

SEQUENCE SPACES OF CONTINUOUS FUNCTIONS

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1. Introduction. The familiar sequence spaces c_0 and l^p , $0 < p < \infty$, may conveniently be understood as spaces of continuous functions that vanish at infinity on the locally compact space \mathbf{N} of positive integers, or as spaces of continuous functions on the one-point compactification of \mathbf{N} . On the other hand, to study l^∞ as a space of continuous functions on a compact space (rather than continuous bounded functions on \mathbf{N}) requires considerably more effort; the appropriate compact space is $\beta\mathbf{N}$, the Stone-Ćech compactification of \mathbf{N} , and it may be obtained variously by methods of set theory, topology, or analysis.

If instead of looking at sequences of scalars we look at sequences whose entries are taken from some fixed Banach space E , the study of c_0 and l^p , $0 < p < \infty$, becomes only marginally more involved. There is a compact space X (for instance, the closed unit ball of the dual space of E , in the weak* topology) such that E may be regarded as a space of continuous functions on X , and members of the sequence spaces can be interpreted as continuous functions on the one-point compactification of $\mathbf{N} \times X$. However, viewing l^∞ as a space of continuous functions now often involves significant new complexities which can have important reverberations elsewhere in analysis. It is this situation that we propose to explore here.

By a *Banach function space* on a compact Hausdorff space X , we mean a Banach space lying in $C(X)$ which separates the points of X , has norm dominating the uniform norm, and (for convenience) contains the constant functions. Here point separation means that if x_1 and x_2 are distinct points of X , then $f(x_1) \neq f(x_2)$ for some function f in the function space, and norm domination means that there is a positive constant c such that the function space norm on f is at least $c\|f\|_\infty$, where $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$. Scalars may be real or complex.

If E is a Banach function space on X , we denote by $l^\infty(\mathbf{N}, E)$, or more

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compactly \tilde{E} , the Banach space of all bounded sequences $\tilde{f} = \langle f_k \rangle_{k=1}^\infty$ of functions f_1, f_2, \dots , in E under componentwise operations and the norm $\|\tilde{f}\| = \sup\{\|f_k\| : k \in \mathbf{N}\} < \infty$. Since every sequence $\tilde{f} = \langle f_k \rangle_{k=1}^\infty$ in $C(X)^\sim$ can be considered as a bounded continuous function on the product space $\mathbf{N} \times X$ defined by $\tilde{f}(n, x) = f_n(x)$ for $(n, x) \in \mathbf{N} \times X$, it has a unique extension, also denoted by \tilde{f} , to a continuous function defined on $\tilde{X} = \beta(\mathbf{N} \times X)$, the Stone-Ćech compactification of $\mathbf{N} \times X$. Hence $C(X)^\sim = C(\tilde{X})$ and thus \tilde{E} can be considered as a Banach space lying in $C(\tilde{X})$. It is far from obvious that \tilde{E} separates the points of \tilde{X} and so is a Banach function space on \tilde{X} ; indeed, this is generally not the case and, as we shall see, having this property can be very important.

The systematic study of \tilde{E} was begun by Alain Bernard and colleagues in the late 1960's and early 70's in connection with aspects of Banach algebra theory related to the corona theorem. We will describe these developments, dealing with naturality properties of Banach function algebras, in Section 2. It turns out that the main application of \tilde{E} has been in a very different direction, initiated by Bernard in a series of articles culminating in [2], showing that very often the affine functions are the only ones that operate on a real Banach function space, especially if the function space consists of the real part of the functions in a (complex) uniform algebra. In Section 3 we will discuss how this line of developments proceeds.

To our knowledge, some of the material in Section 2 has not appeared in print before, though no claim to originality is being made.

2. Banach function algebras and uniform naturality. By a *Banach function algebra* on a compact Hausdorff space X we mean a complex Banach function space on X that is also a Banach algebra with pointwise multiplication. The norm domination requirement here is redundant, since Gelfand theory tells us that, given the rest, it holds automatically with $c = 1$. A Banach function algebra on X is a *uniform algebra* on X if the norm of the algebra coincides with the supremum norm on X .

Standard examples are provided by Gelfand transform algebras. Let A be a commutative complex Banach algebra with the identity element 1 and M_A its maximal ideal space, or spectrum. M_A is the

weak* compact subset of the closed unit ball of the dual space of A consisting of those nonzero linear functionals ϕ on A that are *multiplicative*: $\phi(a_1 a_2) = \phi(a_1)\phi(a_2)$ for elements a_1, a_2 of A . The Gelfand transform algebra of A is the algebra \hat{A} of functions \hat{f} from M_A into \mathbf{C} , where \hat{f} is the Gelfand transform of $f \in A$ defined by $\hat{f}(\phi) = \phi(f)$ for all $\phi \in M_A$. \hat{A} is a Banach function algebra under the norm $\|u\|_{\hat{A}} = \inf\{\|f\|_A : \hat{f} = u\}$. If A is a Banach function algebra on X , then X can be thought of as a compact subset of M_A with $x \in X$ identified with the functional $\phi_x \in M_A$ given by $\phi_x(f) = f(x)$.

Suppose A is a Banach function algebra on X . Then \tilde{A} is a Banach algebra of continuous functions on \tilde{X} , and a fundamental problem is this: If we know M_A , to what extent can we identify $M_{\tilde{A}}$? We begin with a lemma, which depends on two general facts. If A is a commutative algebra with an identity element, then every proper ideal of A is contained in a maximal (proper) ideal of A ; and if A is a commutative complex Banach algebra with an identity element, then the maximal ideals of A are precisely the kernels of the functionals in M_A .

Lemma 2.1. *Let A be a commutative complex Banach algebra with the identity element 1, and let Y be a subset of M_A . Then the following are equivalent:*

(1) Y is dense in M_A .

(2) For any $\delta > 0$ and $f_1, \dots, f_n \in A$ with $\sum_{i=1}^n |\hat{f}_i| \geq \delta$ on Y , there exist $g_1, \dots, g_n \in A$ such that $\sum_{i=1}^n f_i g_i = 1$.

Proof. The proof follows the by now familiar lines of that for H^∞ (see, e.g., [8]).

(1) implies (2). Let $\delta > 0$ and $f_1, \dots, f_n \in A$ be given with $\sum_{i=1}^n |\hat{f}_i| \geq \delta$ on Y . Since Y is dense in M_A , $\sum_{i=1}^n |\hat{f}_i| \geq \delta$ on M_A , and hence all the functions $\hat{f}_1, \dots, \hat{f}_n$ cannot vanish simultaneously at any element of M_A . Thus the ideal generated by f_1, \dots, f_n is equal to A . Therefore, $\sum_{i=1}^n f_i g_i = 1$ for some $g_1, \dots, g_n \in A$.

(2) implies (1). Assume that Y is not dense in M_A . Let $\phi \in M_A \setminus \overline{Y}$, where \overline{Y} is the closure of Y . Since \hat{A} separates the points of M_A , for each $y \in \overline{Y}$, $f_y \in A$ exists such that $\hat{f}_y(\phi) = 0$ and $\hat{f}_y(y) = 2$. The

open sets $U_y = \{\psi \in M_A : |\widehat{f}_y(\psi)| > 1\}$ for all $y \in \overline{Y}$ form an open cover of the compact set \overline{Y} , and hence we can choose $y_1, \dots, y_n \in \overline{Y}$ such that $\overline{Y} \subset \cup_{i=1}^n U_{y_i}$. Then $\sum_{i=1}^n |\widehat{f}_{y_i}| \geq 1$ on Y . If $\sum_{i=1}^n f_{y_i} g_i = 1$ for some $g_1, \dots, g_n \in A$, then we have

$$1 = \phi\left(\sum_{i=1}^n f_{y_i} g_i\right) = \sum_{i=1}^n \widehat{f}_{y_i} \widehat{g}_i(\phi) = 0,$$

which is impossible, and therefore, for any $g_1, \dots, g_n \in A$, we have $\sum_{i=1}^n f_{y_i} g_i \neq 1$. \square

Definition 2.2. A commutative complex Banach algebra A with an identity element is said to be *natural* on a subset Y of M_A if A satisfies one of the equivalent conditions of Lemma 2.1.

It is clear for a Banach function algebra A on X that, if \tilde{A} is natural on \tilde{X} , then A is natural on X , that is, $M_{\tilde{A}} = \tilde{X}$ implies $M_A = X$. To make this more precise even when \tilde{A} does not separate the points of \tilde{X} , define a natural inclusion map $\tau : \mathbf{N} \times M_A \rightarrow M_{\tilde{A}}$ by

$$\tau(n, \phi)(\tilde{f}) = \tau(n, \phi)(\langle f_k \rangle_{k=1}^\infty) = \phi(f_n) = \widehat{f}_n(\phi)$$

for $\tilde{f} = \langle f_k \rangle_{k=1}^\infty \in \tilde{A}$. The map τ is a homeomorphism of $\mathbf{N} \times M_A$ into $M_{\tilde{A}}$. Condition (2) of the following theorem is a version of condition (2) of the above lemma, with estimates added.

Theorem 2.3. *Let A be a Banach function algebra on X , and let Y be a subset of M_A . Then the following are equivalent:*

- (1) $\tau(\mathbf{N} \times Y)$ is dense in $M_{\tilde{A}}$.
- (2) For any $n \in \mathbf{N}$ and $0 < \delta < 1$, there is a constant $C(n, \delta) > 0$ such that for any $f_1, \dots, f_n \in A$ with $\|f_i\| \leq 1$ for $1 \leq i \leq n$ and $\sum_{i=1}^n |\widehat{f}_i| \geq \delta$ on Y , there exist $g_1, \dots, g_n \in A$ satisfying $\|g_i\| \leq C(n, \delta)$ for $1 \leq i \leq n$ and $\sum_{i=1}^n f_i g_i = 1$ on X .

Proof. (1) implies (2). Suppose that $\tau(\mathbf{N} \times Y)$ is dense in $M_{\tilde{A}}$, and assume that A fails to satisfy condition (2). Then there exist $n \in \mathbf{N}$ and $0 < \delta < 1$ such that, for each $k \in \mathbf{N}$, we can choose

$f_{1k}, f_{2k}, \dots, f_{nk} \in A$ with $\|f_{ik}\| \leq 1$ for $1 \leq i \leq n$ and $\sum_{i=1}^n |\widehat{f_{ik}}| \geq \delta$ on Y such that

$$(*) \quad \text{for any } g_{1k}, \dots, g_{nk} \in A \text{ with } \sum_{i=1}^n f_{ik}g_{ik} = 1 \text{ on } X,$$

$$\max_{1 \leq i \leq n} \|g_{ik}\| > k.$$

Put $\tilde{f}_i = \langle f_{ik} \rangle_{k=1}^\infty$ for $1 \leq i \leq n$. Then $\tilde{f}_i \in \tilde{A}$, and

$$\sum_{i=1}^n |\hat{f}_i(\tau(k, \phi))| = \sum_{i=1}^n |\tau(k, \phi)(\tilde{f}_i)| = \sum_{i=1}^n |\widehat{f_{ik}}(\phi)| \geq \delta$$

for all $(k, \phi) \in \mathbf{N} \times Y$ and hence $\sum_{i=1}^n |\hat{f}_i(\psi)| \geq \delta$ for all $\psi \in \tau(\mathbf{N} \times Y)$.

Since $\tau(\mathbf{N} \times Y)$ is dense in $M_{\tilde{A}}$, by Lemma 2.1 we can choose $\tilde{g}_1, \dots, \tilde{g}_n \in \tilde{A}$ such that $\sum_{i=1}^n \tilde{f}_i \tilde{g}_i = 1$ on \tilde{X} , where $\tilde{g}_i = \langle g_{ik} \rangle_{k=1}^\infty$. Thus, for every $k \in \mathbf{N}$,

$$\sum_{i=1}^n f_{ik}g_{ik}(x) = \sum_{i=1}^n \tilde{f}_i \tilde{g}_i(k, x) = 1$$

for all $x \in X$. Then, by (*), for every $k \in \mathbf{N}$ we have $\max\{\|g_{ik}\| : 1 \leq i \leq n\} > k$. Hence, for some $1 \leq i_0 \leq n$, $\tilde{g}_{i_0} = \langle g_{i_0k} \rangle_{k=1}^\infty \notin \tilde{A}$ since $\|\tilde{g}_{i_0}\| = \sup_{k \in \mathbf{N}} \|g_{i_0k}\| \geq \sup_{k \in \mathbf{N}} k$, which is a contradiction.

(2) implies (1). Suppose that A satisfies (2). We will show that \tilde{A} is natural on $\tau(\mathbf{N} \times Y)$. Then, by Lemma 2.1, $\tau(\mathbf{N} \times Y)$ is dense in $M_{\tilde{A}}$.

Let $\delta > 0$ and $\tilde{f}_1, \dots, \tilde{f}_n \in \tilde{A}$ be given such that $\sum_{i=1}^n |\hat{f}_i| \geq \delta$ on $\tau(\mathbf{N} \times Y)$. Dividing δ and the \tilde{f}_i by a large positive constant if necessary, we may assume that $\delta < 1$ and that $\|\tilde{f}_i\| \leq 1$ for $1 \leq i \leq n$. Write $\tilde{f}_i = \langle f_{ik} \rangle_{k=1}^\infty$. Then, for every $k \in \mathbf{N}$, we have $\|f_{ik}\| \leq \|\tilde{f}_i\| \leq 1$ for $1 \leq i \leq n$ and $\sum_{i=1}^n |\widehat{f_{ik}}| \geq \delta$ on Y . Thus, for each $k \in \mathbf{N}$, by the assumption we can choose $g_{1k}, \dots, g_{nk} \in A$ such that $\|g_{ik}\| \leq C(n, \delta)$ for $1 \leq i \leq n$ and $\sum_{i=1}^n f_{ik}g_{ik} = 1$ on X . Put $\tilde{g}_i = \langle g_{ik} \rangle_{k=1}^\infty \in \tilde{A}$ for $1 \leq i \leq n$. Then $\sum_{i=1}^n \tilde{f}_i \tilde{g}_i = 1$ on \tilde{X} since

$$\sum_{i=1}^n \tilde{f}_i \tilde{g}_i(k, x) = \sum_{i=1}^n f_{ik}g_{ik}(x) = 1$$

for all $(k, x) \in \mathbf{N} \times X$, and the latter is dense in \tilde{X} . \square

Definition 2.4. A Banach function algebra on X is *uniformly natural* on a subset Y of M_A if A satisfies one of the equivalent conditions of Theorem 2.3.

Notice that the proof of Theorem 2.3 gives more than was promised. If \tilde{A} is natural on $\tau(\mathbf{N} \times Y)$ (condition (1)), then A and \tilde{A} are uniformly natural on Y and $\tau(\mathbf{N} \times Y)$, respectively, with the same constants $C(n, \delta)$.

Why did these ideas arise in the context of Lennart Carleson's corona theorem [5]? This theorem asserts that the open unit disk $\Delta = \{z \in \mathbf{C} : |z| < 1\}$ is dense in the maximal ideal space of $H^\infty = H^\infty(\Delta)$, the algebra of all bounded holomorphic functions on Δ ; in other words, H^∞ is natural on Δ . In fact, Carleson proved that H^∞ is *uniformly natural* on Δ . Now a more tractable cousin of H^∞ is the disk algebra $A(\overline{\Delta})$, consisting of all continuous functions on the closed unit disk $\overline{\Delta} = \{z \in \mathbf{C} : |z| \leq 1\}$ that are holomorphic on Δ . It is not hard to see (Proposition 2.5 below) that H^∞ is uniformly natural on Δ precisely if $A(\overline{\Delta})$ is uniformly natural on $\overline{\Delta}$ (so a corollary of Carleson's theorem is uniform naturality of $A(\overline{\Delta})$ on $\overline{\Delta}$). It is well known and not difficult to see that $A(\overline{\Delta})$ is natural on $\overline{\Delta} : M_{A(\overline{\Delta})} = \overline{\Delta}$. Suppose one could find a "soft" proof that $\tau(\mathbf{N} \times \overline{\Delta})$ is dense in $M_{A(\overline{\Delta})}$. This would make $A(\overline{\Delta})$ uniformly natural on $\overline{\Delta}$, hence H^∞ would be uniformly natural on Δ . A possible by-product of such a development is this: The "soft" proof might well apply to polydisk algebras as well (a polydisk is a finite product of disks), and since Proposition 2.5 certainly does, the as yet open question of whether Δ^n is dense in the maximal ideal space of $H^\infty(\Delta^n)$ would be answered in the affirmative.

Proposition 2.5. $A(\overline{\Delta})$ is uniformly natural on $\overline{\Delta}$ if and only if H^∞ is uniformly natural on Δ .

Proof. Suppose first that $A = A(\overline{\Delta})$ is uniformly natural on $\overline{\Delta}$. Given $0 < \delta < 1$ and $n \in \mathbf{N}$, let M be a positive constant that can serve as $C(n, \delta)$ for A . Suppose $f_1, \dots, f_n \in H^\infty$ are such that $\|f_i\|_\infty \leq 1$ for

$1 \leq i \leq n$ and $\sum_{i=1}^n |f_i| \geq \delta$ on Δ . For $0 < r < 1$ and $1 \leq i \leq n$, let $f_{ir}(z) = f_i(rz)$, so $f_{ir} \in A$, $\|f_{ir}\|_\infty \leq 1$, and $\sum_{i=1}^n |f_{ir}| \geq \delta$ on $\overline{\Delta}$. By assumption there exist $g_{ir}, \dots, g_{nr} \in A$ such that $\|g_{ir}\|_\infty \leq M$ and $\sum_{i=1}^n f_{ir}g_{ir} = 1$ on $\overline{\Delta}$. For each i , $f_{ir} \rightarrow f_i$ pointwise (and uniformly on compact sets) on Δ as $r \rightarrow 1^-$. By a normal families argument, we may find a sequence $\langle r_k \rangle$, $0 < r_k < 1$, such that as $k \rightarrow \infty$, $r_k \rightarrow 1^-$ and $g_{ir_k} \rightarrow g_i$ pointwise (again, uniformly on compact sets) for appropriate functions $g_1, \dots, g_n \in H^\infty$. Then $\|g_i\|_\infty \leq M$ and $\sum_{i=1}^n f_i g_i = 1$ on Δ . Thus H^∞ is uniformly natural on Δ .

Conversely, suppose H^∞ is uniformly natural on Δ . Given $0 < \delta < 1$ and $n \in \mathbf{N}$, let M work as $C(n, \delta)$ for H^∞ . Fix $M' > M$. Take $\eta > 0$ so small that $nM\eta < 1$ and $M/(1 - nM\eta) \leq M'$. Suppose $f_1, \dots, f_n \in A$ satisfy $\|f_i\|_\infty \leq 1$ and $\sum_{i=1}^n |f_i| \geq \delta$ on $\overline{\Delta}$. By assumption, there exist $h_1, \dots, h_n \in H^\infty$ such that $\|h_i\|_\infty \leq M$ and $\sum_{i=1}^n f_i h_i = 1$ on Δ . Using uniform continuity of the f_i , take r , $0 < r < 1$, so close to 1 that $|f_i(rz) - f_i(z)| < \eta$ for $z \in \overline{\Delta}$ and $1 \leq i \leq n$. Let $F(z) = \sum_{i=1}^n f_i(z)h_i(rz)$ for $z \in \overline{\Delta}$. Then $F \in A$ and $|F(z) - 1| = |\sum_{i=1}^n [f_i(z) - f_i(rz)]h_i(rz)| < nM\eta$, so $|F| > 1 - nM\eta > 0$ on $\overline{\Delta}$. For $1 \leq i \leq n$, let $g_i(z) = h_i(rz)/F(z)$, so $g_i \in A$, $\|g_i\|_\infty \leq M'$, and $\sum_{i=1}^n f_i g_i = 1$ on $\overline{\Delta}$. We conclude that A is uniformly natural on $\overline{\Delta}$. \square

Again, more has been proven than was advertised. If δ and n are fixed, any constant that can be $C(n, \delta)$ for $A(\overline{\Delta})$ also works as $C(n, \delta)$ for H^∞ , while anything strictly larger than a $C(n, \delta)$ for H^∞ can serve as $C(n, \delta)$ for $A(\overline{\Delta})$.

In the best of all worlds, $A(\overline{\Delta})$ would be uniformly natural because of a general result that *all* uniform algebras (or even Banach function algebras) are uniformly natural on their maximal ideal spaces. This question was raised by Walter Rudin [3, p. 347]. If A is a uniform algebra on $X = M_A$, then do there exist appropriate constants $C(A, n, \delta)$ for all $0 < \delta < 1$ and $n \in \mathbf{N}$? If so, can these constants be chosen to be independent of the algebra A ? It is easy to see that an affirmative answer to the first question implies an affirmative answer to the second. For, suppose that for some n and δ , no $C(n, \delta)$ exists that works for all uniform algebras A . Then, for each $k \in \mathbf{N}$, there are a uniform algebra A_k on $X_k = M_{A_k}$ and functions $f_{k1}, \dots, f_{kn} \in A_k$ such that $\|f_{ki}\|_\infty \leq 1$ and $\sum_{i=1}^n |f_{ki}| \geq \delta$ on X_k , and whenever $\sum_{i=1}^n f_{ki}g_i = 1$

for some $g_1, \dots, g_n \in A_k$, it follows that $\max\{\|g_i\|_\infty : 1 \leq i \leq n\} > k$. Let X denote the one-point compactification of the disjoint union of the X_k , and let A consist of the continuous functions on X whose restriction to each X_k belongs to A_k . It is routine to check that A is a uniform algebra on X whose maximal ideal space coincides with X and that no constant can serve as $C(n, \delta)$.

Unfortunately, the answer to Rudin's question is negative, as shown by an ingenious example produced by Jean-Pierre Rosay [12].

Example 2.6 [12]. Let $Y = \{(\zeta_1, \zeta_2) \in \mathbf{C}^2 : 1/2 \leq \max\{|\zeta_1|, |\zeta_2|\} \leq 1\}$ and let $X = \{(\zeta_1, \zeta_2) \in Y : |\zeta_1 + \zeta_2| \geq 1/4\}$. We endow Y (and perforce X) with an equivalence relation \sim : $(\zeta_1, \zeta_2) \sim (\zeta'_1, \zeta'_2)$ if and only if $\zeta_1 = \pm\zeta'_1$ and $\zeta_2 = \pm\zeta'_2$. Note that every point of Y is equivalent to at least one point of X , and likewise for $\overset{\circ}{Y}$ and $\overset{\circ}{X}$, the interior of Y and X , respectively, in \mathbf{C}^2 . Thus, any function f on X (or $\overset{\circ}{X}$) that is constant on each equivalence class extends uniquely to such a function \tilde{f} on Y (or $\overset{\circ}{Y}$), and any continuous or holomorphic character of f will be enjoyed by \tilde{f} as well. We shall need a special case of Hartog's theorem from several complex variables: Every holomorphic function on $\overset{\circ}{Y}$ has a unique holomorphic extension to Δ^2 .

Let B denote the uniform algebra on X consisting of continuous functions that are holomorphic on $\overset{\circ}{X}$. The elements z_1 and z_2 of B are the coordinate functions: $z_1(\zeta_1, \zeta_2) = \zeta_1$ and $z_2(\zeta_1, \zeta_2) = \zeta_2$. $z_1 + z_2$ is invertible in B , so $\widehat{z}_1 + \widehat{z}_2 = z_1 + z_2$ can never vanish on M_B ; thus, \widehat{z}_1 and \widehat{z}_2 do not vanish simultaneously, whence \widehat{z}_1^2 and \widehat{z}_2^2 cannot vanish simultaneously on M_B , whence $|\widehat{z}_1^2| + |\widehat{z}_2^2| \geq \delta$ on M_B for some constant δ , $0 < \delta < 1$. Of course, $\|z_1^2\|_\infty = \|z_2^2\|_\infty = 1$.

Take an increasing sequence $\langle E_k \rangle$ of finite subsets of X whose union is dense in X such that each E_k is closed under \sim in X . If $p \in E_k$ and $q \in X$ satisfy $p \sim q$, then $q \in E_k$. Let B_k denote the subalgebra of B consisting of functions f such that $f(p) = f(q)$ whenever p and q are equivalent points of E_k . Of course, z_1^2 and z_2^2 belong to B_k . General Banach algebra theory shows that M_{B_k} is nothing but M_B with finitely many point identifications: two points of M_B are identified if and only if they are equivalent points of E_k . As a consequence, $|\widehat{z}_1^2| + |\widehat{z}_2^2| \geq \delta$

on M_{B_k} .

We shall now show that no single constant can serve as $C(2, \delta)$ for all the B_k . Indeed, suppose M were such a constant. For each k there would be g_{1k} and g_{2k} in B_k such that $\|g_{jk}\|_\infty \leq M$ and $z_1^2 g_{1k} + z_2^2 g_{2k} = 1$ on X . By a normal families argument, on passing to a subsequence if necessary, we may assume that there are holomorphic functions g_1 and g_2 on $\overset{\circ}{X}$ such that $g_{1k} \rightarrow g_1$ and $g_{2k} \rightarrow g_2$ pointwise (uniformly on compacta) on $\overset{\circ}{X}$ as $k \rightarrow \infty$. Evidently, $z_1^2 g_1 + z_2^2 g_2 = 1$ on $\overset{\circ}{X}$, and $g_1(p) = g_1(q)$, $g_2(p) = g_2(q)$ whenever p and q are equivalent points of $\overset{\circ}{X}$. Then g_1, g_2 have holomorphic extensions \tilde{g}_1, \tilde{g}_2 to $\overset{\circ}{Y}$ that still satisfy $z_1^2 \tilde{g}_1 + z_2^2 \tilde{g}_2 = 1$ on $\overset{\circ}{Y}$. This relation persists on Δ^2 if \tilde{g}_1 and \tilde{g}_2 are extended holomorphically, and this contradicts the fact that $z_1^2 = z_2^2 = 0$ at $(0, 0)$. \square

How does one prove that a Banach function algebra is uniformly natural? An easy example is $C(X)$. If $\|f_i\|_\infty \leq 1$ and $\sum_{i=1}^n |f_i| \geq \delta > 0$ on X , let $F = \sum_{i=1}^n |f_i|^2 \geq n^{-1} \delta^2$ and $g_i = F^{-1} f_i$. Then $\|g_i\|_\infty \leq n \delta^{-2} = C(n, \delta)$ and $\sum_{i=1}^n f_i g_i = 1$. By mimicking this argument, we can treat several other examples.

A Banach function space E on X is said to be *self-adjoint* if it is closed under conjugation: $f \in E$ implies $\bar{f} \in E$. An easy application of the closed graph theorem shows that then the real-linear mapping $f \mapsto \bar{f}$ on E is continuous; of course, if the norm is the uniform norm, this conclusion is immediate. A Banach function algebra A on X is *inverse-closed* if whenever $f \in A$ and f vanishes nowhere on X , it follows that $1/f \in A$. If A is a Banach function algebra on X , that is both self-adjoint and inverse-closed, then A is natural on X ; if f_1, \dots, f_n are functions in A that have no common zero in X , then $F = \sum_{i=1}^n f_i \bar{f}_i \in A$ and F has no zeros in X , so $g_i = F^{-1} \bar{f}_i \in A$ and $\sum_{i=1}^n f_i g_i = 1$. To obtain a version of this that includes estimates, we first make a definition.

Definition 2.7. A Banach function algebra A on X is *uniformly inverse-closed* (on X) if, whenever $0 < \delta < 1$, there is a positive constant $C(\delta)$ such that if $f \in A$ satisfies $\|f\| \leq 1$ and $|f| \geq \delta$ on X , it follows that there is a $g \in A$ such that $\|g\| \leq C(\delta)$ and $fg = 1$ on X .

Theorem 2.8. *Let A be a Banach function algebra on X . Suppose that A is both self-adjoint and uniformly inverse-closed on X . Then A is uniformly natural on X .*

Proof. Take a positive constant M such that $\|\bar{f}\| \leq M\|f\|$ for every $f \in A$. Suppose $0 < \delta < 1$ and $n \in \mathbf{N}$ are given. Let $f_1, \dots, f_n \in A$ satisfy $\|f_i\| \leq 1$ for $1 \leq i \leq n$ and $\sum_{i=1}^n |f_i| \geq \delta$ on X . Then $F = n^{-1} \sum_{i=1}^n f_i \bar{f}_i$ belongs to A and satisfies $\|F\| \leq 1$, $F \geq \delta' = n^{-2}\delta^2$ on X , so $F^{-1} \in A$ and $\|F^{-1}\| \leq C(\delta')$. If $g_i = n^{-1}F^{-1}\bar{f}_i$, then $g_i \in A$, $\|g_i\| \leq n^{-1}C(\delta')M$ for $1 \leq i \leq n$ and $\sum_{i=1}^n f_i g_i = 1$ on X . \square

The interpretation via sequence algebras is similar to Theorem 2.3 and has a similar but simpler proof. We merely state the result, for which no application will be exhibited here.

Theorem 2.9. *Let A be a Banach function algebra on X . Then the following are equivalent:*

- (1) *Whenever $\tilde{f} \in \tilde{A}$ and $|\tilde{f}| \geq \delta$ on $\mathbf{N} \times X$ for some $\delta > 0$, it follows that $\tilde{f}^{-1} \in \tilde{A}$.*
- (2) *A is uniformly inverse-closed.*

We conclude this section with two examples. Note that in both cases the last assertion is that $\tau(\mathbf{N} \times X)$ is dense in $M_{\tilde{A}}$, rather than that $M_{\tilde{A}} = \tilde{X}$. The reason is that, for reasons we will see in the next section, \tilde{A} definitely *does not* separate the points of \tilde{X} ; thus, $M_{\tilde{A}}$ is actually a certain quotient space of \tilde{X} .

2.10. The algebra $\text{Lip}_\alpha(\mathbf{T})$, $0 < \alpha \leq 1$. Let \mathbf{T} be the unit circle in the complex plane, and let $A = \text{Lip}_\alpha(\mathbf{T})$ be the subspace of $C(\mathbf{T})$ consisting of the functions f for which

$$\sup_{\substack{t \in \mathbf{T} \\ h \neq 0}} \frac{|f(t+h) - f(t)|}{|h|^\alpha} < \infty$$

with the norm

$$\|f\| = \|f\|_\infty + \sup_{\substack{t \in \mathbf{T} \\ h \neq 0}} \frac{|f(t+h) - f(t)|}{|h|^\alpha}.$$

Then A is a self-adjoint Banach function algebra on \mathbf{T} under pointwise multiplication and addition which is inverse-closed, and $M_A = \mathbf{T}$.

Now let $f \in A$ be such that $\|f\| \leq 1$ and $|f| \geq \delta > 0$ on \mathbf{T} . Take $C(\delta) = \delta^{-2}$. Then since $\|f^{-1}\|_\infty \leq \delta^{-2}\|f\|_\infty$ and

$$\sup_{\substack{t \in \mathbf{T} \\ h \neq 0}} \frac{|f^{-1}(t+h) - f^{-1}(t)|}{|h|^\alpha} \leq \delta^{-2} \sup_{\substack{t \in \mathbf{T} \\ h \neq 0}} \frac{|f(t+h) - f(t)|}{|h|^\alpha},$$

we have that $f^{-1} \in A$ and $\|f^{-1}\| \leq \delta^{-2}\|f\| \leq \delta^{-2} = C(\delta)$. Thus, A is uniformly inverse-closed and therefore $\tau(\mathbf{N} \times \mathbf{T})$ is dense in $M_{\bar{A}}$.

2.11 The algebra $C^{(m)}[0, 1]$. Let $m \in \mathbf{N}$, and let $A = C^{(m)}[0, 1]$ be the Banach function algebra of complex-valued continuous functions on the closed unit interval $[0, 1]$ which are m -times continuously differentiable with the norm

$$\|f\| = \sum_{k=0}^m \frac{\|f^{(k)}\|_\infty}{k!}.$$

Then A is self-adjoint and natural on $[0, 1]$. Moreover, A is uniformly inverse-closed by taking $C(\delta) = \sum_{k=0}^m (\lambda_k/k!)$ where $\lambda_k = \lambda_k(\delta, 1)$ are chosen as in the lemma below. Thus, A is uniformly natural on $[0, 1]$, and therefore $\tau(\mathbf{N} \times [0, 1])$ is dense in $M_{\bar{A}}$.

Lemma. *Let $0 < a < b$ be two real numbers. For each integer $k \geq 0$, there exist a positive constant $\lambda_k = \lambda_k(a, b)$ and a real polynomial in $k + 1$ variables, $P_k = P_k(\zeta_0, \zeta_1, \dots, \zeta_k)$ such that if $m \in \mathbf{N}$ and if $h \in C^{(m)}[0, 1]$ satisfies $|h| \geq a$ on $[0, 1]$ and $\|h\| \leq b$, then for $k = 0, 1, \dots, m$ and $u = 1/h \in C^{(m)}[0, 1]$ we have*

$$u^{(k)} = \frac{d^k u}{dt^k} = u^{k+1} P_k(h, h', \dots, h^{(k)}),$$

$$\|u^{(k)}\|_\infty \leq \lambda_k.$$

Proof. Induction provides the P_k with $P_0(\zeta_0) = 1$,

$$P_{k+1}(\zeta_0, \zeta_1, \dots, \zeta_{k+1}) \\ = -(k+1)\zeta_1 P_k(\zeta_0, \dots, \zeta_k) + \sum_{j=0}^k \zeta_1 \zeta_{j+1} \frac{\partial P_k}{\partial \zeta_j}(\zeta_0, \dots, \zeta_k).$$

Then take $\lambda_k = \lambda_k(a, b) = a^{-(k+1)} \sup\{|P_k(\zeta_0, \dots, \zeta_k)| : \sum_{j=0}^k (|\zeta_j|/j!) \leq b\}$. \square

3. The ultraseparation property. Consider a Banach function algebra A on $X = M_A$. In Section 2 we studied the map $\tau : \mathbf{N} \times X \rightarrow M_{\tilde{A}}$ given by $\tau(n, \phi)(\tilde{f}) = \widehat{f_n}(\phi)$ where $\tilde{f} = \langle f_k \rangle_{k=1}^\infty \in \tilde{A}$ and saw that one formulation of A being uniformly natural on X is that $\tau(\mathbf{N} \times X)$ is dense in $M_{\tilde{A}}$. This can be slightly reformulated if we note that τ has a continuous extension $\tilde{\tau}$ from $\tilde{X} = \beta(\mathbf{N} \times X)$ into $M_{\tilde{A}}$. Thus A is uniformly natural on X if and only if $\tilde{\tau} : \tilde{X} \rightarrow M_{\tilde{A}}$ is onto. It is now natural to ask: What happens if $\tilde{\tau}$ is one-to-one? This possibility can be formulated so as to apply to Banach function *spaces* as well as algebras and gives us the following fundamental concept introduced by Bernard.

Definition 3.1. A Banach function space E on X is said to be *ultraseparating* on X if \tilde{E} , regarded as a Banach space of continuous functions on \tilde{X} , separates the points of \tilde{X} .

Various characterizations of this property have been proven, but we shall ignore them. Instead, we shall try to indicate some of the ways in which ultraseparability can be used.

For a more comprehensive view of the “state of the art” in the late 1960’s and early 70’s, when Bernard was creating and applying these ideas, we call the reader’s attention to Robert B. Burckel’s monograph [4]. The proofs below are substantially those of Bernard. In particular, the use of sequence spaces in the proofs of Bernard’s generalization of the Hoffman-Wermer theorem, Theorem 3.4 and of Wermer’s theorem 3.5, is completely different from earlier proofs of similar results. The key tool for many of the applications is the following result, for which scalars may be real or complex.

Bernard's lemma 3.2 [1], [2]. *Let E be a Banach function space on X . If \tilde{E} , regarded as a Banach space of continuous functions on \tilde{X} , is uniformly dense in $C(\tilde{X})$, then $E = C(X)$ (and consequently, $\tilde{E} = C(\tilde{X})$).*

Proof. For simplicity let N denote the norm on E and $\| \cdot \|$ the uniform norm on the appropriate space. We first prove existence of a positive integer k such that whenever $u \in C(X)$ and $\|u\| \leq 1$, there is an $f \in E$ such that $N(f) \leq k$ and $\|f - u\| \leq 1/2$. Indeed, if no such k exists, then for each $k \in \mathbf{N}$ there is a $u_k \in C(X)$ such that $\|u_k\| \leq 1$ and $N(f) > k$ whenever $f \in E$ satisfies $\|f - u_k\| \leq 1/2$. Consider $\tilde{u} = \langle u_k \rangle_{k=1}^\infty \in C(X)^\sim = C(\tilde{X})$. By the hypothesized density there is $\tilde{f} = \langle f_k \rangle_{k=1}^\infty \in \tilde{E}$ such that $\|\tilde{f} - \tilde{u}\| \leq 1/2$. This says that, for each $k \in \mathbf{N}$, $\|f_k - u_k\| \leq 1/2$, hence $N(f_k) > k$. This in turn contradicts the fact that the sequence $\langle N(f_k) \rangle$ is bounded because $\tilde{f} \in \tilde{E}$. Renormalizing, we have shown that there is a positive constant M that satisfies

$$(**) \quad \begin{array}{l} \text{If } u \in C(X), \text{ then there is an } f \in E \text{ such} \\ \text{that } N(f) \leq M\|u\| \text{ and } \|f - u\| \leq 2^{-1}\|u\|. \end{array}$$

Now let $u \in C(X)$. Let $u_1 = u$ and take $f_1 \in E$ such that $N(f_1) \leq M\|u_1\|$ and $\|u_2\| \leq 2^{-1}\|u_1\|$ where $u_2 = u_1 - f_1$. Then take $f_2 \in E$ satisfying $N(f_2) \leq M\|u_2\|$ and $\|u_3\| \leq 2^{-1}\|u_2\|$ where $u_3 = u_2 - f_2$. Continuing in this fashion using (**), we obtain sequences $u = u_1, u_2, \dots$ in $C(X)$ and f_1, f_2, \dots in E with $u_{k+1} = u_k - f_k$, $N(f_k) \leq M\|u_k\|$, and $\|u_{k+1}\| \leq 2^{-1}\|u_k\|$. By induction $\|u_k\| \leq 2^{1-k}\|u\|$, hence $N(f_k) \leq 2^{1-k}M\|u\|$. The series $\sum_{k=1}^\infty f_k$ then converges in E , say to $F \in E$, with $N(F) \leq \sum_{k=1}^\infty 2^{1-k}M\|u\| = 2M\|u\|$. Then also this series converges to F uniformly on X . But $\|u - \sum_{j=1}^k f_j\| = \|u_{k+1}\|$ which tends to 0 as $k \rightarrow \infty$, so the series also converges uniformly to u . Thus, $u = F \in E$. \square

The following corollary shows that, as anticipated, the algebras in examples 2.10 and 2.11 are not ultraseparating.

Corollary 3.3. *Let A be a Banach function algebra on X . If A is both self-adjoint and ultraseparating on X , then $A = C(X)$.*

Proof. The ultraseparation hypothesis shows that \tilde{A} is a Banach function algebra on \tilde{X} . Because the conjugation mapping on A is continuous, \tilde{A} is also self-adjoint on \tilde{X} . The Stone-Weierstrass theorem implies that \tilde{A} is uniformly dense in $C(\tilde{X})$, and then Bernard's lemma gives $A = C(X)$. \square

Perhaps the first application of ultraseparability was Bernard's extension of the Hoffman-Wermer theorem [9]. The Hoffman-Wermer theorem asserts that if A is a uniform algebra on X and if

$$\begin{aligned} \operatorname{Re} A &= \{\operatorname{Re} f : f \in A\} \\ &= \{u \in C_{\mathbf{R}}(X) : u + iv \in A \text{ for some } v \in C_{\mathbf{R}}(X)\} \end{aligned}$$

is uniformly closed, then $A = C(X)$. A number of authors (initially [14]) later showed that if A is a uniform algebra on X , if K is a nonempty closed subset of X , and if $(\operatorname{Re} A)|_K$, the space of restrictions to K of functions from $\operatorname{Re} A$, is uniformly closed, then $A|_K = C(K)$. If we observe that $A|_K$, the restriction algebra, is actually a Banach function algebra on K in an appropriate quotient norm, and that $(\operatorname{Re} A)|_K = \operatorname{Re}(A|_K)$, then we see that this is in turn a particular case of Bernard's result.

Theorem 3.4 [1], [2]. *Let A be a Banach function algebra on X . Suppose that $\operatorname{Re} A$ is uniformly closed. Then $A = C(X)$.*

Proof. Let B denote the uniform closure of A . Then B is a uniform algebra on X and $\operatorname{Re} B = \operatorname{Re} A$ so, by the Hoffman-Wermer theorem $B = C(X)$, hence $\operatorname{Re} A = C_{\mathbf{R}}(X)$. Letting N denote the norm in A and $\|\cdot\|$ the uniform norm on X , the open mapping theorem applied to the continuous real-linear surjection $f \mapsto \operatorname{Re} f$ from A to $\operatorname{Re} A = C_{\mathbf{R}}(X)$ shows that there is a positive constant M such that whenever $u \in C_{\mathbf{R}}(X)$, there is an $f \in A$ for which $\operatorname{Re} f = u$ and $N(f) \leq M\|u\|$. Now suppose $\tilde{u} = \langle u_k \rangle_{k=1}^{\infty} \in C_{\mathbf{R}}(X)^{\sim} = C_{\mathbf{R}}(\tilde{X})$. For each $k \in \mathbf{N}$ there is an $f_k \in A$ that satisfies $\operatorname{Re} f_k = u_k$ and $N(f_k) \leq M\|u_k\|$. Thus $\tilde{f} = \langle f_k \rangle_{k=1}^{\infty} \in \tilde{A}$ and $\operatorname{Re} \tilde{f} = \tilde{u}$. This gives $\operatorname{Re} \tilde{A} = C_{\mathbf{R}}(\tilde{X})$ so, as in the beginning of the proof, the uniform closure of \tilde{A} is all of $C(\tilde{X})$, i.e., \tilde{A} is dense in $C(\tilde{X})$. Then Bernard's lemma gives $A = C(X)$. \square

The main applications of ultraseparating function spaces have been to the symbolic calculus on real parts of Banach function spaces, especially uniform algebras. From our point of view, the first result along these lines was published by John Wermer in 1963:

Theorem 3.5 [15]. *Let A be a uniform algebra on X . If $\operatorname{Re} A$ is closed under multiplication, then $A = C(X)$.*

The special hypothesis here, that uv belongs to $\operatorname{Re} A$ whenever u and v both do, can be rephrased to say that $\phi(t) = t^2$ operates on $\operatorname{Re} A$ in the sense that the composition $\phi \circ u \in \operatorname{Re} A$ whenever $u \in \operatorname{Re} A$. Now if E is any Banach function space, then every affine function $\phi(t) = at + b$ surely operates on E . In view of Wermer's theorem (and to be sure, others which are important but do not lie directly on the path we are following), it is natural to ask *which* continuous functions operate on *which* Banach function spaces, and in *what* manner (e.g., boundedness conditions)? A number of results in this vein are proved in Bernard's seminal paper [2], including the following: If A is a uniform algebra on X and if $|u| \in \operatorname{Re} A$ whenever $u \in \operatorname{Re} A$, that is, $\phi(t) = |t|$ operates on $\operatorname{Re} A$, then $A = C(X)$. All this made the following conjecture natural.

Conjecture. *Let A be a uniform algebra on X , let I be an interval of real numbers, and let ϕ be a real-valued function on I that is not the restriction to I of an affine function on \mathbf{R} . If ϕ operates on $\operatorname{Re} A$, in the sense that $\phi \circ u \in \operatorname{Re} A$ whenever $u \in \operatorname{Re} A$ and u has range in I , then $A = C(X)$.*

In [13] (see also [10] and Theorem 2.5.6 in [11]) the results of Wermer and Bernard on this conjecture were extended to several situations, most notably to that in which I contains no nondegenerate subinterval on which ϕ is the restriction of an affine function. Finally, in [7] Hatori completed the proof of the conjecture in general. Substantial later work has extended these developments in various directions, but we shall content ourselves with showing how Bernard's methods apply to give a proof of Wermer's theorem quite different from Wermer's original one.

Proof of Theorem 3.5 [2]. Let $\| \cdot \|$ be the uniform norm, and let N be the norm on $\operatorname{Re} A$ as a quotient space of A : $N(u) = \inf \{ \|f\| : f \in A, \operatorname{Re} f = u \}$. As in Wermer's paper, the closed graph theorem shows that multiplication is continuous on $\operatorname{Re} A$: there is a constant $M > 0$ such that $N(uv) \leq MN(u)N(v)$ whenever $u, v \in \operatorname{Re} A$. This is proved in two stages: a first application of the closed graph theorem shows that for each $u \in \operatorname{Re} A$ the linear mapping $v \mapsto uv$ from $\operatorname{Re} A$ to itself is a *bounded* linear mapping $L(u)$, and a second application shows that the linear mapping $u \mapsto L(u)$ from $\operatorname{Re} A$ to the space of bounded linear operators on $\operatorname{Re} A$ is bounded.

The immediate and crucial consequence of the existence of M is that $(\operatorname{Re} A)^\sim = \operatorname{Re}(\tilde{A})$ is actually closed under multiplication. So if we can show that $(\operatorname{Re} A)^\sim$ separates the points of \tilde{X} , i.e., that $\operatorname{Re} A$ is ultraseparating on X , then the Stone-Weierstrass theorem will imply that $(\operatorname{Re} A)^\sim$ is uniformly dense in $C_{\mathbf{R}}(X)^\sim = C_{\mathbf{R}}(\tilde{X})$; Bernard's lemma will then give $\operatorname{Re} A = C_{\mathbf{R}}(X)$, and the Hoffman-Wermer theorem will yield $A = C(X)$.

It suffices, then, to prove that $\operatorname{Re} \tilde{A}$ separates the points of \tilde{X} or, equivalently, that \tilde{A} does. Now the Stone-Weierstrass theorem forces $\operatorname{Re} A$ to be uniformly dense in $C_{\mathbf{R}}(X)$ (a uniform algebra with this property is said to be *Dirichlet*). Therefore, if K and L are disjoint compact subsets of X , $u \in \operatorname{Re} A$ exists such that $u < 0$ on K , $u > 1$ on L and $u < 2$ on X ; taking $v \in \operatorname{Re} A$ such that $u + iv \in A$ and setting $f = \exp(u + iv)$, we have $f \in A$, $|f| < 1$ on K , $|f| > e$ on L , and $\|f\| < e^2$. Suppose now that ϕ and ψ are distinct points in \tilde{X} . Let U and V be disjoint compact neighborhoods of ϕ and ψ , respectively. For each $k \in \mathbf{N}$, $K_k = \{x \in X : \tau(k, x) \in U\}$ and $L_k = \{x \in X : \tau(k, x) \in V\}$ are disjoint compact subsets of X , so there is an $f_k \in A$ that satisfies $|f_k| < 1$ on K_k , $|f_k| > e$ on L_k and $\|f_k\| < e^2$. Then $\tilde{f} = \langle f_k \rangle_{k=1}^\infty$ belongs to \tilde{A} , $|\tilde{f}| < 1$ on $\cup_k \tau(\{k\} \times K_k)$ and $|\tilde{f}| > e$ on $\cup_k \tau(\{k\} \times L_k)$. ϕ belongs to the closure in \tilde{X} of the first of these unions, and ψ belongs to the closure of the second. Therefore, $|\tilde{f}(\phi)| \leq 1$ and $|\tilde{f}(\psi)| \geq e$ and in particular $\tilde{f}(\phi) \neq \tilde{f}(\psi)$. This completes the proof. \square

We shall close with a few remarks on the preceding proof and how to extend it. There are two ways in which the fact that the operating function is $\phi(t) = t^2$ is used. First, from the fact that ϕ operates on

$\operatorname{Re} A$ it is deduced that ϕ operates on $(\operatorname{Re} A)^\sim$. Second, the Stone-Weierstrass theorem is applied twice. For more general ϕ , appropriate variants of these steps must be found. Typically, the Baire category theorem is used to show that composition by ϕ takes a dense subset B_0 of some ball B in $\operatorname{Re} A$ (or, in some cases, in an appropriate subspace of $\operatorname{Re} A$) into a *bounded* subset of $\operatorname{Re} A$. It follows that ϕ “operates” from a small part of $(\operatorname{Re} A)^\sim$, namely those sequences whose members come from B_0 , into $(\operatorname{Re} A)^\sim$ and so, by the continuity of ϕ (which can be deduced from the hypotheses in the conjecture), composition by ϕ carries sequences from B into the *uniform closure* of $(\operatorname{Re} A)^\sim$. Is this enough to make $(\operatorname{Re} A)^\sim$ uniformly dense in $C_R(\tilde{X})$? Obviously not, since at one extreme, all the functions in B might have range contained in some interval on which ϕ is affine! In [13], an extra hypothesis is inserted in Theorem 2 to avoid precisely this difficulty, but generally considerable effort must be expended to apply the category argument in such a manner that the resulting ball B is well located.

Even supposing this has been accomplished, formidable technical difficulties may remain. Since $\operatorname{Re} A$ (and perforce $(\operatorname{Re} A)^\sim$) is not known to be closed under multiplication, Stone-Weierstrass is unavailable. Fortunately, de Leeuw and Katznelson provided a viable substitute [6]: If E is a subspace of $C_R(X)$ that contains the constant functions and separates the points of X , and if ϕ is a continuous real-valued function on some interval I such that ϕ is not the restriction to I of an affine function, then E is uniformly dense in $C_R(X)$ if ϕ “operates” on E in the sense that $\phi \circ u \in E$ whenever $u \in E$ has range contained in I . (Actually, this is the *real* version of the theorem in [6], in which E is a *complex* function space and ϕ is defined on an open set of *complex* numbers). In our situation, with $E = \operatorname{Re} A$, it gives density of $\operatorname{Re} A$ in $C_R(X)$ immediately, so the first application of Stone-Weierstrass in the proof of Theorem 3.5 is covered. Unfortunately, the second – density of $(\operatorname{Re} A)^\sim$ in $C_R(\tilde{X})$ – is much harder because ϕ may operate on too small a subset of $(\operatorname{Re} A)^\sim$. To get a general idea of what must be done, let’s examine the proof of the de Leeuw–Katznelson theorem.

Here is a sketch of the proof, following Bernard [2, Appendices]. Assume E is uniformly closed. For a well-chosen C^∞ function k of small support near zero, the convolution $\phi_1 = \phi * k$ will be close enough to ϕ to be nonaffine, while it still operates on E , $\phi_1 \circ u = \int k(t)\phi \circ (u - t) dt$, and is C^∞ . Because ϕ_1 is not affine ϕ_1'' is not identically zero; with some

linear changes of variable, we can assume $\phi_1(t) = t^2 + t^2\varepsilon(t)$ where $\varepsilon(t)$ tends to zero with t . If $u \in E$, $n^2\phi_1 \circ (u/n) \rightarrow u^2$, so $u^2 \in E$. Now Stone-Weierstrass applies.

To apply all this in *our* situation, let V denote the uniform closure of $(\operatorname{Re} A)^\sim$ in $C_R(\tilde{X})$, and let H be its “multiplier algebra” consisting of those $\tilde{u} \in C_R(\tilde{X})$ for which $\tilde{u}\tilde{v} \in V$ whenever $\tilde{v} \in V$ (it suffices to consider $\tilde{v} \in (\operatorname{Re} A)^\sim$). H is a uniformly closed subalgebra of $C_R(\tilde{X})$ that contains the constant functions and is contained in V . If H separates the points of \tilde{X} , then $H = C_R(\tilde{X})$ so $V = C_R(\tilde{X})$. If ϕ_1 is as in the proof sketched above, then ϕ_1 carries sequences from our selected ball B into V . If now \tilde{u} is such a sequence and if $\tilde{v} \in (\operatorname{Re} A)^\sim$ is arbitrary, the difference quotients $(\phi_1 \circ (\tilde{u} + t\tilde{v}) - \phi_1 \circ \tilde{u})/t$ belong to V for small nonzero t and tend to $(\phi_1' \circ \tilde{u})\tilde{v}$ as $t \rightarrow 0$, so $(\phi_1' \circ \tilde{u})\tilde{v} \in V$. Thus $\phi_1' \circ \tilde{u} \in H$, and the problem becomes to find enough functions of this type to separate the points of \tilde{X} . Even this can't *always* be done in any obvious way, but after various reductions, something very much like it *can*, ultimately yielding a proof of the full conjecture.

The investigation of functions operating on general Banach function spaces continues to flourish, and the methods we have discussed continue to play a prominent role. The “ultraseparation” property is of some interest in its own right. We close with a problem in this area.

Problem. *If A_1 and A_2 are ultraseparating uniform algebras on X_1 and X_2 , respectively, and if A is the uniformly closed linear span on $X = X_1 \times X_2$ (topological product) of the functions $f_1 \otimes f_2(x_1, x_2) = f_1(x_1)f_2(x_2)$ where $f_i \in A_i$, $i = 1, 2$, must A be ultraseparating on X ?*

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