ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 31, Number 2, Summer 2001

ON JACOBI'S THEOREM IN HAMILTON-JACOBI THEORY

HANS SAMELSON

ABSTRACT. Jacobi's theorem states that a complete integral of the Hamilton partial differential equation for a given Hamiltonian determines in a simple way all the trajectories of the Hamiltonian flow. It is usually proved by appealing to the theory of canonical transformations. Our approach consists in noting a fact which is actually at the center of the existing proofs, whose proof is just a simple differentiation, and which doesn't seem to have been noticed so far: Given a one-parameter family of solutions of the Hamilton-Jacobi differential equation, its partial derivative with respect to the parameter is an integral for the corresponding field curves. Jacobi's theorem is an immediate consequence of this, without any further computation.

We recall: A Hamiltonian is a function H of $2n+1$ real variables q_i, p_i, t with $i = 1, \dots, n$, defined in some open set D in $\mathbf{R}^{2n+1}$. (We shall use notations like q for the point $(q_1, q_2, \dots, q_n)$ and H_q for the sequence of partial derivatives $(H_{q_1}, \dots, H_{q_n})$).

A trajectory or extremal (for H) is a curve $(q(t), p(t), t)$ in $\mathbf{R}^{2n+1}$, defined on some t-interval, that lies in D and satisfies the canonical or Hamilton equations

$$dq/dt = H_p(q(t), p(t), t), \qquad dp/dt = -H_q(q(t), p(t), t). \tag{0.1}$$

There is the associated Lagrangian, a function L of $2n+1$ variables $q_1, q_2, \dots, q_n,\ \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n, t$. First one maps the original domain D to a domain D' in $(q, \dot{q}, t)$-space by the map (assumed to be a diffeomorphism) given by the identity on q and t and

$$\dot{q} = H_p(q, p, t). \tag{0.2}$$

Then one defines L by the relation

$$L(q, \dot{q}, t) + H(q, p, t) = \sum p_i \cdot \dot{q}_i. \tag{0.3}$$

Support by NSF DMS 91-02078 is acknowledged.
Received by the editors on October 26, 1998.

(The inverse map to (0.2) has $p = L_{\dot{q}}$.) The H-trajectories, or rather their projections into (q,t)-space, are the curves for which the first variation of $\int L\,dt$ (with fixed endpoints) vanishes.

A function $S = S(q,t)$, defined in some suitable open set in (q,t)-space, is called a field function, if it is a solution of the Hamilton-Jacobi partial differential equation

$$S_t(q,t) + H(q, S_q(q,t), t) = 0. \tag{0.4}$$

(The argument of H is assumed to be a point of D.)

The family of curves in (q,t)-space satisfying the first order ordinary differential system

$$dq/dt = H_p(q(t), S_q(q(t),t), t) \tag{0.5}$$

is called the (Mayer) field associated to S.

The curves $(q(t), p(t), t)$ with $p(t) = S_q(q(t),t)$ are then H-trajectories. (The simple proof uses the derivatives with respect to q_i of equation (0.4)).

We come to our main device; it is at the heart of all proofs of Jacobi's theorem.

Main lemma. *Let $S(q,u,t)$ (or $S^u(q,t)$) be a family of field functions (solutions of (0.4)) depending on a real parameter u. Then the partial derivative $\partial/\partial u S(q,u,t)$ $(= S_u(q,u,t))$ is constant along any curve of the field associated with S^u, i.e., it is an "integral" of the system (0.5).*

Proof. We differentiate the relation (0.4) (which by assumption holds for all S^u) with respect to u, getting

$$\partial/\partial u S_t + \sum H_{p_i} \cdot \partial/\partial q_i S_u = 0. \tag{0.6}$$

In particular this holds along the field curves $(q,t) = (q(t),u,t)$ for S^u. With relation (0.5) it transforms into

$$\partial/\partial t S_u + \sum \partial/\partial q_i S_u \cdot dq_i/dt = 0. \tag{0.7}$$

And that says

$$d/dtS_u = 0$$

along the field curves of S^u. □

Remark. The lemma has a geometrical meaning; it says that a certain variation of a field curve, constructed from the family S^u, has first variation equal to 0.

First, it is a standard and basic fact that for any two points $Q^1 = (q^1, t^1)$, $Q^0 = (q^0, t^0)$ on a field curve for a field function S the difference of the values of S at the two points equals the integral of the Lagrangian L along the field curve from Q^0 to Q^1 ("action integral").

Let $\Gamma^{\bar{u}}$, for a given $\bar{u}$, be a field curve to $S^{\bar{u}}$ from Q^0 to Q^1. The promised variation assigns to any u near $\bar{u}$ the field curve of S^u through Q^0, extended to its point of intersection with the level surface of S^u through Q^1. Thus the initial end point is fixed under the variation, but the final one is not. (This construction assumes that the differential $dS^{\bar{u}}$ does not vanish on the tangent vector to $\Gamma^{\bar{u}}$ at Q^1).

One can show that the final end point curve is transversal in the usual sense (Hilbert invariant form) to $\Gamma^{\bar{u}}$ at Q^1. Thus, for this variation, the endpoint contributions to the first variation vanish, and so does of course the integral term; and so the first variation, i.e., the u-derivative of the action integral in this family of curves at $\bar{u}$ is equal to 0. But by an earlier remark this action integral equals $S(q^1, u, t^1) - S(q^0, u, t^0)$ and so $S_u(q^1, \bar{u}, t^1) - S_u(q^0, \bar{u}, t^0)$ vanishes.

We come to Jacobi's theorem.

Let $S = S(q, a, t) = S^a(q, t)$ be a function defined in some open set in (q, a, t)-space; here a means $(a_1, \ldots, a_n)$. (Thus S is an n-parameter family of functions of (q, t)).

S is called a complete integral of the Hamilton-Jacobi equation (0.4) (at a given point) if

(1) for each fixed a the function S^a satisfies equation (0.4) and

(2) the functional determinant $\det(\partial^2/\partial q_i \partial a_j S(q, a, t)$ is not 0 at the point.

Condition (2) means that near the point one can solve the equations

$$p = S_q(q, a, t) \tag{0.8}$$

for a in terms of (q, p, t) and similarly the equations

$$S_a(q, a, t) = b \tag{0.9}$$

for q in terms of (a, b, t). (Here $b = (b_1, \dots, b_n)$ is a new set of n variables). More globally one would require that equations (0.8) (together with the identity on q and t) define a diffeomorphism ϕ' of some open set E in (q, a, t)-space with an open set E' in (q, p, t)-space (of course contained in the original set D there), and similarly equations (0.9) define a diffeomorphism ϕ'' of E with an open set E'' in (a, b, t)-space.

Jacobi's theorem. *For each* (a, b) *a solution curve* $q = q(a, b, t)$ *of the equations* (0.9) (*with* (a, b, t) *in* E''), *together with* $p = S_q(q(a, b, t), a, t)$, *is an* H*-trajectory. Conversely, for every* H*-trajectory* $(q(t), p(t), t)$ *in* E' *there is an* a *such that* $S_a(q(t), a, t)$ *is constant, equal to some* b, *with* (a, b, t) *in* E''.

Thus the trajectories are "parametrized" by a and b; they are obtained by solving the equations $S_a(q, a, t) = b$ for q in terms of a, b and t.

Proof. As described above, a field curve $q(t)$ for S^a defines a trajectory (by putting $p(t) = S_q(q(t), a, t)$). Every trajectory (in E') appears this way because our map ϕ' from E to E' is a diffeomorphism. (One uses here of course the usual uniqueness, existence, etc., theorems for solutions of a system of ordinary differential equations under the appropriate differentiability conditions).

On the other hand, by the main lemma, the "integrals" $S_a(q, a, t)$ are equal to constants b along any field curve $q(t)$ to S^a. Again, since the map ϕ'' from E to E'' is a diffeomorphism, this correspondence between the field curves for all the possible a-values and the "lines" (a, b, t) for fixed (a, b) is bijective. Putting the two together, the diffeomorphism $\phi' \circ (\phi'')^{-1}$ from E'' to E' gives a bijective correspondence of the set of lines (a, b, t) for fixed (a, b) in E'' with the set of trajectories in E'. □

Note. The representation of the trajectories by the t-lines in (a, b, t)-space here is different from the standard representation as the t-lines in (q_0, p_0, t)-space with q_0, p_0 the initial values for some t_0.

After this was written the author noticed that the proof given here is related to (but simpler than) the two proofs appearing on pages 368–372 in [**1**].

REFERENCES

1. M. Giaquinta-St. Hildebrandt, *Calculus of variations* II, Springer-Verlag, Berlin, New York, 1996.

Department of Mathematics, Stanford University, Stanford, CA 94305
E-mail address: samelson@math.stanford.edu