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THE SHARP BOUND FOR SOME COEFFICIENT FUNCTIONAL WITHIN THE CLASS OF HOLOMORPHIC BOUNDED FUNCTIONS AND ITS APPLICATIONS

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ABSTRACT. We determine the sharp bound of the functional $|c_3 + pc_1c_2 + qc_1^3|$ for any given real numbers p and q within the class of holomorphic and bounded functions $\omega(z) = c_1z + c_2z + \dots$, $|\omega(z)| < 1$, $|z| < 1$, which have real coefficients. Applications are given.

1. Let Ω denote the class of holomorphic functions of the form

$$\omega(z) = c_1z + c_2z^2 + \cdots \tag{1}$$

in the unit disk $|z| < 1$ which satisfies the condition $|\omega(z)| < 1$, $|z| < 1$.

In [**3**] the sharp bound for the functional was given by

$$\Psi(\omega) = |c_3 + pc_1c_2 + qc_1^3| \tag{2}$$

where p and q are arbitrary but fixed real numbers.

Namely, the following theorem holds.

Theorem A. *If $\omega \in \Omega$, then for any real numbers p and q, the following sharp estimate holds*:

$$\Psi(\omega) \leq H(p, q) \tag{3}$$

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where
(4)
$$H(p,q)=\begin{cases} 1 & \text{for } (p,q)\in D_1\cup D_2\\ |q| & \text{for } (p,q)\in \cup_{k=3}^{7} D_k\\ \frac{2}{3}(|p|+1)\Big(\frac{|p|+1}{3(|p|+1+q)}\Big)^{1/2} & \text{for } (p,q)\in D_8\cup D_9\\ \frac{1}{3}q\Big(\frac{p^2-4}{p^2-4q}\Big)\Big(\frac{p^2-4}{3(q-1)}\Big)^{1/2} & \text{for } (p,q)\in D_{10}\cup D_{11}\setminus\{\pm 2,1\}\\ \frac{2}{3}(|p|-1)\Big(\frac{|p|-1}{3(|p|-1-q)}\Big)^{1/2} & \text{for } (p,q)\in D_{12} \end{cases}.$$

The extremal functions, up to the rotation, have the form:

$$\omega(z)=z^3,\quad \omega(z)=z,\quad \omega(z)=\omega_1(z)=z(t_1-z)/(1-t_1z),$$
$$t_1=\left(\frac{|p|+1}{3(|p|+1+q)}\right)^{1/2}$$
$$\omega(z)=\omega_0(z)=z\frac{[(1-\lambda)\varepsilon_2+\lambda\varepsilon_1]-\varepsilon_1\varepsilon_2 z}{1-[(1-\lambda)\varepsilon_1+\lambda\varepsilon_2]z},$$
$$|\varepsilon_1|=|\varepsilon_2|=1$$
$$\varepsilon_1=t_0-e^{-i\varphi_0/2}(a\mp b),\quad \varepsilon_2=-e^{-i\varphi_0/2}(ia\pm b),$$
(5)
$$a=t_0\cos\frac{\varphi_0}{2},\quad b=\sqrt{1-t_0^2\sin^2\frac{\varphi_0}{2}},\quad \lambda=\frac{b\pm a}{2b}$$
$$t_0=\left[\frac{2q(p^2+2)-3p^2}{3(q-1)(p^2-4q)}\right]^{1/2},$$
$$\cos\frac{\varphi_0}{2}=\frac{p}{2}\left[\frac{q(p^2+8)-2(p^2+2)}{2q(p^2+2)-3p^2}\right]$$
$$\omega(z)=\omega_2(z)=z\frac{t_2+z}{1+t_2z},\quad t_2=\left(\frac{|p|-1}{3(|p|-1-q)}\right)^{1/2}.$$

The sets D_k, $k=1,2,\dots,12$ are defined as follows:

$$D_1:=\left\{(p,q):|p|\le\frac{1}{2},\ |q|\le 1\right\},$$
$$D_2:=\left\{(p,q):\frac{1}{2}\le|p|\le 2,\ \frac{4}{27}(|p|+1)^3-(|p|+1)\le q\le 1\right\},$$

$$
\begin{aligned}
D_3 &:= \left\{(p,q) : |p| \le \frac{1}{2},\ q \le -1\right\},\\
D_4 &:= \left\{(p,q) : |p| \ge \frac{1}{2},\ q \le -\frac{2}{3}\,(|p|+1)\right\},\\
D_5 &:= \{(p,q) : |p| \le 2,\ q \ge 1\},\\
D_6 &:= \left\{(p,q) : 2 \le |p| \le 4,\ q \ge \frac{1}{12}\,(p^2+8)\right\},\\
D_7 &:= \left\{(p,q) : |p| \ge 4,\ q \ge \frac{2}{3}\,(|p|-1)\right\},\\
D_8 &:= \Bigg\{(p,q) : \frac{1}{2} \le |p| \le 2,\\
&\qquad -\frac{2}{3}\,(|p|+1) \le q \le \frac{4}{27}\,(|p|+1)^3 - (|p|+1)\Bigg\},\\
D_9 &:= \left\{(p,q) : |p| \ge 2,\ -\frac{2}{3}\,(|p|+1) \le q \le \frac{2|p|(|p|+1)}{p^2+2|p|+4}\right\},\\
D_{10} &:= \left\{(p,q) : 2 \le |p| \le 4,\ \frac{2|p|(|p|+1)}{p^2+2|p|+4} \le q \le \frac{1}{12}\,(p^2+8)\right\},\\
D_{11} &:= \left\{(p,q) : |p| \ge 4,\ \frac{2|p|(|p|+1)}{p^2+2|p|+4} \le q \le \frac{2|p|(|p|-1)}{p^2-2|p|+4}\right\},\\
D_{12} &:= \left\{(p,q) : |p| \ge 4,\ \frac{2|p|(|p|-1)}{p^2-2|p|+4} \le q \le \frac{2}{3}\,(|p|-1)\right\}.
\end{aligned} \tag{6}
$$

As one can see from the formulae (5) all the extremal functions except $\omega_0(z)$ have real coefficients. Therefore, if one considers the problem of the sharp bound for the functional (2) within the subclass $\Omega^r \subset \Omega$ consisting of functions $\omega(z)$ which have real coefficients, the bound will be different if $(p,q) \in D_{10} \cup D_{11} \setminus \{\pm 2, 1\}$, where the extremal function is $\omega_0(z)$.

The corresponding sharp result is presented in Theorem 1. We also give the applications for the class of holomorphic bounded and nonvanishing functions in the unit disk as well as for the class of bounded starlike functions, where we apply some deep result of Barnard and Lewis [**1**].

Let us mention finally that our Theorem A and Theorem 1 are very useful for other classes of holomorphic functions which are defined by subordination.

Theorem 1. *If $\omega \in \Omega^r$, then for any real numbers p and q the following sharp estimate holds*

$$\Psi(\omega) \leq G(p,q) \tag{7}$$

where

(8)

$$G(p,q)=\begin{cases} 1 & \textit{for } (p,q)\in D_1 \cup D_2 \\ |q| & \textit{for } (p,q)\in \cup_{k=3}^{7} D_k \\ \frac{2}{3}\,(|p|+1)\Big(\frac{|p|+1}{3(|p|+1+q)}\Big)^{1/2} & \textit{for } (p,q)\in D_8\cup D_9^0\cup D_9^{00} \\ 1+\frac{1}{27}\frac{(p^2-4)^3}{(p^2-4q)^2} & \textit{for } (p,q)\in D_{10}^0\cup D_{11}^0 \setminus \{\pm 2, 1\} \\ \frac{2}{3}\,(|p|-1)\Big(\frac{(|p|-1)}{3(|p|-1-q)}\Big)^{1/2} & \textit{for } (p,q)\in D_{12}^0\cup D_{12}^{00} \end{cases}.$$

The sets $D_9^0, D_9^{00}, D_{10}^0, D_{11}^0, D_{12}^0, D_{12}^{00}$ are defined as follows:

(9)

$$\begin{aligned}
D_9^0 &= \Big\{(p,q) : 2 \leq |p| \leq p_0,\ -\frac{2}{3}\,(|p|+1) \leq q \leq q_0(p)\Big\},\\
D_9^{00} &= \Big\{(p,q) : |p| \geq p_0,\ -\frac{2}{3}\,(|p|+1) \leq q \leq 2\frac{p^2-1}{p^2+3}\Big\},\\
D_{10}^0 &= \Big\{(p,q) : 2 \leq |p| \leq 4,\ q_0(p) \leq q \leq \frac{1}{12}\,(p^2+8)\Big\},\\
D_{11}^0 &= \Big\{(p,q) : 4 \leq |p| \leq p_0,\ q_0(p) \leq q \leq -\frac{1}{12}\,(|p|-1)(p^2-4|p|-8)\Big\},\\
D_{12}^0 &= \Big\{(p,q) : 4 \leq |p| \leq p_0,\ -\frac{1}{12}\,(|p|-1)(p^2-4|p|-8)\\
&\qquad\qquad \leq q \leq \frac{2}{3}\,(|p|-1)\Big\},\\
D_{12}^{00} &= \Big\{(p,q) : |p| \geq p_0,\ 2\,\frac{p^2-1}{p^2+3} \leq q \leq \frac{2}{3}\,(|p|-1)\Big\}.
\end{aligned}$$

The number $p_0 \in (4.51, 4.52)$ is the root of the equation $p^3-4p^2-5p+12=0$, and the function $q=q_0(p)$ is the unique root of the equation ($t=p^2-4q$):

$$81t^3 - 3(p-2)(2p^2+19p+26)t^2 + (p^2-4)^2(p^2+2p-8)t - (p^2-4)^4 = 0. \tag{10}$$

The extremal function for $(p,q) \in D_{10}^0 \cup D_{11}^0$ has the form

$$\omega(z) = \omega_0^r(z) = z\frac{x_0 + (1/2)px_0z + z^2}{1+(1/2)px_0z + x_0z^2}, \qquad x_0 = \frac{2}{3}\,\frac{p^2-4}{p^2-4q}. \tag{11}$$

Proof. We can restrict ourselves to the case $p > 2$ and $(p,q) \in D_{10}^+ \cup D_{11}^+ \setminus \{2,1\}$, where

$$\begin{aligned} D_{10}^+ &= \left\{(p,q) : 2 \le p \le 4,\ \frac{2p(p+1)}{p^2+2p+4} \le q \le \frac{1}{12}\,(p^2+8)\right\} \\ D_{11}^+ &= \left\{(p,q) : p \ge 4,\ \frac{2p(p+1)}{p^2+2p+4} \le q \le \frac{2p(p-1)}{p^2-2p+4}\right\}. \end{aligned} \tag{12}$$

Note that in the case under consideration we have $(p^2-4q) > 0$.

In order to find the maximum of (2) we will use the Caratheodory inequalities [**3**], which for the class Ω^r take the form:

$$\begin{aligned} -1 \le c_1 &\le 1 \\ -(1-c_1^2) \le c_2 &\le 1-c_1^2 \\ -(1-c_1^2)^2 + c_2^2 - c_1c_2^2 &\le c_3(1-c_1^2) \\ &\le (1-c_1^2)^2 - c_2^2 - c_1c_2^2. \end{aligned} \tag{13}$$

If $c_1 = \pm 1$, then $\Psi(\omega) = |q|$. Putting $c_1 = x \in (-1,1)$, $c_2 = y \in [-(1-x^2), (1-x^2)]$ we have by (13) the following inequality:

$$(c_3 + pc_1c_2 + qc_1^3) \le 1 - x^2 - \frac{y^2}{1-x} + pxy + qx^3.$$

Denoting

$$\Phi(x,y) := -\frac{1}{1-x}\,y^2 + pxy + (1-x^2+qx^3), \tag{14}$$

we easily find

$$\max_{-(1-x^2)\le y\le 1-x^2} \Phi(x,y) = \begin{cases} \Phi(x,-(1-x^2)) := H_1(x) & \text{if } y_0 \le -(1-x^2) \\ \Phi(x,y_0) := H_2(x) & \text{if } -(1-x^2) \le y_0 \le 1-x^2 \\ \Phi(x,1-x^2) := H_3(x) & \text{if } y_0 \ge 1-x^2 \end{cases} \tag{15}$$

where

$$y_0 = \frac{1}{2}\,px(1-x).$$

Taking into account that $p > 2$, we obtain from (15):

$$\begin{gathered} H_1(x) = (q+p+1)x^3 - (p+1)x, \\ \text{if} \quad -1 \le x \le \frac{-2}{p+2} \quad \text{and} \quad p > 2; \end{gathered} \tag{16}$$

$$\begin{gathered} H_2(x) = \left(q - \frac{1}{4}\,p^2\right)x^3 + \left(\frac{1}{4}\,p^2 - 1\right)x^2 + 1, \\ \text{if} \quad \frac{-2}{p+2} \le x \le 1 \quad \text{and} \quad p \in (2,4], \\ \text{and} \quad \frac{-2}{p+2} \le x \le \frac{2}{p-2} \quad \text{and} \quad p \ge 4; \end{gathered} \tag{17}$$

$$H_3(x) = (q-p+1)x^3 + (p-1)x, \quad \frac{2}{p-2} \le x \le 1, \quad p \ge 4. \tag{18}$$

After the determination of the maximal values of $H_1(x), H_2(x)$ and $H_3(x)$ in the corresponding intervals for x, and taking into account that $(p,q) \in D_{10}^+ \cup D_{11}^+ \setminus \{2,1\}$, we find

$$\begin{gathered} \max_{-1\le x\le -2/(p+2)} H_1(x) = H_1\left(-\sqrt{\frac{p+1}{3(q+p+1)}}\,\right) \\ = \frac{2}{3}\,(p+1)\sqrt{\frac{p+1}{3(p+1+q)}}, \\ \text{for all } (p,q) \in D_{10}^+ \cup D_{11}^+; \end{gathered} \tag{19}$$

(20)
$$\max_{-2/(p+2)\le x\le 1} H_2(x) = H_2\left(x_0 = \frac{2}{3}\,\frac{p^2-4}{p^2-4q}\right) = 1 + \frac{1}{27}\,\frac{(p^2-4)^3}{(p^2-4q)^2},$$
$$\text{for} \quad p \in (2,4] \quad \text{and} \quad (p,q) \in D_{10}^{+} \setminus \{2,1\};$$

(21)
$$\max_{-2/(p+2)\le x\le 2/(p-2)} H_2(x) = \begin{cases} H_2\Big(\dfrac{2}{3}\,\dfrac{p^2-4}{p^2-4q}\Big) & \text{for } (p,q)\in D_{11}^{+} \text{ and} \\ & q \le -\dfrac{1}{12}(p-1)(p^2-4p-8) \\ H_2\Big(\dfrac{2}{p-2}\Big) & \text{for } (p,q)\in D_{11}^{+} \text{ and} \\ & q \ge -\dfrac{1}{12}(p-1)(p^2-4p-8), \end{cases}$$

(22)
$$\max_{2/(p-2)\le x\le 1} H_3(x) = \begin{cases} H_3\Big(\dfrac{2}{p-2}\Big) & \text{for } (p,q)\in D_{11}^{+} \text{ and} \\ & q \le -\dfrac{1}{12}(p-1)(p^2-4p-8) \\ H_3\Big(\sqrt{\dfrac{p-1}{3(p-1-q)}}\,\Big) & \text{for } (p,q)\in D_{11}^{+} \text{ and} \\ \quad = \dfrac{2}{3}\,(p-1)\sqrt{\dfrac{p-1}{3(p-1-q)}} & q \ge -\dfrac{1}{12}\,(p-1)(p^2-4p-8) \end{cases}.$$

One can observe that $H_2(2/(p-2)) = H_3(2/(p-2))$. Then, comparing (21) and (22) we find that for $p \ge 4$ and $(p,q) \in D_{11}^{+}$, we have

(23)
$$\max\{\max H_2(x), \max H_3(x)\} = \begin{cases} \dfrac{2}{3}\,(p-1)\sqrt{\dfrac{p-1}{3(p-1-q)}} & \text{if } q \ge -\dfrac{1}{12}(p-1)(p^2-4p-8) \\ 1 + \dfrac{1}{27}\,\dfrac{(p^2-4)^3}{(p^2-4q)^2} & \text{if } q \le -\dfrac{1}{12}\,(p-1)(p^2-4p-8) \end{cases}.$$

Now we have to compare (23) and the maximal value of $H_1(x)$, i.e.,

$$\frac{2}{3}\,(p+1)\sqrt{\frac{p+1}{3(p+1+q)}} \quad \text{for } p \geq 4, (p,q) \in D_{11}^{+},$$

and maximal value of $H_1(x)$ with

$$1+\frac{1}{27}\,\frac{(p^2-4)^3}{(p^2-4q)^2} \quad \text{for } p \in (2,4] \text{ and } (p,q) \in D_{10}^{+}.$$

The inequality

$$\frac{2}{3}\,(p-1)\sqrt{\frac{p-1}{3(p-1-q)}} \geq \frac{2}{3}\,(p+1)\sqrt{\frac{p+1}{3(p+1+q)}}$$

holds for $(p,q) \in D_{11}^{+}$ and $q \geq -(1/12)(p-1)(p^2-4p-8)$ if and only if $p \geq p_0$ and $q \geq 2[(p^2-1)/(p^2+3)]$.

The simple considerations prove that the equation

$$1+\frac{1}{27}\,\frac{(p^2-4)^3}{(p^2-4q)^2} = \frac{2}{3}\,(p+1)\sqrt{\frac{p+1}{3(p+1+q)}} \tag{24}$$

has for fixed $p \in (2,4]$ and $q \in [(2p(p+1))/(p^2+2p+4); (1/12)(p^2+8)]$ as well as for fixed $p \in (4,p_1)$ and $q \in [(2p(p+1))/(p^2+2p+4), -(1/12)(p-1)(p^2-4p-8)]$ exactly one solution $q = q_0(p)$ (the number, p_1, $p_1 > p_0$ is the unique root of the equation $(2p(p+1))(p^2+2p+4)^{-1} = -(1/12)(p-1)(p^2-4p-8)$.

After some calculations one can reduce the equation (24) to the equation of the third degree (10) where we put $t = (p^2-4q)$.

The number $p_0 \in (4.51; 4.52)$ being the root of the equation $p^3-4p^2-5p+12 = 0$ is this particular value of p for which the curves $q = q_0(p)$, $q = 2(p^2-1)/(p^2+3)$ and $q = -(1/12)(p-1)(p^2-4p-8)$ meet.

Taking into account (19), (20) and (23), we get (8).

The extremal function for $p, q \in (D_{10}^{0} \cup D_{11}^{0})$ has the form

$$\begin{aligned} \omega(z) &= \omega_0^r(z) = c_1^{(r)}z + c_2^{(r)}z^2 + c_3^{(r)}z^3 + \cdots \\ &= z\left(\frac{\tau - z}{1-\tau z}\right)\left(\frac{\bar{\tau} - z}{1-\bar{\tau} z}\right), \quad |\tau| < 1, \end{aligned} \tag{25}$$

where

$$\begin{cases} c_1^{(r)} = x_0 = \dfrac{2}{3}\, \dfrac{p^2-4}{p^2-4q} \\ c_2^{(r)} = \dfrac{1}{2}\, p x_0 (1-x_0) \\ c_3^{(r)} = 1 - x_0^2 - \dfrac{y_0^2}{1-x_0} = 1 - x_0^2 - \dfrac{1}{4}\, p^2 x_0^2 (1-x_0). \end{cases}$$

Comparing the coefficients in (25) we can determine the value of τ and the final form (11) of the extremal function follows. This ends the proof of Theorem 1. □

2. (a) Now we will apply Theorem 1 to get the sharp bound, depending on $t = -\log a_0$, for $|a_3|$ within the class B_0^r of holomorphic functions in $|z| < 1$ of the form

$$f(z) = e^{-t} + a_1 z + a_2 z^2 + \dots\,, \quad t > 0, |z| < 1, \tag{26}$$

which satisfy the condition $0 < |f(z)| < 1$, $|z| < 1$, and have real coefficients.

It is well known that

$$f \in B_0^r \Longleftrightarrow f(z) = \exp\left\{ -t\, \frac{1-\omega(z)}{1+\omega(z)} \right\}, \quad \omega \in \Omega^r. \tag{27}$$

Comparison of the coefficients in (27) together with (26) and (1) implies

$$\begin{cases} a_1 = 2te^{-t} c_1 \\ a_2 = 2te^{-t} (c_2 + (t-1) c_1^2) \\ a_3 = 2te^{-t} \Big\{ c_3 + 2(t-1) c_1 c_2 + \dfrac{1}{3}\, (2t^2 - 6t + 3) c_1^3 \Big\}. \end{cases} \tag{28}$$

Theorem 2. *If $f \in B_0^r$, then we have the following sharp estimates*

(29)
$$|a_3| \le \begin{cases} 2te^{-t} & \text{for } t \in (0, t_1] \\ \dfrac{2\sqrt{2}}{3}\, e^{-t}(2t-1)^{3/2} & \text{for } t \in [t_1, t_2] \\ \dfrac{2\sqrt{2}}{3}\, te^{-t}(2t-3)^{3/2}(-t^2+6t-6)^{-1/2} & \text{for } t \in [t_2, t_3] \\ \dfrac{2}{3}\, te^{-t}(2t^2-6t+3) & \text{for } t \in [t_3, +\infty). \end{cases}$$

The numbers $t_1 = 1.65\ldots$, $t_2 = 3.30\ldots$, $t_3 = 3.82\ldots$ are the roots of the equations

$$\begin{cases} 16t^3 - 33t^2 + 12t - 2 = 0 \\ 8t^4 - 40t^3 + 50t^2 - 18t + 3 = 0 \\ 2t^2 - 10t + 9 = 0, \end{cases} \tag{30}$$

respectively.

Proof. By the formula (28) the estimate of $|a_3|$ is equivalent to the value of the bound of functional (2) for $\omega \in \Omega^r$ with $p = 2(t-1)$ and $q = (1/3)(2t^2 - 6t + 3)$.

Therefore, the extremal values for $|a_3|$ follow from (8). For the determination of the numbers t_1, t_2 and t_3 we have to find the points of intersection of the parabola

$$q = \frac{1}{6}\,(p^2 - 2p - 2), \quad p \ge -2 \tag{31}$$

and the boundary curves of the sets D_k given by (9). The parabola (31) starts at the point $(-2, 1)$ and intersects the curves:

$$q = \frac{4}{27}\,(p+1)^3 - (p+1);$$
$$q = 2\,\frac{p^2-1}{p^2+3} \quad \text{and} \quad q = \frac{2}{3}\,(p-1).$$

The curve $q = 2(p^2-1)/(p^2+3)$ intersects with parabola (31) for $p = \hat{p} \in (4.6; 4.7)$.

The above results together with the estimates for B_0 given in [**4**] imply (29) and (30), which ends the proof.

Remark. One can observe by comparing the estimates (29) for the class B_0^r with those for the class B_0 [**4**] that they are the same except for the interval $3.30\ldots \le t \le 3.47\ldots$.

(b) An interesting application of Theorem A is the coefficient problem within the class of bounded univalent and starlike functions in the unit disk $|z| < 1$.

For fixed $M > 1$, let S_M^* denote the class of holomorphic, univalent functions in $|z| < 1$ of the form:

$$F(z) = z + a_2 z^2 + a_3 z^3 + \cdots, |z| < 1, \tag{32}$$

which satisfy the conditions

$$|F(z)| < M \quad \text{and} \quad \operatorname{Re} \frac{zF'(z)}{F(z)} > 0 \quad \text{for } |z| < 1.$$

For this class of functions very little is known about the coefficient's estimate.

Namely, we have sharp bound

$$|a_2| \le 2(1-a), \quad a = \frac{1}{M} \in (0,1)$$

and the sign of equality holds for the so-called Pick functions $P_M(z)$

$$P_M(z) = z + A_2 z^2 + A_3 z^3 + \ldots, |z| < 1 \tag{33}$$

which maps the disk $|z| < 1$ onto the disk $|w| < M$ minus the segment $[-M, -M(2M - 1 - 2\sqrt{M(M-1)})]$.

In [**1**] Barnard and Lewis proved a very interesting result that, if $f \in S_M^*$, then

$$\log \frac{F(z)}{z} \prec \log \frac{P_M(z)}{z}, \tag{34}$$

that is, $\log(F(z)/z) = \log(P_M(\omega(z))/\omega(z))$, $\omega \in \Omega$.

Comparison of the coefficients in (34) together with (1) and (32) implies

$$\begin{cases} a_2 = A_2 c_1 \\ a_3 = A_2 c_2 + A_3 c_1^2 \\ a_4 = A_4 c_1^3 + 2A_3 c_1 c_2 + A_2 c_3, \end{cases} \tag{35}$$

where

$$\begin{cases} A_2 = 2(1-a) \\ A_3 = (1-a)(3-5a) \\ A_4 = 2(1-a)(2-8a+7a^2), \end{cases} \tag{36}$$
$$a = \frac{1}{M} \in (0,1).$$

Therefore, we have

Theorem 3. *If $F \in S_M^*$, then we have the following estimates*

$$|a_3| \le \begin{cases} (1-a)(3-5a) & \text{for } a \in (0,(1/5)] \\ 2(1-a) & \text{for } a \in [(1/5),1] \end{cases} \tag{37}$$

(38)
$$|a_4| \le \begin{cases} 2(1-a)(2-8a+7a^2) & \text{for } a \in \left(0, \frac{7}{59}\right] \\ \dfrac{2\frac{5}{3}^{3/2}(1-a)(1-5a)^{3/2}(2-8a+7a^2)}{\sqrt{(1-7a)}\,(1+3a)} & \text{for } a \in \left[\frac{7}{59}, a_1^*\right] \\ \dfrac{4\sqrt{3}}{9}\,(1-a)^{1/2}(4-5a)^{3/2}(6-7a)^{-1/2} & \text{for } a \in [a_1^*, a_2^*] \\ 2(1-a) & \text{for } a \in [a_2^*, 1) \end{cases},$$

where $a_1^* \in (0.12, 0.13)$, $a_2^* \in (0.23, 0.24)$ are the unique roots of the equations

$$175a^3 - 305a^2 + 148a - 14 = 0$$
$$500a^3 - 1011a^2 + 609a - 94 = 0,$$

respectively.

Proof. The estimate (37) was given in [**1**] and follows from (35) and (36) by direct calculations. In order to obtain (38) we apply Theorem A and the result of Barnard and Lewis (34).

The use of the formulae (35) and (36) and the investigation of the points of intersections yield the curve

$$q = \frac{1}{25}(7p^2 - 2p - 7), \quad p \geq -2,$$

with boundaries of D_k given by (6), ending the proof of Theorem A. □

Remark. It seems that the bounds (37) and (38) are sharp for $M \geq 5$ and $M \geq (59/7)$, respectively.

(c) Another interesting class of functions for which the coefficient problem is still unsolved is the class of α-strongly starlike functions, $0 < \alpha \leq 1$, e.g., [**2, 5**].

Namely, we say that a holomorphic function F of the form (32) is α-strongly starlike, $0 < \alpha \leq 1$, $F \in S^*\langle a \rangle$, if it satisfies the condition

$$\left| \arg \frac{zF'(z)}{F(z)} \right| < \alpha \frac{\pi}{2}, \quad |z| < 1.$$

It is well known that

$$|a_2| \leq 2\alpha \quad \text{and} \quad |a_3| \leq \begin{cases} \alpha & \text{for } \alpha \in \left(0, \frac{1}{3}\right] \\ 3\alpha^2 & \text{for } \alpha \in \left[\frac{1}{3}, 1\right], \end{cases}$$

and these bounds are sharp.

Applying Theorem A, we obtain

Theorem 4. *If $F \in S^*\langle \alpha \rangle$, $\alpha \in (0, 1]$, then the following sharp estimates hold*

$$|a_4| \leq \begin{cases} \frac{2}{3}\alpha & \text{for } \alpha \in \left(0, \sqrt{\frac{2}{17}}\right] \\ \frac{2\alpha}{9}(17\alpha^2 + 1) & \text{for } \alpha \in \left[\sqrt{\frac{2}{17}}, 1\right] \end{cases}. \tag{39}$$

The extremal functions are given by the equations

$$\frac{zF'(z)}{F(z)} = \left(\frac{1+z^3}{1-z^3}\right)^{\alpha} \quad \textit{and} \quad \frac{zF'(z)}{F(z)} = \left(\frac{1+z}{1-z}\right)^{\alpha},$$

respectively.

Remark. The estimates (38) and (39) were already obtained in [**6**].

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