1(X)$.

A nondegenerate proper subcontinuum A of a continuum X is said to be *terminal in* X provided that, for each subcontinuum B of X such that $B \cap A \neq \varnothing$, either $A \subset B$ or $B \subset A$. Thus A is terminal in X if and only if for each subcontinuum B of X, the condition $B \cap A \neq \varnothing \neq B \setminus A$ implies $A \subset B$, see [**16**, p. 361].

For $n \in \mathbf{N}$, an *n-od* means a continuum X for which there exists $A \in C(X)$ such that $X \setminus A$ has at least n components. A 3-od is also called a *triod*. An *n-cell* means a space homeomorphic to $\mathbf{I}^n$. If D is a 2-cell in a space X and $h : \mathbf{I}^2 \to D$ is a homeomorphism, then the *relative interior* of D in X is defined as the set

$$\mathsf{int}_X(D) = D \setminus h(\mathrm{bd}_{\mathbf{R}^2}(\mathbf{I}^2)).$$

The reader is referred to [**10**] and [**17**] for terms not defined here.

A continuum X is said to be *irreducible* (between points p and q, which are called *points of irreducibility* of X) provided that $p, q \in X$ and no proper subcontinuum of X contains both p and q. A continuum X which is irreducible between p and q is said to be *of type* λ (see [**13**, p. 197]), provided that there is a monotone mapping $g : X \to \mathbf{I}$ such that

$$g(p) = 0, \qquad g(q) = 1 \quad \text{and} \quad \mathrm{int}_X(g^{-1}(t)) = \varnothing \text{ for each } t \in \mathbf{I}.$$

The mapping g is said to be *canonical*, and the sets $T_t = g^{-1}(t)$ (where $t \in \mathbf{I}$) are called *layers* (or *tranches*) of X, see [**13**, p. 199]. In particular, $g^{-1}(0)$ and $g^{-1}(1)$ are called *end layers* of X.

For an irreducible continuum X of type λ, put $L_0 = T_0$, $R_1 = T_1$ and

$$L_t = \cup\{T_s : s \in [0,t)\} = g^{-1}([0,t)) \quad \text{for each } t \in (0,1],$$
$$R_t = \cup\{T_s : s \in (t,1]\} = g^{-1}((t,1]) \quad \text{for each } t \in [0,1).$$

A layer T_t of an irreducible continuum X of type λ is said to be a *layer of cohesion* provided that $T_t \subset (\mathrm{cl}_X(L_t)) \cap (\mathrm{cl}_X(R_t))$ (see [**12**, p. 260]). Note that the above inclusion can be replaced by the equality (see [**12**, p. 260]; cf. also [**13**, p. 201]). It is known that the family of all layers of X which are not layers of cohesion is at most countable, [**12**, p. 261] and [**13**, p. 201].

Let p and q be points of irreducibility of an irreducible continuum X of type λ. Assume that each layer T_t of X is a layer of cohesion. We then say that the continuum X is:

— *of type* $\lambda\diamond$ provided that $T_0 = \{p\}$ and $T_1 = \{q\}$;

— *of type* $\lambda\triangleleft$ provided that $T_0 = \{p\}$ and T_1 is nondegenerate;

— *of type* $\lambda\square$ provided that T_0 and T_1 are nondegenerate.

The concept of a continuum of type $\lambda\diamond$ has been introduced in [**8**, p. 212]; irreducible continua with degenerate (end) layers, and properties of their layers of cohesion were studied, e.g., in [**6**] and [**18**].

It will be shown in the next two sections of the paper that if an arc-like continuum X is of type $\lambda\diamond$, $\lambda\triangleleft$, or $\lambda\square$, then the family $\Im(X)$ can contain infinitely many elements, or it has at most two elements, or consists of exactly one member, respectively. To prove this, some auxiliary results (shown in [**1**], [**2**], [**4**] and [**8**]) are needed. For the reader's convenience, they are collected below.

It is shown in [**19**, p. 177] that if a continuum X contains an n-od, then the hyperspace $C(X)$ contains an n-cell. The opposite implication (for $n \geq 2$) is proved in [**7**, p. 64]. Thus, the following equivalence is true.

Theorem 2.1 (Illanes, Rogers). *Let X be a continuum, and let $n \geq 2$. The hyperspace $C(X)$ contains an n-cell if and only if X contains an n-od.*

Theorem 2.2 [**1**, Theorem 3]. *Let X be a continuum such that the hyperspace $C(X)$ contains a 2-cell $\mathcal{D}$, and let a point $p \in X$ satisfy*

$\{p\} \in \mathsf{int}_{C(X)}(\mathcal{D})$. *Then, for each* $\varepsilon > 0$, *there is a triod* T *in* X *such that* $H(T, \{p\}) < \varepsilon$.

Theorem 2.3 [**1**, Lemma 2]. *Let* X *be a continuum and* $E \in C(X)$. *Then each arc component of* $C(X) \setminus \{E\}$ *contains a singleton of* X.

Theorem 2.4 [**1**, Lemma 4], compare [**16**, p. 361]. *Let a continuum* E *be terminal in a continuum* X, *let* $F \in C(E) \setminus \{E\}$ *and* $G \in C(X)$ *be such that* $G \setminus E \neq \varnothing$. *Then each mapping* $\gamma : \mathbf{I} \to C(X)$ *with* $\gamma(0) = F$ *and* $\gamma(1) = G$ *satisfies* $E \in \gamma(\mathbf{I})$.

Theorem 2.5 [**8**, p. 212]. *Let a continuum* X *irreducible between points* p *and* q *be of type* $\lambda\diamond$, *let* A *be an arc with end points* a *and* b, *and* Y *be the one-point union of* A *and* X *with the points* b *and* p *identified. Then* $C(X) \approx C(Y)$.

Theorem 2.6 [**2**, Theorem 3.1]. *Let a continuum* X *be such that*:

(2.6.1) X *is irreducible between points* p *and* q;

(2.6.2) $C(\{p\}, X)$ *and* $C(\{q\}, X)$ *are arcs in* $C(X)$;

(2.6.3) *whenever a continuum* $X' = P \cup X \cup Q$ *is obtained by joining to* X *two disjoint arcs* P *and* Q *such that* $P \cap X = \{p\}$ *and* $Q \cap X = \{q\}$, *where* p *and* q *are end points of* P *and* Q, *respectively, it follows that* $X \approx X'$.

Let M *be an arc with end points* p *and* q *such that* $X \cap M = \{p, q\}$. *If* $Y = X \cup M$, *then* $C(X) \approx C(Y)$.

Theorem 2.7 [**1**, Lemma 6]. *Let a subcontinuum* R *be terminal in a continuum* X. *Let a continuum* Y *be such that* $C(X) \approx C(Y)$, *and let* $h : C(Y) \to C(X)$ *be a homeomorphism. Then* $R \notin h(F_1(Y))$.

Theorem 2.8 [**1**, Lemma 16]. *Let continua* X *and* Y *have homeomorphic hyperspaces* $C(X)$ *and* $C(Y)$ *with a homeomorphism* $h : C(Y) \to C(X)$, *and let* $y \in Y$. *Then there is no* 2*-cell* $\mathcal{D}$ *in* $C(X)$ *such that* $h(\{y\}) \in \mathsf{int}_{C(X)}(\mathcal{D}) \cap \mathrm{int}_{C(X)}(\mathcal{D})$.

Theorem 2.9 [**2**, Theorem 4.2]. *Let X be a continuum and $A, B \in C(X)$ be such that $A \subsetneq B$. If*

(2.9.1) $A = A_1 \cup A_2$ *for some* $A_1, A_2 \in C(A) \setminus \{A\}$,

(2.9.2) $B = B_1 \cup B_2$ *for some* $B_1, B_2 \in C(B) \setminus \{B\}$,

(2.9.3) $A_1 \subsetneq B_1$, $A_2 \subsetneq B_2$ *and* $A_1 \cap A_2 = B_1 \cap B_2 \in C(X)$,

then there is a 2*-cell* $\mathcal{D}$ *in* $C(X)$ *such that* $A \in \mathsf{int}_{C(X)}(\mathcal{D})$.

Theorem 2.10 [**4**, Lemma 2.7]. *Let $X = V_1 \cup R_1 \cup R_2 \cup V_2$ be a continuum such that $V_1 \cup R_1$ and $V_2 \cup R_2$ are compactifications of the disjoint rays $V_1 = [p, \infty)$ and $V_2 = (-\infty, q]$, respectively, such that R_1 is the remainder of $V_1 \cup R_1$ and R_2 is the remainder of $V_2 \cup R_2$, and $V_1 \cap R_2 = \varnothing = V_2 \cap R_1$. If A is a nondegenerate subcontinuum of X such that $p, q \notin A$ and $A \cap (V_1 \cup V_2) \neq \varnothing$, then there is a* 2*-cell $\mathcal{D}$ in $C(X)$ such that $A \in \mathsf{int}_{C(X)}(\mathcal{D})$.*

Theorem 2.11 [**2**, Theorem 4.4]. *Let $X = V \cup ce \cup cb$ be a continuum such that*:

(2.11.1) $V = [a, \infty)$ *is a ray*,

(2.11.2) ce *and* cb *are arcs*,

(2.11.3) *the arc ce is the remainder of the compactification $V \cup ce$ of V*,

(2.11.4) $cb \cap (V \cup ce) = \{c\}$,

and let an arc ab satisfy $ab \cap X = \{a, b\}$. Then $\Im(X) = \{X, X \cup ab\}$.

For given continua X and $K \in C(X)$, define

(2.12) $K^+ = \{x \in X$: there is a mapping $\alpha : \mathbf{I} \to C(X) \setminus \{K\}$ such that $\alpha(0) = \{x\}$ and $\alpha(1) = X\}$,

(2.13) $K^- = \{x \in X$: for each mapping $\alpha : \mathbf{I} \to C(X)$ such that $\alpha(0) = \{x\}$ and $\alpha(1) = X$, we have $K \in \alpha(\mathbf{I})\}$.

Clearly, $K = (K \cap K^+) \cup (K \cap K^-)$ and $X \setminus K \subset K^+$, whence $K^- \subset K$. Therefore, $K = (K \cap K^+) \cup K^-$.

The following lemmas will be needed in the next section.

Lemma 2.14. *Let X be a continuum and $K \in C(X)$. If $x \in K^+$ and $A \in C(X)$ is such that $x \in A$ and $K \setminus A \neq \varnothing$, then $A \subset K^+$.*

Proof. To show the inclusion, take $y \in A$. Let $\alpha_1, \alpha_2 : \mathbf{I} \to C(X)$ be order arcs from $\{y\}$ to A and from $\{x\}$ to A, respectively (see [**10**, p. 112]). Since $x \in K^+$, there is a mapping $\alpha_0 : \mathbf{I} \to C(X) \setminus \{K\}$ such that $\alpha_0(0) = \{x\}$ and $\alpha_0(1) = X$. Define $\alpha : \mathbf{I} \to C(X)$ by

$$\alpha(t) = \begin{cases} \alpha_1(3t) & \text{if } t \in \left[0, \dfrac{1}{3}\right], \\ \alpha_2(2-3t) & \text{if } t \in \left[\dfrac{1}{3}, \dfrac{2}{3}\right], \\ \alpha_0(3t-2) & \text{if } t \in \left[\dfrac{2}{3}, 1\right]. \end{cases}$$

It is easy to see that α is a well-defined mapping such that $\alpha(0) = \{y\}$, $\alpha(1) = X$ and $K \notin \alpha(\mathbf{I})$. This shows the conclusion. ◻

For a continuum X, a point $x \in X$ and a subcontinuum K of X with $x \in K$, let $\kappa(x, K)$ stand for the composant of the continuum K to which x belongs.

Lemma 2.15. *Let X be a continuum and $K \in C(X)$. The following implications are true*:

(a) *if $x \in K^+$, then $\kappa(x, K) \subset K^+$*;

(b) *if $x \in K^-$, then $\kappa(x, K) \subset K^-$.*

Proof. To show (a), take $y \in \kappa(x, K)$. Then there exists $A \in C(K) \setminus \{K\}$ such that $x, y \in A$. Thus $A \subset K^+$ by Lemma 2.14, hence $y \in K^+$. The proof for (b) is similar. ◻

Lemma 2.16. *Let X be a continuum and $K \in C(X)$. If K is decomposable, then either $X = K^+$ or $K = K^-$.*

Proof. Since K is decomposable, there is a point $x \in K$ such that $\kappa(x, K) = K$. Then the conclusion follows from Lemma 2.15 and the fact that $X \setminus K \subset K^+$. ◻

3. Chainable continua X with infinite $\Im(X)$. We start this section with the following two lemmas which concern the structure of X and of $C(X)$ for an irreducible continuum X of type λ.

Lemma 3.1. *Let an irreducible continuum X be of type λ. If T_t is a layer of cohesion for some $t \in \mathbf{I}$, then*

(3.1.1) *T_t is terminal in X or it is a singleton*;

(3.1.2) *the set $\Gamma = C(X) \setminus C(T_t)$ is an arc component of $C(X) \setminus \{T_t\}$.*

Proof. If T_t is degenerate, the conclusion holds by [**16**, p. 358]. So assume that T_t is nondegenerate. Let $g : X \to \mathbf{I}$ be the canonical mapping, and let $T_t = g^{-1}(t)$ be a layer of cohesion.

To show (3.1.1), consider a subcontinuum K of X such that $K \cap T_t \neq \varnothing$ and that K is not a subcontinuum of T_t. Then $g(K) \neq \{t\}$. Denote $g(K) = [s_0, s_1]$ with $s_0 < s_1$ and note that one of the two numbers s_0 and s_1 (but not both) can be equal to t. Then the union $g^{-1}([0, s_0]) \cup K \cup g^{-1}([s_1, 1])$ is a continuum that contains the points of irreducibility of X, whence it equals X. It follows that

$$g^{-1}((s_0, s_1)) \subset K. \tag{3.1.3}$$

Since T_t is a layer of cohesion, for each point $x \in T_t$, there exists a sequence $t_n \in (s_0, s_1)$ tending to t and such that the sequence of layers $g^{-1}(t_n)$ converges to an element of $C(X)$ and $x \in \operatorname{Lim} g^{-1}(t_n)$. Then $g^{-1}(t_n) \subset K$ by (3.1.3), whence $\operatorname{Lim} g^{-1}(t_n) \subset K$, and thus $x \in K$. Therefore, $T_t \subset K$, so T_t is terminal in X.

To verify (3.1.2), observe that Γ is arcwise connected by [**16**, p. 358] and note that $\Gamma \subset C(X) \setminus \{T_t\}$ according to the definition of Γ. Thus there exists an arc component Δ of $C(X) \setminus \{T_t\}$ such that $\Gamma \subset \Delta$. We will show that $\Gamma = \Delta$. Indeed, suppose on the contrary that $\Gamma \subsetneq \Delta$, and fix elements $A \in \Delta \setminus \Gamma$ and $B \in \Gamma$. Then $A \in C(T_t) \setminus \{T_t\}$ and, since $B \in \Gamma$, we have $B \setminus T_t \neq \varnothing$. Take a mapping $\gamma : \mathbf{I} \to \Delta$ with $\gamma(0) = A$ and $\gamma(1) = B$. Applying Theorem 2.4 (with $E = T_t$), we conclude that $T_t \in \gamma(\mathbf{I})$ which contradicts the fact that $\Delta \subset C(X) \setminus \{T_t\}$. The proof is complete. $\square$

Lemma 3.2. *Let a continuum X of type λ having p and q as points* *of irreducibility be such that each layer T_t of X is a layer of cohesion, and let $g : X \to \mathbf{I}$ be the canonical mapping. Then*

(3.2.1) *for each $A \in C(X)$ either $A \subset T_t$ for some $t \in \mathbf{I}$ or A is of the form $A = g^{-1}([s,t])$, where $0 \le s < t \le 1$;*

(3.2.2) *for each $t \in \mathbf{I} \setminus \{0\}$ the continuum $C(T_t, \mathrm{cl}_X(L_t))$ is an order arc from T_t to $\mathrm{cl}_X(L_t)$, and for each $t \in \mathbf{I} \setminus \{1\}$ the continuum $C(T_t, \mathrm{cl}_X(R_t))$ is an order arc from T_t to $\mathrm{cl}_X(R_t)$;*

(3.2.3) *the union $C(T_0, X) \cup C(T_1, X)$ is an arc in $C(X)$ with end points T_0 and T_1;*

(3.2.4) *for each $t \in \mathbf{I} \setminus \{0,1\}$, the continuum $C(T_t, X)$ is a 2-cell in $C(X)$;*

(3.2.5) *if $t \in \mathbf{I} \setminus \{0,1\}$, then for each $K \in C(T_t, X) \setminus \{T_t\}$ such that $p, q \notin K$, there is a 2-cell $\mathcal{D}$ in $C(X)$ for which $K \in \mathfrak{int}_{C(X)}(\mathcal{D}) \cap \mathrm{int}_{C(X)}(\mathcal{D})$.*

Proof. To show (3.2.1), take any $A \in C(X)$ and assume that A is not contained in a single layer of X. Then $g(A) = [s,t]$ for some $s, t \in \mathbf{I}$ with $s < t$. For any $r \in [s,t]$, $A \cap T_r \ne \varnothing$ and A is not contained in T_r. Then $T_r \subset A$ by (3.1.1) of Lemma 3.1. Thus, $g^{-1}([s,t]) \subset A$. Since $A \subset g^{-1}([s,t])$, we conclude that $A = g^{-1}([s,t])$ and (3.2.1) follows.

To prove (3.2.2), take $t \in \mathbf{I}$. By (3.2.1), $C(T_t, \mathrm{cl}_X(L_t)) = \{g^{-1}([s,t]) : 0 \le s \le t\}$, which clearly is an order arc. The proof that $C(T_t, \mathrm{cl}_X(R_t))$ is an order arc is similar.

To show (3.2.3), note that, by (3.2.2), the continua $C(T_0, X)$ and $C(T_1, X)$ are order arcs in $C(X)$ from T_0 and T_1 to X, respectively. Clearly, $C(T_0, X) \cap C(T_1, X) = \{X\}$, whence (3.2.3) follows.

To prove (3.2.4), take $t \in \mathbf{I} \setminus \{0,1\}$. By (3.2.1), $C(T_t, X) = \{g^{-1}([r,s]) : 0 \le r \le t \le s \le 1\}$. This implies that the function $h_t : C(T_t, X) \to [0,t] \times [t,1]$ defined by

(3.2.6) $$h_t(A) = (\min g(A), \max g(A)) \quad \text{for each } A \in C(T_t, X)$$

is a homeomorphism. Thus, (3.2.4) has been shown.

Finally, to show (3.2.5), take $t \in \mathbf{I} \setminus \{0,1\}$ and $K \in C(T_t, X) \setminus \{T_t\}$ such that $p, q \notin K$. By (3.2.1),

(3.2.7) $$K = g^{-1}([r,s]), \quad \text{where } 0 < r \le t \le s < 1 \text{ and } r < s.$$

Fix a number $t_0 \in (r, s)$. Let $h : C(X) \to \mathbf{I} \times \mathbf{I}$ be defined by

$$h(A) = (\min g(A), \max g(A)) \quad \text{for each } A \in C(X).$$

Then h is continuous. Let h_{t_0} be as in (3.2.6). Note that $h_{t_0} = h|C(T_{t_0}, X)$ and that h_{t_0} is a homeomorphism between $C(T_{t_0}, X)$ and $[0, t_0] \times [t_0, 1]$. Then $\mathcal{D} = C(T_{t_0}, X)$ is a 2-cell such that $K \in \mathtt{int}_{C(X)}(\mathcal{D})$ by (3.2.7). Since $h^{-1}((0, t_0) \times (t_0, 1))$ is an open subset of $C(X)$ containing K, we conclude that $K \in \mathrm{int}_{C(X)}(\mathcal{D})$. This completes the proof of the lemma. □

Given a continuum X irreducible between points p and q of type λ, let P and M be arcs such that p is an end point of P, p and q are the end points of M, $P \cap X = \{p\}$ and $M \cap X = \{p, q\}$. Then define

$$\mathtt{L}\,(X) = X \cup P \quad \text{and} \quad \mathtt{V}\,(X) = X \cup M. \tag{3.3}$$

The next lemma generalizes the above quoted main result of [**8**], namely Theorem 2.5.

Lemma 3.4. (a) *If a continuum X is of type $\lambda\diamond$ or of type $\lambda\triangleleft$, then*

$$C(X) \approx C(\mathtt{L}\,(X)). \tag{3.4.1}$$

(b) *If a continuum X is of type $\lambda\diamond$, then*

$$C(X) \approx C(\mathtt{L}\,(X)) \approx C(\mathtt{V}\,(X)). \tag{3.4.2}$$

Proof. If X is of type $\lambda\diamond$, then (3.4.1) is just a reformulation of Theorem 2.5. Its proof, presented in [**8**, pp. 212–213], does not require the assumption that both end layers of X are degenerated. It uses only the fact that one end layer of X, viz. T_0, is a point. Thus (a) follows.

To show (b), choose points $a, b \in M$ such that $p < a < b < q$ in the natural order $<$ on M from p to q. It follows from (a) that $C(X) \approx C(X \cup pa)$. Since the continuum $X \cup pa$ is of type $\lambda\diamond$, Theorem 2.5 implies that $C(X \cup pa) \approx C(X \cup pa \cup qb)$. By Theorem

2.6 we get $C(X \cup pa \cup qb) \approx C(\mathtt{V}(X))$. Thus $C(X) \approx C(\mathtt{V}(X))$, as needed. $\square$

For an irreducible continuum X of type λ and a canonical mapping $g : X \to \mathbf{I}$, consider the sets

$$N(X) = \{t \in \mathbf{I} : g^{-1}(t) \text{ is nondegenerate}\}, \tag{3.5}$$

$$D(X) = \mathbf{I} \setminus N(X). \tag{3.6}$$

The theorem below is the key result of this section.

Theorem 3.7. *Let a continuum X of type $\lambda\diamond$ be different from an arc, and let an arc $ab \subset \mathtt{V}(X)$ satisfy $ab \cap g^{-1}(N(X)) = \varnothing$. Then the continuum $Y = \mathtt{V}(X) \setminus (ab \setminus \{a, b\})$ is of type $\lambda\diamond$ and $C(X) \approx C(Y)$.*

Proof. Since each nondegenerate layer of X is in Y and since X is not an arc, $X \cap Y \neq \varnothing$. Observe that $X \cap Y$ has at most two components.

If $X \cap Y$ has two components, then by (3.2.1) of Lemma 3.2, it can be written in the form

$$X \cap Y = g^{-1}([0, t_1]) \cup g^{-1}([t_2, 1]),$$

where $0 \leq t_1 < t_2 \leq 1$. In this case $M \subset Y$. Therefore, $Y = (X \cap Y) \cup M = g^{-1}([0, t_1]) \cup M \cup g^{-1}([t_2, 1])$, and we can assume that $a \in g^{-1}(t_1)$ and $b \in g^{-1}(t_2)$. Thus the layers T_{t_1} and T_{t_2} are degenerate. Hence Y is of type $\lambda\diamond$, and $\mathtt{V}(Y) \approx \mathtt{V}(X)$.

If $X \cap Y$ is connected, then again by (3.2.1) of Lemma 3.2, it can be written in the form

$$X \cap Y = g^{-1}([t_1, t_2]),$$

where $0 \leq t_1 < t_2 \leq 1$. Observe that again the layers T_{t_1} and T_{t_2} are degenerate. Hence the continuum $X \cap Y$ is of type $\lambda\diamond$. So Y can be obtained by attaching to $X \cap Y$ either none, or one, or two disjoint arcs at the points of irreducibility of $X \cap Y$, i.e., at the singletons $g^{-1}(t_1)$ and $g^{-1}(t_2)$. Therefore, Y is of type $\lambda\diamond$, and again $\mathtt{V}(Y) \approx \mathtt{V}(X)$.

To complete the proof, note that according to Lemma 3.4 (b), $C(X) \approx C(\mathtt{V}(X)) \approx C(\mathtt{V}(Y)) \approx C(Y)$. $\square$

The following corollary answers in the positive Question 5 of [**8**, p. 215].

Corollary 3.8. *There exists an arc-like continuum X of type $\lambda\diamond$ for which the families $\Im(X)$ and $\Im(\mathtt{V}(X))$ are infinite and contain countably many arc-like continua of type $\lambda\diamond$.*

Proof. In the unit square $[0,1]\times[0,1]$, define $p=(0,0)$, $q=(1,0)$, and let a continuum X be irreducible between p and q such that if $g: X \to \mathbf{I}$ means the canonical mapping, then

(3.8.1) $g^{-1}(0)=\{p\}$ and $g^{-1}(1)=\{q\}$;

(3.8.2) $g^{-1}(1/(n+1))=\{1/(n+1)\}\times[0,1/(n+1)]$ for each $n\in\mathbf{N}$;

(3.8.3) $g^{-1}((1/(n+1),1/n))$ is a homeomorphic copy of the real line $\mathbf{R}$ such that $g^{-1}([1/(n+1),1/n])$ is a compactification of $\mathbf{R}$ with $g^{-1}(1/(n+1))\cup g^{-1}(1/n)$ as the (nonconnected) remainder, for each $n\in\mathbf{N}$;

(3.8.4) $g^{-1}([1/(n+1),1/n])$ and $g^{-1}([1/(m+1),1/m])$ are not homeomorphic if $n\neq m$.

Then X is an arc-like continuum of type $\lambda\diamond$, and g can be seen as the orthogonal projection of X onto $[0,1]$.

For each $n\in\mathbf{N}$, choose a positive number ε_n so small that

$$\frac{1}{n+1}<\frac{1}{n+1}+\varepsilon_n<\frac{1}{n}-\varepsilon_n<\frac{1}{n},$$

and define

$$Y_n=\mathtt{V}(X)\setminus g^{-1}\left(\left(\frac{1}{n+1}+\varepsilon_n,\frac{1}{n}-\varepsilon_n\right)\right).$$

Then each Y_n is an arc-like continuum of type $\lambda\diamond$, $C(X)\approx C(Y_n)$ for each $n\in\mathbf{N}$ according to Theorem 3.7, and by property (3.8.4) no two of them are homeomorphic. Further, according to Lemma 3.4 (b), we have $C(X)\approx C(\mathtt{V}(X))$, so $C(\mathtt{V}(X))\approx C(Y_n)$ for each $n\in\mathbf{N}$. Thus each of the families $\Im(X)$ and $\Im(\mathtt{V}(X))$ contains all continua Y_n, and thus it contains countably many arc-like continua of type $\lambda\diamond$. $\square$

Remark 3.9. Since, for each arc-like continuum X of type $\lambda\diamond$ the continuum $\mathtt{V}(X)$ is circle-like, it follows from Corollary 3.8 that there are arc-like continua and there are circle-like continua which have neither unique hyperspace nor almost unique hyperspace.

Let $\{A_\sigma : \sigma \in \Sigma\}$ be an uncountable family of arc-like continua no two of which are homeomorphic. Replacing in the above construction of the continuum X the nondegenerate layer $g^{-1}(1/2)$ by A_σ, we get an arc-like continuum X_σ of type $\lambda\diamond$ with the property as in Corollary 3.8. Thus the corollary can be strengthened as follows.

Corollary 3.10. *There exists an uncountable family* $\{X_\sigma : \sigma \in \Sigma\}$ *of arc-like continua of type* $\lambda\diamond$*, no two of which are homeomorphic such that for each* $\sigma \in \Sigma$ *the families* $\Im(X_\sigma)$ *and* $\Im(\mathtt{V}(X_\sigma))$ *are infinite, and they contain countably many arc-like continua of type* $\lambda\diamond$*.*

Questions 3.11. Does there exist (a) an arc-like, (b) a circle-like continuum X such that the family $\Im(X)$ is uncountable?

Questions 3.12. Does there exist (a) an arc-like, (b) a circle-like continuum X such that the family $\Im(X)$ contains uncountably many arc-like continua?

4. Irreducible continua of type λ. Let an irreducible continuum X be given. Recall that the sets $N(X)$ and $D(X)$ are defined by (3.5) and (3.6), respectively, and put $e_0 = \inf N(X)$. Let us accept the following notation. We will say that a continuum X is *of type* $\lambda^\star$ provided that X is an irreducible continuum of type λ, X is not an arc, each layer of X is a layer of cohesion, and the set $g^{-1}(D(X))$ is dense in X.

The following theorem is the main result of this section (and of the whole paper).

Theorem 4.1. *Let an irreducible continuum* X *be of type* $\lambda^\star$*. Then*

(4.1.1) *if* X *is of type* $\lambda\Box$*, then* X *has unique hyperspace*;

(4.1.2) *if* X *is of type* $\lambda\triangleleft$*, then*

(a) *X has unique hyperspace whenever the set $N(X)$ has a minimum (i.e.,* $e_0 \in N(X)$*)*;

(b) $\Im(X)$ *contains exactly two elements, namely*

$$\Im(X) = \{\mathtt{L}\,(X), g^{-1}([e_0, 1])\}$$

whenever the set $N(X)$ *does not have a minimum,* (*i.e.,* $e_0 \in D(X)$);

(4.1.3) *if* X *is of type* $\lambda\diamond$*, then* $\Im(X) \subset \{\mathtt{V}\,(X)\} \cup \{\mathtt{V}\,(X) \setminus (ab \setminus \{a, b\}) :$ *ab is an arc in* $\mathtt{V}\,(X)$ *such that* $ab \cap g^{-1}(N(X)) = \varnothing\}$.

Proof. Let a continuum Y be such that $C(X) \approx C(Y)$ with a homeomorphism $h : C(Y) \to C(X)$. Consider the induced mapping $C(g) : C(X) \to C(\mathbf{I})$ of g which is defined by $C(g)(A) = g(A)$, the image of A under g (see [**10**, pp. 188–189 and 381]). It is well known that a geometric model for $C(\mathbf{I})$ is a triangle (in the plane) with vertices $\{0\}, \{1\}$ and $\mathbf{I}$ whose manifold boundary, denoted by $\partial C(\mathbf{I})$, is given by

$$\partial C(\mathbf{I}) = F_1(\mathbf{I}) \cup C(\{0\}, \mathbf{I}) \cup C(\{1\}, \mathbf{I})$$

(see [**10**, p. 33]). Note that

(4.1.4) $C(g)(A) \in F_1(\mathbf{I})$ if and only if $A \subset T_t$ for some $t \in \mathbf{I}$.

We will show several claims.

(4.1.5) $C(g)(h(F_1(Y))) \subset \partial C(\mathbf{I})$.

To prove (4.1.5), suppose on the contrary that $C(g)(h(\{y\})) \notin \partial C(\mathbf{I})$ for some $y \in Y$. Then $g(h(\{y\}))$ is a nondegenerate subinterval $[s, u]$ of $\mathbf{I}$ which does not contain 0 and 1. Let $t \in (s, u)$. Then $h(\{y\})$ intersects T_t and it is not contained in T_t. By (3.1.1) of Lemma 3.1, $h(\{y\}) \in C(T_t, X) \setminus \{T_t\}$. Then (3.2.5) of Lemma 3.2 implies that there is a 2-cell $\mathcal{D}$ in $C(X)$ for which $h(\{y\}) \in \mathtt{int}_{C(X)}(\mathcal{D}) \cap \mathrm{int}_{C(X)}(\mathcal{D})$. This contradicts Theorem 2.8 and completes the proof of (4.1.5).

(4.1.6) If Γ is an arc component of $C(X) \setminus \{T_t\}$, then $h(F_1(Y)) \cap \Gamma \neq \varnothing$ for each $t \in \mathbf{I}$.

Indeed, note that $h^{-1}(\Gamma)$ is an arc component of $C(Y) \setminus \{h^{-1}(T_t)\}$, whence by Theorem 2.3 we have $h^{-1}(\Gamma) \cap F_1(Y) \neq \varnothing$. Thus the conclusion follows.

(4.1.7) $h(F_1(Y)) \not\subseteq C(T_t)$ for each $t \in \mathbf{I}$.

To show (4.1.7), let $t \in \mathbf{I}$. By (3.1.2) of Lemma 3.1, the set $\Gamma' = C(X) \setminus C(T_t)$ is an arc component of $C(X) \setminus \{T_t\}$, whence $h(F_1(Y)) \cap \Gamma' \neq \varnothing$, and so (4.1.7) holds.

(4.1.8) $h(F_1(Y)) \cap C(T_t) \neq \varnothing$ for each $t \in N(X)$.

To prove (4.1.8), take $t \in N(X)$ and let $x \in T_t$. Then $\{x\} \in C(X) \setminus \{T_t\}$. Let Γ be the arc component of $C(X) \setminus \{T_t\}$ such that $\{x\} \in \Gamma$. By (4.1.6), there exists a point $y \in Y$ such that $h(\{y\}) \in \Gamma$. By (3.1.2) of Lemma 3.1, the set $\Gamma' = C(X) \setminus C(T_t)$ is an arc component of $C(X) \setminus \{T_t\}$. Since $\{x\} \in C(T_t)$, hence $\Gamma \cap \Gamma' = \varnothing$. Therefore $\Gamma \subset C(T_t)$, so $h(\{y\}) \subset T_t$, and so (4.1.8) is proved.

(4.1.9) $T_t \notin h(F_1(Y))$ for each $t \in N(X)$.

To show (4.1.9), take $t \in N(X)$. By (3.1.1) of Lemma 3.1, the layer T_t is terminal in X. Then the conclusion follows from Theorem 2.7.

(4.1.10) $h(Y) \neq T_t$ for each $t \in N(X)$.

To prove (4.1.10), suppose on the contrary that $h(Y) = T_t$ for some $t \in N(X)$. By (3.1.1) of Lemma 3.1, the layer T_t is terminal in X. Then by [**16**, p. 361], the set $C(X) \setminus \{T_t\}$ is not arcwise connected. Then $C(Y) \setminus \{h^{-1}(T_t)\} = C(Y) \setminus \{Y\}$ is not arcwise connected as well, whence by [**16**, p. 104], the continuum Y is indecomposable.

By (4.1.8) and (4.1.9), there is a point $y \in Y$ such that $h(\{y\}) \in C(T_t) \setminus \{T_t\}$. Let Γ be the arc component of $C(X) \setminus \{T_t\}$ with $h(\{y\}) \in \Gamma$. Since $\Gamma' = C(X) \setminus C(T_t)$ is an arc component of $C(X) \setminus \{T_t\}$ according to (3.1.2) of Lemma 3.1, and since $h(\{y\}) \subset T_t$, we have $\Gamma \subset C(T_t)$. For each point $z \in Y$, let $\kappa(z, Y)$ be the composant of Y that contains z. Put $C(\kappa(z, Y)) = \{A \in C(Y) : A \subset \kappa(z, Y)\}$. By [**16**, p. 105], $C(\kappa(z, Y))$ is an arc component of $C(Y) \setminus \{Y\}$. In particular, since $h^{-1}(\Gamma)$ is the arc component of $C(Y) \setminus \{Y\}$ that contains $\{y\}$, we conclude that $C(\kappa(y, Y)) = h^{-1}(\Gamma)$. Hence $h(C(\kappa(y, Y))) \subset C(T_t)$. Since $F_1(\kappa(y, Y))$ is dense in $F_1(Y)$ and $F_1(\kappa(y, Y)) \subset C(\kappa(y, Y))$, passing to the images under h and taking the closure, we obtain $h(F_1(Y)) \subset C(T_t)$, a contradiction to (4.1.7). This shows (4.1.10).

Let $L = \{t \in \mathbf{I} : \{t\} \in C(g)(h(F_1(Y)))\}$. Applying (4.1.4), it can easily be seen that $L = \{t \in \mathbf{I} : h(F_1(Y)) \cap C(T_t) \neq \varnothing\}$, whence by (4.1.8) we infer that

(4.1.11) $N(X) \subset L$.

Put

$$X_0 = g^{-1}(L)$$

and

$$X_1 = \{x \in X_0 : g(x) \in D(X)\} = X_0 \cap g^{-1}(D(X)).$$

By (4.1.5) and by connectedness of $C(g)(h(F_1(Y)))$, it follows that the set L has at most two components.

(4.1.12) $F_1(X_1)\subset h(F_1(Y))$, $X_0 = \mathrm{cl}_X(X_1)$, and $F_1(X_0)\subset h(F_1(Y))$.

To prove the first inclusion of (4.1.12), take $x \in X_1$. Thus $g(x) \in D(X)$ and $h(F_1(Y)) \cap C(T_{g(x)}) \neq \varnothing$, so there exists a point $y \in Y$ such that $h(\{y\}) \subset T_{g(x)} = \{x\}$, whence $\{x\} = h(\{y\})$. Thus the inclusion is shown.

To prove that $X_0 = \mathrm{cl}_X(X_1)$, take a point $x \in X_0$ and suppose that $x \notin \mathrm{cl}_X(X_1)$. By the definition of X_1, if $t = g(x)$, then $x \in T_t$ and T_t is a nondegenerate layer of X. In the case when L contains a nondegenerate interval $[s, u]$ containing t, since T_t is a layer of cohesion, the point x can be approximated by points in the open set $g^{-1}((s, u))$. Since $g^{-1}(D(X))$ is dense in X, each point of $g^{-1}((s, u))$ can be approximated by points $x' \in X$ having the property that $g(x') \in D(X) \cap (s, u) \subset D(X) \cap L$. So $x' \in X_1$. Therefore, $x \in \mathrm{cl}_X(X_1)$, which is a contradiction. This shows that the component of L containing t is degenerate. Since L has at most two components, it either is connected or has the form $L = [0, v] \cup [w, 1]$ with $v < w$. This implies that either $L = \{t\}$, or $t = 0 = v$, or $t = 1 = w$. We will analyze the first two cases; the third one is similar to the second.

Case 1. $L = \{t\}$. If $0 \neq t \neq 1$, we have $h(F_1(Y)) \subset C(T_t)$, contrary to (4.1.7). Therefore, either $t = 0$ or $t = 1$. If $t = 0$, it follows that $C(g)(h(F_1(Y))) \subset C(\{0\}, \mathbf{I}) \cup C(\{1\}, \mathbf{I})$. This implies, using (3.2.1) of Lemma 3.2, that $h(F_1(Y)) \subset C(T_0) \cup C(T_0, X) \cup C(T_1, X)$. By (4.1.7) and (4.1.8) we have

$$h(F_1(Y)) \cap C(T_0) \neq \varnothing \neq h(F_1(Y)) \cap \big(C(T_0, X)) \cup C(T_1, X)\big).$$

Since $h(F_1(Y))$ is connected, there exists a point $y_0 \in Y$ such that $h(\{y_0\}) \in C(T_0) \cap \big(C(T_0, X) \cup C(T_1, X)\big) = \{T_0\}$. This contradicts (4.1.9). If $t = 1$, the argument is the same. This shows that Case 1 is impossible.

Case 2. $t = 0 = v$. Thus $L = \{0\} \cup [w, 1]$, whence $C(g)(h(F_1(Y))) = F_1([w, 1]) \cup C(\{0\}, \mathbf{I}) \cup C(\{1\}, \mathbf{I})$. This implies, using (3.2.1) of Lemma

3.2, that

$$h(F_1(Y)) \subset \big(C(g^{-1}([w,1])) \cup C(T_0,X) \cup C(T_1,X)\big) \cup C(T_0).$$

By (4.1.7) and (4.1.8), we have

$$h(F_1(Y)) \cap C(T_0) \neq \varnothing$$

and

$$h(F_1(Y)) \cap \big(C(g^{-1}([w,1])) \cup C(T_0,X) \cup C(T_1,X)\big) \neq \varnothing.$$

Since $h(F_1(Y))$ is connected, there exists a point $y_0 \in Y$ such that $h(\{y_0\}) \in \big(C(g^{-1}([w,1])) \cup C(T_0,X) \cup C(T_1,X)\big) \cap C(T_0) = \{T_0\}$. This contradicts (4.1.9) and proves that this case is also impossible.

We have proved that $X_0 \subset \mathrm{cl}_X(X_1)$. Since $X_1 \subset X_0$ and X_0 is closed, we infer that $X_0 = \mathrm{cl}_X(X_1)$.

Finally, $F_1(X_0) = F_1(\mathrm{cl}_X(X_1)) = \mathrm{cl}_{C(X)}(F_1(X_1)) \subset h(F_1(Y))$. Therefore, $F_1(X_0) \subset h(F_1(Y))$. This completes the proof of (4.1.12).

(4.1.13) $h(Y) \notin C(T_t)$ for each $t \in N(X)$.

Suppose on the contrary that $h(Y) \in C(T_t)$ for some $t \in N(X)$. Then $h(Y) \in C(T_t) \setminus \{T_t\}$ by (4.1.10). Put $K = h^{-1}(T_t)$ and note that $K \neq Y$. Consider the subsets K^+ and K^- of Y defined by (2.12) and (2.13) with Y in place of X.

We claim that

(4.1.14) if $s \in D(X)$ and $h(\{y\}) = T_s$ for some $y \in Y$, then $y \in K^- \subset K$.

To show (4.1.14), note that since T_t is terminal in X by (3.1.1) of Lemma 3.1, it follows from Theorem 2.4 that every path which joins T_s and $h(Y)$ in $C(X)$ passes through T_t. Thus every path which joins $\{y\}$ and Y in $C(Y)$ passes through $K = h^{-1}(T_t)$. Hence $y \in K^- \subset K$, and (4.1.14) is shown.

Next we claim that

(4.1.15) $K \cap K^+ \neq \varnothing$, and there exists a point $y' \in K^-$ such that $h(\{y'\}) \cap T_t = \varnothing$.

To see this, take $x \in h(Y)$. It follows from (4.1.11) that $t \in L$. So $x \in X_0$. By (4.1.12) there exists a point $y \in Y$ such that

$h(\{y\}) = \{x\}$ and there is a sequence of points $x_n \in X_1$ converging to x. Again, by (4.1.12), for each $n \in \mathbf{N}$, a point $y_n \in Y$ exists such that $\{x_n\} = h(\{y_n\})$. By (4.1.14), since $g(x_n) \in D(X)$, we have $y_n \in K^- \subset K$. Put $y' = y_1$. Since the sequence of points y_n tends to y, we get $y \in K$. Let $\beta : \mathbf{I} \to C(X)$ be an order arc from $\{x\}$ to $h(Y)$. Note that $T_t \notin \beta(\mathbf{I})$. Then $\alpha = h^{-1} \circ \beta : \mathbf{I} \to C(Y)$ is a mapping such that $\alpha(0) = \{y\}$, $\alpha(1) = Y$ and $K \notin \alpha(\mathbf{I})$. Thus, $y \in K^+$ by (2.12). Therefore, $K \cap K^+ \neq \varnothing$, and (4.1.15) holds.

It follows from (4.1.15) and Lemma 2.16 that K is indecomposable. Take $z \in K \cap K^+$, and let $\kappa(z, K)$ be the composant of K that contains z. Further, let y' be as in (4.1.15). By Lemma 2.15 (a) we have $\kappa(z, K) \subset K^+$. So $y' \notin \kappa(z, K)$. Consider a sequence of points $z_n \in \kappa(z, K) \subset K$ tending to y'. We will show that

(4.1.16) $h(\{z_n\}) \subset T_t$ for each $n \in \mathbf{N}$.

Suppose on the contrary that $h(\{z_n\}) \setminus T_t \neq \varnothing$ for some $n \in \mathbf{N}$, and consider a mapping $\alpha : \mathbf{I} \to C(Y) \setminus \{K\}$ such that $\alpha(0) = \{z_n\}$ and $\alpha(1) = Y$. Then $\beta = h \circ \alpha : \mathbf{I} \to C(X)$ is a mapping such that $\beta(0) = h(\{z_n\})$ and $\beta(1) = h(Y)$. Since $h(Y) \subsetneq T_t$ by the assumption and (4.1.10), and since $h(\{z_n\}) \setminus T_t \neq \varnothing$, we have $T_t \in \beta(\mathbf{I})$ by Theorem 2.4. Hence $K \in \alpha(\mathbf{I})$, a contradiction. Thus (4.1.16) is proved.

By (4.1.16) we have $h(\{y'\}) \subset T_t$, which contradicts the choice of y'. Therefore, (4.1.13) is established.

(4.1.17) $B \in C(T_t)$ implies $h^{-1}(B) \in C(h^{-1}(T_t))$ for each $t \in N(X)$.

To show (4.1.17) let $t \in N(X)$ and $B \in C(T_t)$. If $B = T_t$, the conclusion is obvious. Assume then that $B \subsetneq T_t$. By (4.1.13) we have $h(Y) \setminus T_t \neq \varnothing$. Thus by Theorem 2.4 each mapping $\beta : \mathbf{I} \to C(X)$ such that $\beta(0) = B$ and $\beta(1) = h(Y)$ satisfies $T_t \in \beta(\mathbf{I})$, whence it follows that each mapping $\alpha : \mathbf{I} \to C(Y)$ such that $\alpha(0) = h^{-1}(B)$ and $\alpha(1) = Y$ satisfies $h^{-1}(T_t) \in \alpha(\mathbf{I})$. This leads to $h^{-1}(B) \subset h^{-1}(T_t)$, because otherwise an order arc from $h^{-1}(B)$ to Y in $C(Y)$ would not contain $h^{-1}(T_t)$. Hence (4.1.17) follows.

(4.1.18) $h(F_1(Y)) \cap C(T_t) = F_1(T_t)$ for each $t \in N(X)$.

To show (4.1.18), let $t \in N(X)$. So, $t \in L$ by (4.1.11). Then $F_1(T_t) \subset F_1(X_0) \cap C(T_t) \subset h(F_1(Y)) \cap C(T_t)$ by (4.1.8) and (4.1.12). Put $K = h^{-1}(T_t)$ and suppose on the contrary that there exists a point

$y_0 \in Y$ such that $h(\{y_0\}) \in C(T_t) \setminus F_1(T_t)$. Then $\{y_0\} \subset K$ by (4.1.17). Fix a point $x_0 \in T_t \subset X_0$. By (4.1.12), $x_0 \in \mathrm{cl}_X(X_1)$. Since $x_0 \notin X_1$, either $x_0 \in \mathrm{cl}_X(X_1 \cap g^{-1}([0,t)))$ or $x_0 \in \mathrm{cl}_X(X_1 \cap g^{-1}((t,1]))$. Let $\beta : \mathbf{I} \to C(X)$ be an order arc from T_t to $g^{-1}([0,t])$ in the first case, or to $g^{-1}([t,1])$ in the second one. Then, by the choice of β and by (3.2.1) of Lemma 3.2, we have

(4.1.19) $\beta(r) \cap X_1 \neq \varnothing \quad$ for each $r \in (0,1]$.

Define a mapping $\varphi : \mathbf{I} \to C(Y)$ by

$$\varphi(s) = \bigcup \{\cup h^{-1}\big(F_1(\beta(r))\big) : r \in [0,s]\} \quad \text{for each } s \in \mathbf{I}.$$

Note that $\varphi(0) = \cup h^{-1}(F_1(T_t))$. By (4.1.17) we have $h^{-1}(F_1(T_t)) \subset C(K)$; thus $\varphi(0) \subset K$. Since $F_1(T_t) \subset h(F_1(Y))$, we have $\varphi(0) = \{y \in Y : h(\{y\}) \in F_1(T_t)\}$. Since $h(\{y_0\}) \notin F_1(T_t)$, it follows that $y_0 \notin \varphi(0)$. Then $y_0 \in K \setminus \varphi(0)$. By continuity of φ, there is $s > 0$ small enough such that $y_0 \notin \varphi(s)$. Note that $\varphi(0) \subset \varphi(s)$ just by the definition of φ. We claim that

(4.1.20) $\varphi(s) \subset K$.

To show this we first prove that

(4.1.21) $\varphi(0) \subset K^-$.

So take $y \in \varphi(0)$ and let $\alpha : \mathbf{I} \to C(Y)$ be a mapping such that $\alpha(0) = \{y\}$ and $\alpha(1) = Y$. Then $\gamma = h \circ \alpha : \mathbf{I} \to C(X)$ is a mapping satisfying $\gamma(0) = h(\{y\}) \in F_1(T_t)$ and $\gamma(1) = h(Y)$. Since $\gamma(1) \setminus T_t \neq \varnothing$, Theorem 2.4 gives $T_t \in \gamma(\mathbf{I})$, whence $K \in \alpha(\mathbf{I})$. Therefore, $y \in K^-$ and thus (4.1.21) holds.

Now suppose, contrary to claim (4.1.20), that there is a point $y \in \varphi(s) \setminus K$. By the definition of K^+ (see (2.12) with Y in place of X), we have $Y \setminus K \subset K^+$, so $y \in K^+$. Since $K \setminus \varphi(s) \neq \varnothing$ and $y \in \varphi(s)$, Lemma 2.14 implies $\varphi(s) \subset K^+$. Hence $\varphi(0) \subset K^+$ contrary to (4.1.21). Thus (4.1.20) is proved.

Further, the condition $s > 0$ implies that $T_t \subsetneq \beta(s)$, so $g(\beta(s)) \neq \{t\}$. By (4.1.19) there is an $x \in \beta(s) \cap X_1$. By (4.1.12) there is a point $y \in Y$ such that $h(\{y\}) = \{x\}$. By the definition of $\varphi(s)$, we have $y \in \varphi(s) \setminus \varphi(0)$. Since $\varphi(0) \subset K^-$ and $\varphi(s)$ is a proper subcontinuum of K (compare (4.1.20)), which contains y, it follows by Lemma 2.14 that $y \in K^-$. However, since $x \notin T_t$, by (4.1.13) there is an arc $\alpha : \mathbf{I} \to C(X)$ from $\{x\}$ to $h(Y)$ satisfying $T_t \notin \alpha(\mathbf{I})$. Thus $\gamma = h^{-1} \circ \alpha : \mathbf{I} \to C(Y)$ is a mapping such that $\gamma(0) = \{y\}$, $\gamma(1) = Y$ and $K \notin \gamma(\mathbf{I})$. Hence $y \in K^+$, a contradiction. This shows the equality in (4.1.18).

(4.1.22) $h(F_1(Y)) = F_1(X_0) \cup \big(h(F_1(Y)) \cap \big(C(T_0, X) \cup C(T_1, X)\big)\big)$.

To prove (4.1.22), let $y \in Y$. By (4.1.5), $C(g)(h(\{y\})) \in \partial C(\mathbf{I}) = F_1(\mathbf{I}) \cup C(\{0\}, \mathbf{I}) \cup C(\{1\}, \mathbf{I})$. If $C(g)(h(\{y\})) = [0, v]$ for some $v > 0$, then by (3.2.1) of Lemma 3.2 we get $h(\{y\}) = g^{-1}([0, v]) \in C(T_0, X)$. If $C(g)(h(\{y\})) = [u, 1]$ for some $u < 1$, we get $h(\{y\}) \in C(T_1, X)$. Now consider the case that $C(g)(h(\{y\}))$ is a singleton $\{t\}$. Then $t \in L$. If $t \in D(X)$, then $h(\{y\}) = g^{-1}(t) \in F_1(X_1) \subset F_1(X_0)$; and if $t \in N(X)$, by (4.1.18), $h(\{y\}) \in F_1(T_t) \subset F_1(X_0)$. This completes the proof of one inclusion in (4.1.22), that the left member is contained in the right. The opposite inclusion follows from (4.1.12).

Now we are ready to prove the theorem. We will analyze case by case. Let

$$\mathcal{U} = C(T_0, X) \cup C(T_1, X).$$

By (3.2.3) of Lemma 3.2, the union $\mathcal{U}$ is an arc in $C(X)$ with the endpoints T_0 and T_1.

To show (4.1.1), let X be of type $\lambda\square$. Then the arc $\mathcal{U}$ does not intersect $F_1(X)$. By (4.1.18), $h(F_1(Y)) \cap F_1(T_0) \neq \varnothing$. By (4.1.22), since $F_1(Y)$ is connected, we infer that $h(F_1(Y)) = F_1(X_0)$. By (4.1.5), $C(g)\big(h(F_1(Y))\big)$ is a subcontinuum of $\partial C(\mathbf{I}) \cap C(g)(F_1(X_0)) \subset F_1(\mathbf{I})$. By (4.1.8), since X is of type $\lambda\square$, we have $\{0\}, \{1\} \in C(g)\big(h(F_1(Y))\big)$. This implies that $C(g)\big(h(F_1(Y))\big) = F_1(\mathbf{I})$. Thus $L = \mathbf{I}$, $X_0 = X$ and $h(F_1(Y)) = F_1(X)$. Therefore, $Y \approx X$ and X has unique hyperspace. This shows (4.1.1).

To show (4.1.2), assume that X is of type $\lambda\triangleleft$.

By (4.1.9), $T_1 \notin h(F_1(Y))$. Then there exists a subarc $\mathcal{W}$ of $\mathcal{U}$ such that $T_1 \notin \mathcal{W}$, $T_0 \in \mathcal{W}$ and $h(F_1(Y)) \cap \mathcal{U} \subset \mathcal{W}$. Therefore, $h(F_1(Y)) \subset \mathcal{W} \cup F_1(X)$ (see (4.1.22)), $\mathcal{W} \cup F_1(X) \approx \mathsf{L}\,(X)$, $\mathcal{W} \cup F_1(X)$ is a continuum of type $\lambda\triangleleft$ and $h(F_1(Y))$ contains all the nondegenerate layers of $F_1(X)$ (see (4.1.18)). By (3.2.1) of Lemma 3.2, we have $F_1(g^{-1}([e_0, 1])) \subset h(F_1(Y))$.

(a) If $N(X)$ has a minimum, then $e_0 \in N(X)$ and $0 < e_0$. If

$F_1(g^{-1}([e_0,1]))$ is a proper subcontinuum of $h(F_1(Y))$, since $h(F_1(Y))$ is a subcontinuum of $\mathcal{W} \cup F_1(X)$, it follows that $h(F_1(Y)) \approx \mathcal{W} \cup F_1(X) \approx F_1(X)$. Therefore, $Y \approx X$. If $F_1(g^{-1}([e_0,1])) = h(F_1(Y))$, we consider two subcases. If $e_0 = 1$, then $h(F_1(Y)) = F_1(T_1) \subset C(T_1)$, a contradiction to (4.1.7). If $e_0 < 1$, then $h(F_1(Y))$ is homeomorphic to $g^{-1}([e_0,1])$ which is a continuum of type $\lambda\square$. By (4.1.1), the continuum Y has unique hyperspace, thus $g^{-1}([e_0,1]) \approx Y \approx X$. This is absurd, since X is of type $\lambda\triangleleft$. Consequently, $F_1(g^{-1}([e_0,1])) \neq h(F_1(Y))$. Therefore, $Y \approx X$, so in this case X has unique hyperspace.

(b) If $N(X)$ does not have a minimum, $e_0 \in D(X)$. Then $g^{-1}([e_0,1])$ is a continuum of type $\lambda\triangleleft$, and each subcontinuum of $\mathcal{W} \cup F_1(X)$ which contains $g^{-1}([e_0,1])$ is homeomorhpic either to $g^{-1}([e_0,1])$ or to $\mathcal{W} \cup F_1(X) \approx \mathtt{L}(X)$. Therefore $\Im(X) \subset \{\mathtt{L}(X), g^{-1}([e_0,1])\}$. The inverse inclusion follows from Lemma 3.4 (a). The proof of (4.1.2) is complete.

To show (4.1.3), let X be of type $\lambda\diamond$. Then the arc $\mathcal{U}$ intersects $F_1(X)$ in the set $\{T_0, T_1\}$. Hence $F_1(X) \cup \mathcal{U} \approx \mathtt{V}(X)$. By (4.1.22), $h(F_1(Y)) \subset F_1(X) \cup \mathcal{U}$. If $h(F_1(Y)) = F_1(X) \cup \mathcal{U}$, then $Y \approx \mathtt{V}(X)$. Thus we may assume that $h(F_1(Y)) \neq F_1(X) \cup \mathcal{U}$. In order to prove (4.1.3), since $F_1(X) \cup \mathcal{U} \approx \mathtt{V}(X)$, we need to show that there is an arc ab in $F_1(X) \cup \mathcal{U}$ such that $(F_1(X) \cup \mathcal{U}) \setminus h(F_1(Y)) = ab \setminus \{a,b\}$ and $ab \cap g^{-1}(N(X)) = \varnothing$. Since L has at most two components, we distinguish two subcases.

Subcase 1. L has two components. Since $h(F_1(Y))$ is connected, L is of the form $L = [0,u] \cup [v,1]$ where $0 \leq u < v \leq 1$, and $\mathcal{U} \subset h(F_1(Y))$. It follows from (4.1.11) that $(u,v) \cap N(X) = \varnothing$. Then $X_0 = g^{-1}([0,u]) \cup g^{-1}([v,1])$, so $h(F_1(Y)) = F_1(X_0) \cup \mathcal{U}$. Hence Y is of type λ. If $u \in N(X)$, Y is of type either $\lambda\triangleleft$ or $\lambda\square$ (depending on $v \in D(X)$ or $v \in N(X)$, respectively). By (4.1.1) and (4.1.2), since $C(X) \approx C(Y)$, we conclude that X is of type $\lambda\triangleleft$ or of type $\lambda\square$. This contradiction shows that $u \in D(X)$. Similarly, $v \in D(X)$, so $[u,v] \subset D(X)$. Therefore, $F_1(g^{-1}([u,v]))$ is a subarc of $F_1(X) \cup \mathcal{U}$, $h(F_1(Y)) = (F_1(X) \cup \mathcal{U}) \setminus F_1(g^{-1}((u,v)))$ and $F_1(g^{-1}([u,v])) \cap F_1(g^{-1}(N(X))) = \varnothing$. This completes the analysis of Subcase 1.

Subcase 2. L is connected. By (4.1.11), and since $F_1(X_0) = \mathrm{cl}_{C(X)}(F_1(X_1))$ (see (4.1.12)), we infer that L is nondegenerate. So it is of the form $L = [u,v]$, where $0 \leq u < v \leq 1$. Proceeding as in Subcase 1, we conclude that $u, v \in D(X)$ and $\mathbf{I} \setminus L \subset D(X)$. Therefore, $\mathrm{cl}_{C(X)}\big((F_1(X) \cup \mathcal{U}) \setminus F_1(X_0)\big)$ is an arc containing $(F_1(X) \cup \mathcal{U}) \setminus h(F_1(Y))$. Since $F_1(g^{-1}(N(X))) \subset h(F_1(Y))$, we conclude that $(F_1(X) \cup \mathcal{U}) \setminus h(F_1(Y))$ is of the form $ab \setminus \{a, b\}$ where ab is an arc in $F_1(X) \cup \mathcal{U}$ with $ab \cap F_1(g^{-1}(N(X))) = \varnothing$. This complete the analysis of Subcase 2.

Thus (4.1.3) is shown, and the proof of the theorem is complete. □

Remark 4.2. It follows from Theorems 3.7 and 4.1 that, under the assumption of Theorem 4.1, if the continuum X is of type $\lambda\diamond$, then the family $\Im(X)$ consists of all mutually nonhomeomorphic members of the family

$$\{\mathtt{V}(X)\} \cup \{\mathtt{V}(X) \setminus (ab \setminus \{a, b\}) : ab \text{ is an arc in } \mathtt{V}(X) \\ \text{such that } ab \cap g^{-1}(N(X)) = \varnothing\}.$$

Now we will present some consequences of Theorem 4.1. They generalize certain results obtained by the first-named author in [**2**]. To formulate these consequences, we introduce some notation.

Let Ξ denote the class of all continua Z such that there exists a continuum X of type $\lambda^\star$ and of type $\lambda\diamond$ (simultaneously) having the property that $Z \approx \mathtt{V}(X)$.

Theorem 4.3. *The class Ξ is C-determined.*

Proof. Take two members $\mathtt{V}(X)$ and $\mathtt{V}(Y)$ of Ξ such that $C(\mathtt{V}(X)) \approx C(\mathtt{V}(Y))$. Since the continuum X is of type $\lambda\diamond$ by assumption, it follows from Lemma 3.4 (b) that $C(\mathtt{V}(X)) \approx C(X)$, whence $C(\mathtt{V}(Y)) \approx C(X)$. Then, by Theorem 4.1, the continuum $\mathtt{V}(Y)$ is homeomorphic either to $\mathtt{V}(X)$ or to a continuum of the form $\mathtt{V}(X) \setminus (ab \setminus \{a, b\})$ for an arc $ab \subset \mathtt{V}(X)$ such that $ab \cap g^{-1}(N(X)) = \varnothing$. Continua of this form are of type $\lambda\diamond$ by Theorem 3.7. Since the continuum $\mathtt{V}(Y)$ is not irreducible (in particular it is not of type $\lambda\diamond$), it is not homeomorphic to any continuum of the considered form. Therefore $\mathtt{V}(Y) \approx \mathtt{V}(X)$, and the proof is complete. □

A continuum X is called a *double compactification* provided that

$$X = V_1 \cup R \cup V_2,$$

where $V_1 \cup R$ and $V_2 \cup R$ are two compactifications of disjoint rays V_1 and V_2 with endpoints p and q, respectively, and with a common nondegenerate remainder R. Note that each double compactification is a continuum irreducible between p and q, it is of type $\lambda\diamond$, the set $N(X)$ is a singleton $t_0 \in (0,1)$, and $g^{-1}(t_0) = R$ is the only nondegenerate layer, which is a layer of cohesion. Let Φ denote the class of all continua Z such that there exists a double compactification X having the property that $Z \approx \mathtt{V}(X)$. Thus $\Phi \subset \Xi$, whence the following result is a corollary to Theorem 4.3.

Corollary 4.4 [**2**, Theorem 3.5]. *The class Φ is C-determined.*

Nadler asks (see [**16**, p. 33]) if the class of all circle-like continua is C-determined. The next three results give partial answers to this question.

Let Φ_c stand for the class of all circle-like members of Φ. Corollary 4.4 implies the next one.

Corollary 4.5 [**2**, Theorem 3.6]. *The class Φ_c is C-determined.*

Let Ψ_c be the class of arcwise connected circle-like continua. The following theorem is shown in [**2**].

Theorem 4.6 [**2**, Theorems 4.12 and 4.13]. *The class Ψ_c is C-determined. Moreover, if $Z \in \Psi_c$ and Y is a circle-like continuum such that $C(Z) \approx C(Y)$, then $Y \in \Psi_c$.*

Let Ξ_c denote the class of all circle-like members of Ξ. Combining Theorems 4.6 and 4.3, we have the next result.

Theorem 4.7. *The class $\Psi_c \cup \Xi_c$ is C-determined.*

5. Final remarks and questions. The following remark reports relations of Theorem 4.1 to earlier results obtained by the first-named author.

Remark 5.1. Conclusion (4.1.1) of Theorem 4.1 generalizes Theorem 6 of [**1**]. Conclusion (4.1.2) generalizes Theorem 4 of [**1**]. And conclusion (4.1.3) generalizes Theorem 3.4 of [**2**].

Questions 5.2. Theorem 4.1 deals with continua X of type $\lambda^\star$. These continua satisfy, by their definition, the following two assumptions:

1) each layer T_t of X is a layer of cohesion;

2) the set $g^{-1}(D(X))$ is dense in X.

Is each of the assumptions 1) and 2) essential in Theorem 4.1?

There are irreducible continua of type λ which differ much from continua of type $\lambda^\star$ because they do not satisfy the assumptions 1) or 2). For the reader's information, we recall two important examples of such continua.

Example 5.3. Let us consider the V-Λ irreducible continuum described in [**13**, p. 191]. The same continuum is called Cajun accordion in [**20**, p. 343]. The continuum X is of type λ, each layer is nondegenerate, i.e., $D(X) = \varnothing$, each layer of shape V and Λ is *not* a layer of cohesion, while the others are. Thus neither of the two assumptions 1) and 2) is satisfied for this continuum.

Example 5.4. Knaster has constructed in [**11**] an irreducible continuum X of type λ, all of whose layers are nondegenerate and of continuity (thus of cohesion). Thus again $D(X) = \varnothing$, so X does not satisfy the assumption 2), while 1) holds.

We close the paper by showing a collection of continua of type λ but not $\lambda^\star$, and such that for each member X of the collection, the family $\Im(X)$ can be determined by using similar methods as those in the present paper.

Theorem 5.5. *Let a continuum X have the form*

$$X = V_1 \cup ab \cup bc \cup V_2,$$

where

(5.5.1) *$V_1 \cup ab$ is a compactification of the ray V_1 with the arc ab as the remainder*;

(5.5.2) *$V_2 \cup bc$ is a compactification of the ray V_2 with the arc bc as the remainder*;

(5.5.3) *$(V_1 \cup ab) \cap (V_2 \cup bc) = \{b\}$.*

Then $\Im(X) = \{X, \mathtt{V}(X)\}$.

Proof. Let Y be a continuum such that $C(X) \approx C(Y)$, and let $h : C(Y) \to C(X)$ be a homeomorphism. Since X is atriodic by its construction, Theorem 2.1 implies that Y is atriodic, too. Since X is not locally connected, we infer from [**16**, p. 134] that Y is not locally connected.

We claim that

(5.5.4) for no point $y \in Y$ is there a 2-cell $\mathcal{D}$ in $C(X)$ such that $h(\{y\}) \in \mathtt{int}_{C(X)}(\mathcal{D})$.

Indeed, suppose the contrary. Thus $\mathcal{D}' = h^{-1}(\mathcal{D})$ is a 2-cell such that $\{y\} \in \mathtt{int}_{C(Y)}(\mathcal{D}')$. By Theorem 2.2, the continuum Y contains a triod, a contradiction. So (5.5.4) holds.

Denote by p and q the end points of the rays V_1 and V_2, respectively, and put

$$\mathcal{L} = F_1(X) \cup C(\{p\}, X) \cup C(\{q\}, X) \cup C(\{a\}, ab) \cup C(\{c\}, bc).$$

We claim that

(5.5.5) $h(F_1(Y)) \subset \mathcal{L}$.

To show this, take $y \in Y$, put $K = h(\{y\})$ and suppose, on the contrary, that $K \notin \mathcal{L}$. Thus K is a nondegenerate subcontinuum of X such that $p, q \notin K$. Theorem 2.10 and (5.5.4) imply $K \cap (V_1 \cup V_2) = \varnothing$. Hence, $K \subset ab \cup bc$. Since $K \notin F_1(X) \cup C(\{a\}, ab) \cup C(\{c\}, bc)$, we have the following three possibilities:

(i) K is a nondegenerate subcontinuum of $ab \cup bc$ such that $a, c \notin K$;

(ii) $K \in C(ab, ab \cup bc) \setminus \{ab\}$;

(iii) $K \in C(bc, ab \cup bc) \setminus \{bc\}$.

Assume (i). Then K is an arc de. Order $ab \cup bc$ by $<$ so that $a < d < e < c$ and take a point $x \in K \setminus \{d, e\}$. Put in Theorem 2.9:

$$A_1 = dx, \quad A_2 = xe, \quad B_1 = ax, \quad B_2 = xc, \text{ and } B = B_1 \cup B_2,$$

and observe that all the assumptions of the theorem are satisfied. Thereby there is a 2-cell $\mathcal{D}$ in $C(X)$ such that $K \in \mathfrak{int}_{C(X)}(\mathcal{D})$, contrary to (5.5.4). Thus (i) does not hold.

Assume (ii). Then $K = ab \cup be$, where $e \in bc \setminus \{b\}$. Putting in Theorem 2.9:

$$A_1 = ab, \quad A_2 = be, \quad B_1 = V_1 \cup ab, \quad B_2 = V_2 \cup bc, \text{ and } B = B_1 \cup B_2,$$

we again conclude the same contradiction. Thus (ii) is not satisfied.

The argument for (iii) is exactly the same. Therefore, (5.5.5) is provided.

Since $h(F_1(Y))$ is atriodic and not locally connected, it follows from (5.5.5) that $h(F_1(Y))$ is homeomorphic to one of the following continua:

a) $F_1(V_1 \cup ab)$,

b) $F_1(V_1 \cup ab) \cup C(\{a\}, ab)$,

c) $F_1(X)$,

d) $F_1(X) \cup C(\{p\}, X) \cup C(\{q\}, X)$,

e) $F_1(V_1 \cup ab \cup bc)$,

f) $F_1(V_2 \cup bc)$,

g) $F_1(V_2 \cup bc) \cup C(\{c\}, bc)$,

h) $F_1(V_2 \cup ab \cup bc)$.

Assume a). Then $h(F_1(Y))$ is a compactification of a ray with a nondegenerate remainder. Thus Y is of the same form, and by part (a) of (4.1.2) of Theorem 4.1 it has unique hyperspace, whence $X \approx Y$. This implies that X is also a compactification of a ray with a nondegenerate remainder, a contradiction.

Assume b). Then Y is homeomorphic to a continuum $X_0 = V_1 \cup ab \cup aa'$, where $aa' \cap (V_1 \cup ab) = \{a\}$. Then Theorem 2.11 implies that either $X \approx X_0$ or $X \approx X_0 \cup pa'$ where $pa' \cap (V_1 \cup a'b) = \{p, a'\}$. This contradicts the definition of X.

The argument for f) is the same as for a), and one for e), g) and h) is the same as for b). Therefore, either c) or d) is satisfied. Note that

$$F_1(X) \cup C(\{p\}, X) \cup C(\{q\}, X) \approx \mathtt{V}(X).$$

Thus either $Y \approx X$ or $Y \approx \mathtt{V}(X)$, and, by Theorem 2.6, the proof is finished. □

Remark 5.6. Note that each continuum X as described in Theorem 5.5 has the arc $ab \cup bc$ as the only nondegenerate layer, whence $D(X)$ is dense. Since this layer is not of cohesion, assumption 1) mentioned in 5.2 is not satisfied, so Theorem 4.1 cannot be applied.

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