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GENERAL HELICES IN THE THREE-DIMENSIONAL LORENTZIAN SPACE FORMS

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ABSTRACT. We present some Lancret-type theorems for general helices in the three-dimensional Lorentzian space forms which show remarkable differences with regard to the same question in Riemannian space forms. The key point will be the problem of *solving natural equations*. We give a geometric approach to that problem and show that general helices in the three-dimensional Lorentz-Minkowskian space are geodesics either of right general cylinders or of flat B-scrolls. In this sense the anti De Sitter and De Sitter worlds behave as the spherical and hyperbolic space forms, respectively.

1. Introduction. A general helix in the Euclidean space $\mathbf{R}^3$ is a curve which forms a constant angle with a fixed direction in $\mathbf{R}^3$, that is, its tangent indicatrix is a planar curve. The line perpendicular to that plane is called the axis of the general helix. A classical result stated by Lancret in 1802 and first proved by de Saint Venant in 1845 (see [**11**] for details) says that a "curve is a general helix if and only if the ratio of curvature to torsion is constant."

Given a pair of functions, one would like to get an arclength parametrized curve for which that couple works as the curvature and torsion functions. This problem is classically known as the *solving natural equations problem* (see [**11**]). The natural equations for general helices can be integrated in $\mathbf{R}^3$ and in the three-sphere $\mathbf{S}^3$, showing that general helices are geodesics either of right general cylinders or of Hopf cylinders, according to whether the curve lies in $\mathbf{R}^3$ or $\mathbf{S}^3$, respectively (see [**3**] for further details). The hyperbolic space is poor in these kinds of curves because the only general helices are the right circular ones.

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This paper concerns general helices in the three-dimensional Lorentzian space forms. A nonnull curve γ immersed in $\mathbf{L}^3$ is called a general helix if its tangent indicatrix is contained in some plane $\Pi \subset \mathbf{L}^3$. It is well known that Π can be either degenerate or nondegenerate, according to whether the endowed metric is indefinite or Riemannian, respectively. Therefore, we consider both cases separately and call degenerate and nondegenerate general helices, respectively. Then we give a Lancret theorem for general helices in $\mathbf{L}^3$ which formally agrees with the classical one. Actually we prove that "general helices in $\mathbf{L}^3$ correspond with nonnull curves in $\mathbf{L}^3$ for which the ratio of curvature to torsion is constant."

Concerning the behavior of general helices in Euclidean and Lorentzian geometries, we will point out a substantial difference. Whereas general helices in $\mathbf{R}^3$ are geodesics in right general cylinders, we show that general helices in $\mathbf{L}^3$ are geodesics in either right general cylinders or flat B-scrolls, according to whether the general helix is nondegenerate or degenerate (see Theorems 4 and 5), respectively. This nice difference between Euclidean and Lorentzian geometries (from the point of view of the behavior of general helices) confirms once more the important role of B-scroll (see [**1**], [**3**] and [**5**]) in Lorentzian geometries.

To extend the concept of general helix to the three-dimensional De Sitter $\mathbf{S}^3_1$ and anti De Sitter $\mathbf{H}^3_1$ spaces, we use the notion of Killing vector field along a curve in a three-dimensional real space form (see [**9**]). The Lancret theorems in $\mathbf{S}^3_1$ and $\mathbf{H}^3_1$ underline deep differences between pseudospherical and pseudohyperbolic spaces. The pseudohyperbolic case is nicely analogous to the Lorentz-Minkowskian one, whereas in the pseudospherical case there are no general helices. From this point of view, the roles played by nonflat Lorentzian space forms $\mathbf{H}^3_1$ and $\mathbf{S}^3_1$ correspond with those played by nonflat Riemannian space forms $\mathbf{S}^3$ and $\mathbf{H}^3$, respectively (see [**2**]).

Finally, our interest in studying general helices on these backgrounds comes from the interplay between geometry and integrable Hamiltonian systems (see [**3**], [**7**] and [**8**]).

2. Setup. Let $\mathbf{R}_t^{n+2}$ be the $(n+2)$-dimensional pseudo-Euclidean space with index t endowed with the indefinite inner product given by

$$\langle x, y\rangle = -\sum_{i=1}^{t} x_i y_i + \sum_{j=t+1}^{n+2} x_j y_j,$$

where $(x_1, \dots, x_{n+2})$ is the usual coordinate system. Let $\mathbf{S}_\nu^{n+1} = \{x \in \mathbf{R}_\nu^{n+2} : \langle x, x\rangle = 1\}$ and $\mathbf{H}_\nu^{n+1} = \{x \in \mathbf{R}_{\nu+1}^{n+2} : \langle x, x\rangle = -1\}$ be the unit pseudo-sphere and the unit pseudo-hyperbolic space, respectively. They are pseudo-Riemannian hypersurfaces of index ν in $\mathbf{R}_\nu^{n+2}$ and $\mathbf{R}_{\nu+1}^{n+2}$, respectively, with constant sectional curvature $c = +1$ and $c = -1$, respectively. Throughout this paper M will denote $\mathbf{S}_1^3, \mathbf{H}_1^3$ or $\mathbf{L}^3$ according to $c = +1$, $c = -1$ or $c = 0$, respectively, and E will stand for the pseudo-Euclidean space where M is lying.

Let p be a point in M and $v \in T_pM$ a tangent vector. Then v is said to be spacelike, timelike or null according to $\langle v, v\rangle > 0$, $\langle v, v\rangle < 0$ or $\langle v, v\rangle = 0$ and $v \neq 0$, respectively. Notice that the vector $v = 0$ is spacelike. The category into which a given tangent vector falls is called its casual character. These definitions can be generalized for curves as follows. A curve γ in M is said to be spacelike if all of its velocity vectors γ' are spacelike, similarly for timelike and null.

For a better understanding of the next construction we will bring back the notion of cross product in the tangent space T_pM at any point p in M, M being $\mathbf{S}_1^3$ or $\mathbf{H}_1^3$. There is a natural orientation in T_pM defined as follows: an ordered basis $\{X, Y, Z\}$ in T_pM is positively oriented if $\det[pXYZ] > 0$ where $[pXYZ]$ is the matrix with p, X, Y, Z as row vectors. Now let ω be the volume element on M defined by $\omega(X, Y, Z) = \det[pXYZ]$. Then, given $X, Y \in T_pM$, the cross product $X \times Y$ is the unique vector in T_pM such that $\langle X \times Y, Z\rangle = \omega(X, Y, Z)$ for any $Z \in T_pM$. Obviously, $Y \times X = -X \times Y$, and we also have

$$\langle X \times Y, X \times Y\rangle = \langle X, Y\rangle^2 - \langle X, X\rangle\langle Y, Y\rangle.$$

A nonnull curve $\gamma(s)$ in M is said to be a unit speed curve if $\langle \gamma'(s), \gamma'(s)\rangle = \varepsilon$ (ε being $+1$ or -1 according to γ is spacelike or timelike, respectively). A unit speed curve $\gamma(s)$ in M, s being the arclength parameter, is called a Frenet curve if it admits a Frenet frame field $\{T=\gamma', N, B\}$ where $B=T\times N$, satisfying the Frenet equations

$$\begin{aligned}\overline{\nabla}_T T &= \varepsilon_2\kappa N,\\ \overline{\nabla}_T N &= -\varepsilon_1\kappa T - \varepsilon_3\tau B,\\ \overline{\nabla}_T B &= \varepsilon_2\tau N,\end{aligned}$$

where $\varepsilon_1, \varepsilon_2$ and ε_3 denote the causal characters of T, N and B, respectively (in particular, $\varepsilon_i=\pm 1$ and $\varepsilon_1\varepsilon_2\varepsilon_3=-1$), $\overline{\nabla}$ is the semi-Riemannian connection on M and $\kappa=\kappa(s)$ and $\tau=\tau(s)$ are the curvature and the torsion functions of γ, respectively.

The unit tangent vector field $T=\gamma'$ defines a mapping from γ to $Q=\{q\in E:\langle q,q\rangle=\pm 1\}$ which is usually called the *tangent indicatrix* of γ and, from now on, it will also be denoted by T.

Now let $\alpha(s)$ be a null curve in M with Cartan frame $\{A,B,C\}$, i.e., A,B,C are vector fields tangent to M along $\alpha(s)$ satisfying the following conditions:

$$\begin{aligned}&\langle A,A\rangle=\langle B,B\rangle=0, \quad \langle A,B\rangle=-1,\\ &\langle A,C\rangle=\langle B,C\rangle=0, \quad \langle C,C\rangle=1,\end{aligned}$$

and

$$\begin{aligned}&\dot{\alpha}=A,\\ &\dot{A}=\rho C, \quad \rho=\rho(s)\neq 0,\\ &\dot{B}=c\alpha+w_0C, \quad w_0 \text{ being a constant},\\ &\dot{C}=w_0A+\rho B.\end{aligned}$$

If we consider the immersion $X:(s,t)\to\alpha(s)+tB(s)$, then X defines a Lorentz surface, with constant Gaussian curvature $c+w_0^2$ that Graves [**6**] called a *B-scroll*. An easy computation shows that the unit normal vector is given, up to the sign, by $\xi(s,t)=w_0tB(s)+C(s)$.

3. Killing fields. This section is taken from [**9**]. Let $\gamma(t)$ be a nonnull immersed curve in a three-dimensional Lorentzian space form M with sectional curvature c, and let $v(t)=|\gamma'(t)|$ be the speed of γ. Let us consider a variation of γ, $\Gamma=\Gamma(t,z):I\times(-\varepsilon,\varepsilon)\to M$ with $\Gamma(t,0)=\gamma(t)$. In particular, one can choose $\varepsilon>0$ in such a way that all t-curves of the variation have the same causal character as that of γ. Associated with Γ are two vector fields along Γ, $V(t,z) = (\partial\Gamma/\partial z)(t,z)$ and $W(t,z) = (\partial\Gamma/\partial t)(t,z)$. In particular, $V(t) = V(t,0)$ is the variational vector field along γ and $W(t,z)$ is the tangent vector field of the t-curves. We will use the notation $V = V(t,z)$, $v = v(t,z)$, $\kappa = \kappa(t,z)$, etc., with the obvious meanings. Also, if s denotes the arclength parameter of the t-curves, we will write $v(s,z)$, $V(s,z)$, $\kappa(s,z)$, etc., for the corresponding reparametrizations.

A straightforward but long computation allows us to obtain formulas for $(\partial v/\partial z)(t,0)$, $(\partial\kappa^2/\partial z)(t,0)$ and $(\partial\tau^2/\partial z)(t,0)$ which we collect, along with another standard identity in the following lemma.

Lemma 1. *With the above notation, the following assertions hold*:

(1) $[V,W] = 0$;

(2) $(\partial v/\partial z)(t,0) = -\varepsilon_1 g v$, *with* $g = \langle \overline{\nabla}_T V, T\rangle$;

(3) $(\partial\kappa^2/\partial z)(t,0) = 2\varepsilon_2\langle \overline{\nabla}^2_T V, \overline{\nabla}_T T\rangle + 4\varepsilon_1 g\kappa^2 + 2\varepsilon_2\langle R(V,T)T, \overline{\nabla}_T T\rangle$;

(4) $(\partial\tau^2/\partial z)(t,0) = -2\varepsilon_2\langle (1/\kappa)\overline{\nabla}^3_T V - (\kappa'/\kappa^2)\overline{\nabla}^2_T V + \varepsilon_1(\varepsilon_2\kappa + (c/\kappa))\overline{\nabla}_T V - \varepsilon_1(c\kappa'/\kappa^2)V, \tau B\rangle$,

where $\langle\ ,\ \rangle$ *denotes the Lorentzian metric of* M *and* $\kappa' = (\partial\kappa/\partial t)(t,0)$.

Without loss of generality we can assume γ to be arclength parametrized. A vector field $V(s)$ along γ which infinitesimally preserves unit speed parametrization (that means $(\partial v/\partial z)(t,0) = 0$ for a V-variation of γ) is said to be a Killing vector field along γ if this evolves in the direction of V without changing shape, only position. In other words, the curvature and torsion functions of γ remain unchanged as the curve evolves. Hence Killing vector fields along γ are characterized by the equations

$$\frac{\partial v}{\partial z}(t,0) = \frac{\partial \kappa^2}{\partial z}(t,0) = \frac{\partial \tau^2}{\partial z}(t,0) = 0, \tag{3.1}$$

and this is well defined in the sense that it does not depend on the V-variation of γ one chooses to compute the derivatives involved in equation (3.1). In fact we use Lemma 1 and (3.1) to see that V is a Killing vector field along γ if and only if it satisfies the following conditions:

(3.2)

a) $\langle \overline{\nabla}_T V, T\rangle = 0,$

b) $\langle \overline{\nabla}^2_T V, N\rangle + \varepsilon_1 c\langle V, N\rangle = 0,$

c) $\langle (1/\kappa)\overline{\nabla}^3_T V - (\kappa'/\kappa^2)\overline{\nabla}^2_T V + \varepsilon_1(\varepsilon_2\kappa + (c/\kappa))\overline{\nabla}_T V - \varepsilon_1 c(\kappa'/\kappa^2)V,$
$\tau B\rangle = 0.$

In particular, the solutions of (3.2) constitute a six-dimensional linear space.

Now when M is simply connected, since the restriction to γ of any Killing field $\tilde{V}$ of M is a Killing vector field along γ, one concludes from a well-known dimension argument the following lemma.

Lemma 2. *Let M be a complete, simply connected, Lorentzian space form and γ a nonnull immersed curve in M. A vector field V on γ is a Killing vector field along γ if and only if it extends to a Killing field $\tilde{V}$ on M.*

4. General helices in the three-dimensional Lorentz-Minkowski space. Following the classical terminology of the Euclidean geometry (see, for instance, [**10**]) we will say that γ is a *general helix* in $\mathbf{L}^3$ if its tangent indicatrix lies in a plane of $\mathbf{L}^3$. That means that there exists a vector $v \neq 0$ in $\mathbf{L}^3$ which is orthogonal to the acceleration vector field of γ. The straight line generated by v is uniquely determined and will be called the *axis* of γ. In particular, we will say that a general helix is *degenerate* or *nondegenerate* according to whether its axis is null or nonnull, respectively.

It is obvious that nonnull curves in $\mathbf{L}^3$ with zero torsion are examples of nondegenerate general helices. In fact, such a curve lies in a nondegenerate two-plane in $\mathbf{L}^3$, and a unit vector in $\mathbf{L}^3$ orthogonal to this plane works as the axis of the general helix.

Now, given a general helix γ in $\mathbf{L}^3$ with axis v, we can define a translation vector field $\tilde{V}$ in $\mathbf{L}^3$ by $\tilde{V} = v$ for any $p \in \mathbf{L}^3$. Let V be $\tilde{V}$ restricted to γ. Then V defines a Killing vector field along γ with constant length, i.e., $\langle V, V\rangle$ is constant and orthogonal to the acceleration vector field of γ.

Assume now that W is a Killing vector field along a nonnull curve γ with constant length and orthogonal to its normal vector field N. From (3.2a) we can write $W = aT + bB$, a and b being constants. Now use (3.2b) to get $\overline{\nabla}_T W = \lambda N$ where $\lambda = \varepsilon_2(a\kappa + b\tau)$ is also constant. Finally, equation (3.2c) yields $\lambda\tau(\tau/k)' = 0$. From here we consider the following cases.

(i) $\tau \equiv 0$. Then γ is a nondegenerate general helix. It is not difficult to see that $W = B$, unless γ is a circle which will be considered next, and so it extends to a translation vector field $\widetilde{W}$ in $\mathbf{L}^3$.

(ii) *k and τ both are constant.* Then γ is a helix. Now the Killing vector field W is not uniquely determined. In fact, for any couple of constants a and b, the vector field $W(s) = aT + bB$ is a Killing vector field along γ. On the other hand, we can determine a Killing vector field along γ, say $V(s)$, being parallel along γ, and thus it extends to a translation vector field $\tilde{V}(s)$ such that $\tilde{V}(s) = v \in \mathbf{L}^3$. Indeed, just choose a and b such that $a\kappa + b\tau = 0$. Therefore, γ is a nondegenerate general helix unless $\varepsilon_2 = 1$ (which means that N is spacelike or the rectifying plane is Lorentzian anywhere) and $\tau = \pm\kappa$ and then γ is degenerate.

(iii) $\lambda = 0$. Then W is a uniquely determined Killing vector field along γ. Furthermore, it is parallel along γ and extends to a translation vector field $\widetilde{W}$ such that $\tilde{W} = v \in \mathbf{L}^3$. Therefore, γ is a general helix whose axis is v, or W, γ being degenerate when W is null which yields $\varepsilon_2 = 1$ and $\tau = \pm\kappa$.

We will refer to curves in the first two classes as trivial general helices.

Summarizing, we have the following.

Theorem 3 (The Lancret theorem in $\mathbf{L}^3$). *Let γ be a nonnull immersed curve in $\mathbf{L}^3$ with curvature and torsion functions κ and τ, respectively. Then the following statements are equivalent*:

(a) *γ is a general helix in $\mathbf{L}^3$*;

(b) *there exists a constant length Killing vector field V along γ which is orthogonal to the acceleration vector field of γ*;

(c) *there exists a constant r such that $\tau = r\kappa$.*

Moreover, a general helix γ is degenerate if and only if $r = \pm 1$ and *its normal vector field is spacelike. The Killing vector field V in* (b) *is not uniquely determined if γ is a helix* (κ *and* τ *both are constant*)*; however, in this case, V can be uniquely determined, up to constants, once it is chosen parallel along γ;* (*said otherwise, its extended Killing vector field in $\mathbf{L}^3$ is a translation vector field*).

Theorem 4 (Solving natural equation for nondegenerate general helices.) *Let β be a nonnull immersed curve in $\mathbf{L}^3$. Then β is a nondegenerate general helix if and only if it is a geodesic in a right cylinder whose directrix and generatrix are both nonnull.*

Proof. Let v be a unit vector in $\mathbf{L}^3$ and α a unit speed curve in a plane orthogonal to v. The Frenet equations of α are

$$\begin{aligned} \overline{\nabla}_{\overline{T}}\overline{T} &= \delta_2\bar{\kappa}\overline{N}, \\ \overline{\nabla}_{\overline{T}}\overline{N} &= -\,\delta_1\bar{\kappa}\overline{T}, \end{aligned} \tag{4.1}$$

where $\{\overline{T},\overline{N}\}$ is the Frenet frame along α, $\bar{\kappa}$ its curvature function and δ_1,δ_2 the causal characters of $\overline{T}$ and $\overline{N}$, respectively. Notice that the causal character of v is $-\delta_1\delta_2$.

Let us consider the right cylinder $C_{\alpha,v}$ in $\mathbf{L}^3$ generated by α and v which is naturally parametrized as $X(s,t)=\alpha(s)+tv$. It is well known that the geodesics of $C_{\alpha,v}$ are the images under X of straight lines in the (s,t)-plane. Choose such a geodesic $\gamma(s)=\alpha(s)+msv$ where m is a certain constant. Then the translation field $\widetilde{V}$ in $\mathbf{L}^3$ determined by v induces a Killing vector field along γ with constant length and orthogonal to the acceleration vector $\gamma''(s)=\alpha''(s)$. Since v is nonnull, Theorem 3 implies that γ is a nondegenerate general helix.

Conversely, suppose β is a nondegenerate general helix. Then there exists a certain constant r such that $\tau=r\kappa$ (of course κ and τ denote the curvature and torsion functions of β). One can also choose a unit vector, say v, lying on the axis of β. We take a (nondegenerate) plane P in $\mathbf{L}^3$ which is orthogonal to v. Up to congruences in P, there exists a unique curve in P, say α, with curvature function $\bar{\kappa}=|\beta'|\kappa$ and $\delta_2=\varepsilon_2$ (notice that the causal character δ_1 of α is determined by δ_2 and the causal character of v). Let $C_{\alpha,v}$ be the right cylinder generated by α and v. Then it is parametrized by $X(s,t)=\alpha(s)+tv$. Finally we choose the geodesic of $C_{\alpha,v}$ defined by $\gamma(s)=\alpha(s)+msv$ where

$m = \delta_1\varepsilon_3 r$. Then γ is a nonnull geodesic because $\delta_2 = -1$ provided that $m^2 = 1$. Finally it is easy to see that γ and β have the same curvature and torsion functions as well as the same causal characters. This concludes the proof. □

Theorem 5 (Solving natural equation for degenerate general helices.) *Let β be a nonnull immersed curve in $\mathbf{L}^3$. Then β is a degenerate general helix if and only if it is a geodesic in a flat B-scroll in $\mathbf{L}^3$.*

Proof. Let $\alpha(s)$ be a null curve in $\mathbf{L}^3$ with Cartan frame $\{A, B, C\}$ and $S_{\alpha,B}$ the flat B-scroll (i.e., $w_0 = 0$) parametrized by $X(s,t) = \alpha(s)+tB$. We choose a nonnull geodesic of $S_{\alpha,B}$, say $\gamma(u) = \alpha(s(u))+t(u)B(s(u))$. Then the translation field $\widetilde{B}$ in $\mathbf{L}^3$ determined by B induces a Killing vector field along γ, also denoted by B, with constant length and such that $\langle\gamma'(u), B\rangle = -s'(u)$ is constant because the geodesic γ is the image under X of a straight line. Therefore, from Theorem 3, γ is a degenerate general helix in $\mathbf{L}^3$.

To prove the converse, let β be a degenerate general helix that we parametrize with constant speed, say $\langle\beta', \beta'\rangle = p$ constant. From the theorem of Lancret we know that the curvature τ and torsion κ functions of β agree (we can change orientation if necessary) and the acceleration vector field of β is spacelike, i.e., $\varepsilon_2 = 1$. We define the following vector fields

$$\begin{aligned} A &= \frac{|\beta'|}{2}\,(T+B), \\ B &= -\frac{\varepsilon_1}{|\beta'|}\,(T-B), \\ C &= N, \end{aligned}$$

where $\{T, N, B\}$ is the Frenet frame along β, $|\beta'| = \sqrt{\varepsilon_1 p}$ and ε_1 denotes, as usual, the causal character of β.

Let α be a curve in $\mathbf{L}^3$ with tangent vector field A, then α is a null curve in $\mathbf{L}^3$. Furthermore, $\{A, B, C\}$ is a Cartan frame along α with $w_0 = 0$ and $\rho = \kappa|\beta'|$ (see Section 2). Let $S_{\alpha,B}$ be the corresponding flat B-scroll which is parametrized by $X(s,t) = \alpha(s) + tB$. Finally choose the geodesic in $S_{\alpha,B}$ given by $\gamma(s) = \alpha(s) + msB$ where $m = -p/2$. It is not difficult to see that γ and β have the same curvature and torsion functions and also the same causal character, showing that they are congruent in $\mathbf{L}^3$.

5. General helices in nonflat three-dimensional Lorentzian space forms. In order to generalize the notion of general helix to three-dimensional Lorentzian spaces M of nonzero constant curvature, we profit by Theorem 3. A curve γ in M is said to be a general helix if there exists a Killing vector field V along γ with constant length and orthogonal to the acceleration vector field of γ. We will say that V is an axis of the general helix γ. Obvious examples of general helices in M are the following. Curves with torsion vanishing anywhere, where the unit binormal works as an axis. Helices are also general helices, where any vector field chosen in the rectifying plane having constant coordinates relative to T and B runs as an axis.

We can follow the notation and terminology introduced in $\mathbf{L}^3$ to say that zero torsion curves are nondegenerate general helices because the axis B is obviously nonnull. As for curves with both constant curvature and torsion, we know that for any pair of constants a and b the vector field along γ given by $V(s) = aT + bB$ is always a Killing vector field. Of course, when $\varepsilon_2 = -1$, i.e., the rectifying plane is positive definite at any point, all Killing vector fields $V(s)$ are nonnull and we will say that the general helix is nondegenerate. However, if $\varepsilon_2 = 1$, i.e., the rectifying plane is Lorentzian, we have Killing vector fields along γ being either spacelike or timelike or null. It does not allow us to decide if such a general helix is degenerate or not. However, we can determine a unique Killing vector field along the helix by forcing it to be parallel along γ. The helix is said to be *degenerate* or *nondegenerate* according to whether V is null or nonnull, respectively.

Let $\gamma(s)$ be a general helix in M with curvature $\kappa > 0$. Let $V(s)$ be an axis and assume, without loss of generality, that $\langle V, V\rangle = \varepsilon$ where $\varepsilon = -1, 0, 1$. From equation (3.2a) we deduce that

$$V(s) = fT(s) + hB(s) \quad \text{and} \quad \varepsilon = \varepsilon_1 f^2 + \varepsilon_3 h^2, \tag{5.1}$$

for certain constants f and h. By using the Frenet equations of γ, we get

$$\overline{\nabla}_T V = \varepsilon_2 (f\kappa + h\tau) N, \tag{5.2}$$

and

$$\overline{\nabla}_T^2 V = -\varepsilon_1\varepsilon_2\kappa(f\kappa + h\tau)T + \varepsilon_2(f\kappa' + h\tau')N - \varepsilon_2\varepsilon_3\tau(f\kappa + h\tau)B. \tag{5.3}$$

Now from equations (3.2b), (5.1) and (5.3), we deduce that $f\kappa' + h\tau' = 0$, from which we get

$$\tau = b\kappa + a, \tag{5.4}$$

for certain constants a and b. On the other hand, from (5.3), jointly with the Frenet equations of γ, we obtain

$$\overline{\nabla}_T^3 V = -\varepsilon_1\varepsilon_2\lambda\kappa' T - \lambda(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2)N - \varepsilon_2\varepsilon_3\lambda\tau' B, \tag{5.5}$$

where λ stands for the constant $f\kappa + h\tau$. Now equation (3.2c), jointly with equations (5.1)–(5.5), yields

$$\tau(\lambda\tau'\kappa - \lambda\kappa'\tau - ch\kappa') = 0,$$

and then

$$h\kappa'\tau(a^2 + c) = 0. \tag{5.6}$$

In particular, the above equation shows that in the De Sitter space $\mathbf{S}_1^3$, $c = +1$, the only general helices are the trivial ones (see Section 4). So we have proved the following result.

Theorem 6 (The Lancret theorem in the De Sitter space.) *A nonnull immersed curve γ in $\mathbf{S}_1^3$ is a general helix if and only if either*

(1) $\tau \equiv 0$ *and γ is a curve in a totally geodesic surface of $\mathbf{S}_1^3$; or*

(2) *γ is a helix in $\mathbf{S}_1^3$, (i.e., curvature κ and torsion τ are both constant).*

Furthermore, general helices of the first type have only one axis (the binormal) which is parallel and so they are nondegenerate. General helices of the second type have a plane (the rectifying plane) of axes. However, only one of them is parallel. This axis is null and so the general helix is degenerate if and only if $\varepsilon_2 = +1$ (the normal vector is spacelike) and $\tau = \pm\kappa$; otherwise, the helix is nondegenerate.

In the anti De Sitter space, besides the two classes of trivial general helices, we have another one. This new class can be characterized from equations (5.5) and (5.6) where $c = -1$ as the curves in $\mathbf{H}_1^3$ whose curvature and torsion are related by

$$\tau = b\kappa \pm 1,$$

for a certain constant b. These general helices admit only the axis $V = fT + hB$ such that

$$\frac{f}{h} = -b = \frac{1-\tau}{\kappa},$$

whose causal character is

$$\varepsilon = h^2\left(\varepsilon_1 \frac{(\tau-1)^2}{\kappa^2} + \varepsilon_3\right).$$

In particular, a general helix of this type is degenerate if and only if $\varepsilon_2 = 1$ and $b = \pm 1$.

Summarizing, we have shown the following.

Theorem 7 (The Lancret theorem in the anti De Sitter space). *A nonnull immersed curve γ in $\mathbf{H}_1^3$ is a general helix if and only if one of the following statements holds*:

(1) *$\tau \equiv 0$ and γ is a curve in a totally geodesic surface of $\mathbf{H}_1^3$. The curve admits only one axis which agrees with its binormal being nonnull and parallel along the curve. The general helix is nondegenerate*;

(2) *γ is a helix in $\mathbf{H}_1^3$. It admits a plane* (*the rectifying plane*) *of axes but only one is parallel along γ. This parallel axis is null and so γ is degenerate if and only if $\varepsilon_2 = +1$ and $\tau = \pm\kappa$. Otherwise γ is nondegenerate*;

(3) *there exists a certain constant b such that the curvature κ and the torsion τ functions of γ are related by $\tau = b\kappa \pm 1$. The curve admits a unique axis which cannot be parallel along γ. It is null and so γ is degenerate if and only if $b = \pm 1$ and its unit normal vector is spacelike* ($\varepsilon_2 = +1$).

Remark 1. Theorems 6 and 7 should be compared with Theorems 1 and 3 in [**2**], respectively.

Now we are going to solve the natural equations for general helices in M.

In [**4**] we have just constructed a new class of submanifolds in $\mathbf{H}^3_1(-1)$ defined by means of two semi-Riemannian submersions $\pi_s : \mathbf{H}^3_1(-1) \to \mathbf{H}^2_s(-4)$, $s = 0, 1$. By pulling back via π_s a nonnull curve γ in $\mathbf{H}^2_s(-4)$ we get the total horizontal lift of γ, which is an immersed flat surface M_γ in $\mathbf{H}^3_1(-1)$, that will be called the *semi-Riemannian Hopf cylinder associated to* γ. Notice that if $s = 0$, M_γ is a Lorentzian surface, whereas if $s = 1$, M_γ is Riemannian or Lorentzian, according to whether γ is spacelike or timelike, respectively.

Let $\gamma : I \to \mathbf{H}^2_s(-4)$ be a unit speed curve with Frenet frame $\{\overline{T}, \overline{N}\}$ and curvature function $\bar{\kappa}$. Let $\bar{\gamma}$ be a horizontal lift of γ to $\mathbf{H}^3_1(-1)$ with Frenet frame $\{T, N, B\}$, curvature $\kappa = \bar{\kappa} \circ \pi_s$ and torsion $\tau = 1$. Recall that B is nothing but the unit tangent vector field to the fibers along $\bar{\gamma}$. Then the Hopf cylinder M_γ can be orthogonally parametrized by

$$X(t, z) = \begin{cases} \cos(z)\bar{\gamma}(t) + \sin(z)B(t) & \text{when } s = 0, \\ \cosh(z)\bar{\gamma}(t) + \sinh(z)B(t) & \text{when } s = 1. \end{cases}$$

Notice that a unit normal vector field to M_γ into $\mathbf{H}^3_1(-1)$ is obtained from the complete horizontal lift of $\overline{N}$ and it is, of course, N along each horizontal lift of γ. As a consequence, we have that M_γ is a flat surface with mean curvature function α given by $\alpha = (1/2)\kappa$.

Theorem 8 (Solving natural equation for nondegenerate general helices in $\mathbf{H}^3_1(-1)$.) *Let β be a nonnull immersed curve in $\mathbf{H}^3_1$. Then β is a nondegenerate general helix if and only if it is a geodesic in a Hopf cylinder M_γ.*

Proof. Let $\beta(s)$ be an arclength parametrized geodesic in M_γ. Then there exist two constants a and b such that

$$T(s) = \beta'(s) = aX_t + bX_z,$$

with $\varepsilon_1 a^2 + \varepsilon_3 b^2 = \delta_1$, δ_1 being the causal character of β. A direct computation shows that the curvature ρ and the torsion τ of β satisfy

$$\rho = \varepsilon_2 a^2 \kappa + 2ab,$$
$$\tau^2 = \varepsilon_2 \rho^2 - \varepsilon_1 \delta_1 \kappa \rho + 1.$$

It is not difficult to see that $\tau = r\rho \pm 1$, $r = b/a$, showing that β is a general helix. Moreover, if the normal vector N is spacelike, then $r \neq 1$ and then β is nondegenerate.

To prove the converse, let β be a nondegenerate general helix in $\mathbf{H}^3_1(-1)$ with curvature ρ and torsion τ. Then there exists a constant r (with $r \neq \pm 1$ if the normal vector to β is spacelike) such that $\tau = r\rho \pm 1$. We choose $\varepsilon_1 = \pm 1$ and s in $\{0,1\}$ in order for $\delta_1(\varepsilon_1 - (-1)^s r^2)$ to be positive, δ_1 being the causal character of β. Let γ be the unique curve, up to motions, in $\mathbf{H}^2_1(-4)$ with curvature $\bar{\kappa} = \delta_1((-1)^s - \varepsilon_1 r^2)\rho - 2\varepsilon_1(-1)^s r$ and causal character defined by ε_1. Let α be the geodesic in the Hopf cylinder M_γ given by $\alpha(s) = X(as, bs)$ with

$$a^2 = \frac{\delta_1}{\varepsilon_1 - (-1)^s r^2} \quad \text{and} \quad b^2 = r^2 a^2.$$

It is easy to see that β and α have the same curvature and torsion, and also the same causal character, showing that they are congruent.

Theorem 9 (Solving natural equation for degenerate general helices in $\mathbf{H}^3_1(-1)$.) *Let β be a nonnull immersed curve in $\mathbf{H}^3_1$. Then β is a degenerate general helix if and only if it is a geodesic in a flat B-scroll over a null curve.*

Proof. Let $\beta(u)$ be a geodesic of some flat B-scroll $S_{\alpha,B}$ in $\mathbf{H}^3_1(-1)$, i.e., $w_0 = \pm 1$, parametrized by $\beta(u) = \alpha(s(u)) + t(u)B(s(u))$. Then the normal vector to β in $\mathbf{H}^3_1(-1)$ is given by $N(u) = \beta(u) - \alpha(s(u)) + C(s(u))$. From here we obtain that $\overline{\nabla}_T N = T + s'(u)\rho B$. By using the Frenet equations for β we deduce that the vector $(1 + \varepsilon_1\kappa)T + \varepsilon_3 \tau B$ is null, where κ and τ stand for the curvature and torsion of β, respectively. Therefore, N is spacelike and $\tau = \pm\varepsilon_1\kappa \pm 1$, which proves that β is a degenerate general helix.

Conversely, let β be a curve in $\mathbf{H}^3_1(-1)$ with curvature κ and torsion τ satisfying that $\tau = \kappa + \varepsilon_1$ and the normal vector of β is spacelike (the other cases are similar). Let α be the null curve in $\mathbf{H}_1^3(-1)$ given by

$$\alpha(s) = \beta(s) - \frac{1}{2}\, s(T(s) - B(s)),$$

and consider the vector fields

$$\begin{aligned} A(s) &= -\,\frac{\varepsilon_1}{2}\, s\beta(s) + \frac{1}{2}\,(T(s) + B(s)) + \frac{\varepsilon_1}{2}\, sN(s), \\ B(s) &= -\,\varepsilon_1(T(s) - B(s)), \\ C(s) &= -\,\frac{1}{2}\, s(T(s) - B(s)) + N(s). \end{aligned}$$

It is not difficult to see that $\{A, B, C\}$ is a Cartan frame along α with $w_0 = 1$ and $\rho = \tau$. Let $S_{\alpha,B}$ be the B-scroll in $\mathbf{H}_1^3(-1)$ parametrized by $X(s,t) = \alpha(s) + tB(s)$. Then it is clear that $\beta(s) = X(s, -(\varepsilon_1/2)s)$ and so β is a geodesic of that B-scroll.

Remark 2. It is worth noting that in the "if" part in Theorems 8 and 9 we have only used the existence of an axis, not necessarily parallel. Now if γ is a general helix in $\mathbf{H}_1^3(-1)$ with Lorentzian rectifying plane anywhere, then it may have both null and nonnull axes. Therefore, γ is a geodesic in a Hopf cylinder as well as in a flat B-scroll over a null curve.

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