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CONSTANT-SIGN SOLUTIONS OF A SYSTEM OF INTEGRAL EQUATIONS WITH INTEGRABLE SINGULARITIES

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ABSTRACT. We consider the following systems of Fredholm integral equations

$$u_i(t) = \int_0^1 g_i(t,s) f_i(s, u_1(s), u_2(s), \dots, u_n(s))\, ds, \quad t \in [0,1], \quad 1 \le i \le n$$

$$u_i(t) = \int_0^\infty g_i(t,s) f_i(s, u_1(s), u_2(s), \dots, u_n(s))\, ds, \quad t \in [0,\infty), \quad 1 \le i \le n$$

and the system of Volterra integral equations

$$u_i(t) = \int_0^t g_i(t,s) f_i(s, u_1(s), u_2(s), \dots, u_n(s))\, ds, \quad t \in [0,T], \quad 1 \le i \le n,$$

where the nonlinearities f_i, $1 \le i \le n$ may be singular in the independent variable and may also be singular at $u_j = 0$, $j \in \{1, 2, \dots, n\}$. Our aim is to establish criteria such that the above systems have at least one *constant-sign* solution $(u_1, u_2, \dots, u_n)$, i.e., for each $1 \le i \le n$, $\theta_i u_i \ge 0$ where $\theta_i \in \{1, -1\}$ is fixed.

1. Introduction. In this paper we consider three systems of singular integral equations. Specifically we are interested in the following

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systems of Fredholm integral equations

$$(F) \qquad u_i(t) = \int_0^1 g_i(t,s) f_i(s, u_1(s), u_2(s), \dots, u_n(s))\, ds, \quad t \in [0,1], \quad 1 \le i \le n$$

$$(F)_\infty \qquad u_i(t) = \int_0^\infty g_i(t,s) f_i(s, u_1(s), u_2(s), \dots, u_n(s))\, ds, \quad t \in [0,\infty), \quad 1 \le i \le n$$

and the system of Volterra integral equations

$$(V) \qquad u_i(t) = \int_0^t g_i(t,s) f_i(s, u_1(s), u_2(s), \dots, u_n(s))\, ds, \quad t \in [0,T], \quad 1 \le i \le n$$

where $T > 0$ is fixed. The nonlinearities f_i, $1 \le i \le n$ in the above systems may be singular in the independent variable and may also be singular at $u_j = 0$, $j \in \{1, 2, \dots, n\}$.

By using Schauder and Schauder-Tychonoff fixed point theorems, we shall develop existence criteria for a *constant-sign solution* of the above systems. A solution $u = (u_1, u_2, \dots, u_n)$ is said to be of *constant* sign if, for each $1 \le i \le n$, $\theta_i u_i(t) \ge 0$ for t in the respective domain; here $\theta_i \in \{1, -1\}$ is fixed. Note that *positive* solution is a special case of *constant-sign* solution when $\theta_i = 1$ for all $1 \le i \le n$.

There are only a handful of papers in the literature, see [**1**–**10** and the references therein] that tackle particular cases of (F), $(F)_\infty$ and (V), namely, when $n = 1$, $\theta_1 = 1$, and the nonlinearity has the form $f(t, y) = y^{-a}$, $a > 0$. Thus, f is singular only in the dependent variable y. For instance, in [**8, 10**], the following problem that arises in communications, as well as in boundary layer theory in fluid dynamics, is discussed

$$y(t) = \int_0^1 g(t,s) \frac{1}{y(s)}\, ds, \quad t \in [0,1].$$

Karlin and Nirenberg [**6**] have also studied a more general problem

$$y(t) = \int_0^1 g(t,s) \frac{1}{[y(s)]^a}\, ds, \quad t \in [0,1]$$

where $a > 0$ is fixed and g is a nonnegative continuous function on $[0,1] \times [0,1]$.

Our present work uses a new approach to establish new results. In particular, the restrictive conditions in [**6**], namely, (i) $f(t,y)$ is bounded as $y \to \infty$, (ii) g is continuous and bounded, and (iii) $g(t,t) > 0$ for all $t > 0$ are not needed in our theorems. Moreover, we have generalized the problems to (i) *systems*, (ii) *general* form of nonlinearities f_i, $1 \le i \le n$ that can be singular in both independent and dependent variables, (iii) existence of *constant-sign* solutions, which include *positive* solutions as a special case. The paper is outlined as follows. In Section 2 we shall state the necessary fixed point theorems. The existence results for systems (F), (V) and $(F)_\infty$ are presented in Section 3.

2. Preliminaries.

Theorem 2.1 (Schauder fixed point theorem). *Let D be a closed, convex subset of a normed linear space E. Then every compact and continuous map $S : D \to D$ has at least one fixed point.*

Theorem 2.2 (Schauder-Tychonoff fixed point theorem). *Let D be a closed, convex subset of a Fréchet space E. Assume that $S : D \to D$ is continuous, and $S(D)$ is relatively compact in E. Then S has at least one fixed point in D.*

We also require compactness criteria in the various spaces that we work in.

Theorem 2.3 (Arźela-Ascoli theorem). *Let $M \subseteq C[0,T]$. If M is uniformly bounded and equicontinuous, then M is relatively compact in $C[0,T]$.*

Let $BC[0,\infty)$ be the space of bounded continuous functions on $[0,\infty)$, and let

$$C_l[0,\infty) = \Big\{ y \mid y \in BC[0,\infty) \text{ and } \lim_{t\to\infty} y(t) \text{ exists} \Big\}. \tag{2.1}$$

Theorem 2.4 [**4**, p. 62]. *Let* $M \subseteq C_l[0,\infty)$. *Then* M *is compact in* $C_l[0,\infty)$ *if* (a) M *is bounded in* $C_l[0,\infty)$; (b) *the functions in* M *are equicontinuous on any compact interval of* $[0,\infty)$; (c) *the functions in* M *are equiconvergent, i.e., given* $\varepsilon > 0$, *there exists* $T(\varepsilon) > 0$ *such that* $|f(t) - f(\infty)| < \varepsilon$ *for any* $t \geq T(\varepsilon)$ *and* $f \in M$.

3. Main results. In this section we shall present existence results for the systems of integral equations (F), $(F)_\infty$ and (V). Throughout we shall denote $u = (u_1, u_2, \dots, u_n)$, and for $1 \leq j \leq n$,

$$[0,\infty)_j = \begin{cases} [0,\infty) & \text{if } \theta_j = 1, \\ (-\infty, 0] & \text{if } \theta_j = -1. \end{cases} \tag{3.1}$$

System (F). Our first three results are for the system of Fredholm integral equations (F), where the nonlinearities f_i, $1 \leq i \leq n$ may be singular at $u_j = 0$, $j \in \{1, 2, \dots, n\}$ and may also be singular in the independent variable at some set $\Omega \subset [0,1]$ with measure zero. Let the Banach space $B = \{u \mid u \in (C[0,1])^n\}$ be equipped with the norm $\|u\| = \max_{1\leq i\leq n} \sup_{t\in[0,1]} |u_i(t)|$.

Theorem 3.1. *Let* $\theta_i \in \{1, -1\}$, $1 \leq i \leq n$ *be fixed and integers* p, q *be such that* $1 \leq p \leq q \leq \infty$ *and* $1/p + 1/q = 1$. *For each* $1 \leq i \leq n$, *suppose the following conditions are satisfied*:

$$\begin{cases} g_i^t(s) \equiv g_i(t,s) \geq 0 \textit{ for all } t \in [0,1], \textit{ a.e. } s \in [0,1] \textit{ and} \\ g_i^t(s) > 0 \textit{ for a.e. } t \in [0,1], \textit{ a.e. } s \in [0,1]; \end{cases} \tag{3.2}$$

$$\begin{cases} g_i^t(s) \in L^p[0,1] \textit{ for all } t \in [0,1] \textit{ and} \\ \textit{the map } t \to g_i^t \textit{ is continuous from } [0,1] \textit{ to } L^p[0,1]; \end{cases} \tag{3.3}$$

$$\begin{cases} f_i : [0,1] \times (\mathbf{R} \setminus \{0\})^n \to \mathbf{R} \\ \textit{with } t \to f_i(t,u) \textit{ measurable for all } u \in (R \setminus \{0\})^n \\ \textit{and } u \to f_i(t,u) \textit{ continuous for a.e. } t \in (0,1); \end{cases} \tag{3.4}$$

$$\begin{cases} \textit{for any } r_i > 0, \textit{ there exists } \psi_{r_i,i} : [0,1] \to R, \\ \psi_{r_i,i}(t) > 0 \textit{ for a.e. } t \in [0,1], \\ \psi_{r_i,i} \in L^q[0,1] \textit{ such that for all } |u_j| \in (0, r_j],\ 1 \leq j \leq n, \\ \theta_i f_i(t,u) \geq \psi_{r_i,i}(t) \textit{ for a.e. } t \in [0,1]; \end{cases} \tag{3.5}$$

$$(3.6)\qquad \begin{cases} \textit{for any } r_i > 0 \textit{ with } \displaystyle\int_0^1 g_i(t,s)\psi_{r_i,i}(s)\,ds \le r_i \\ \textit{for } t \in [0,1], \textit{ there exists } h_{r_i,i} : [0,1] \to R, \\ h_{r_i,i}(t) \ge 0 \textit{ for a.e. } t \in [0,1], \\ h_{r_i,i} \in L^q[0,1] \textit{ such that} \\ \textit{for all } |u_j| \in \left[\displaystyle\int_0^1 g_j(t,s)\psi_{r_j,j}(s)\,ds, r_j \right],\ 1 \le j \le n, \\ \theta_i f_i(t,u) \le h_{r_i,i}(t) \textit{ for a.e. } t \in [0,1]; \end{cases}$$

$$(3.7)\qquad \begin{cases} \textit{there exists } M_i > 0 \textit{ such that for } t \in [0,1], \\ M_i \ge \displaystyle\int_0^1 g_i(t,s)h_{M_i,i}(s)\,ds \ge \int_0^1 g_i(t,s)\psi_{M_i,i}(s)\,ds. \end{cases}$$

Then, (F) *has a constant-sign solution* $u \in (C[0,1])^n$ *with* $\theta_i u_i(t) > 0$*, almost every* $t \in [0,1]$*,* $1 \le i \le n$*.*

Proof. To begin, we define a closed convex subset of $B = (C[0,1])^n$ as

$$D = \left\{ u \in B \;\Big|\; \int_0^1 g_i(t,s)h_{M_i,i}(s)\,ds \ge \theta_i u_i(t) \ge \int_0^1 g_i(t,s)\psi_{M_i,i}(s)\,ds \right.$$
$$\left. \text{for } t \in [0,1],\ 1 \le i \le n \right\}.$$

Let the operator $S : D \to B$ be defined by

$$(3.8)\qquad Su(t) = (S_1u(t), S_2u(t), \dots, S_nu(t)), \quad t \in [0,1]$$

where

$$(3.9)\qquad S_iu(t) = \int_0^1 g_i(t,s)f_i(s,u(s))\,ds, \quad t \in [0,1], \quad 1 \le i \le n.$$

Clearly, a fixed point of the operator S is a solution of the system (F). Indeed, a fixed point of S obtained in D will be a *constant-sign solution* of the system (F).

First we shall show that S maps D into D. Let $u \in D$. By (3.7) it is clear that

(3.10)
$$M_i \ge \int_0^1 g_i(t,s) h_{M_i,i}(s)\,ds \ge \theta_i u_i(t) \ge \int_0^1 g_i(t,s)\psi_{M_i,i}(s)\,ds > 0, \quad t \in [0,1], \quad 1 \le i \le n.$$

Hence, it follows from (3.5) that

$$\theta_i f_i(t,u) \ge \psi_{M_i,i}(t), \quad \text{a.e.} \quad t \in [0,1], \quad 1 \le i \le n$$

and subsequently

$$\theta_i S_i u(t) = \int_0^1 g_i(t,s)\theta_i f_i(s,u(s))\,ds \ge \int_0^1 g_i(t,s)\psi_{M_i,i}(s)\,ds, \quad t \in [0,1], \quad 1 \le i \le n. \tag{3.11}$$

Also, from (3.6) and (3.10) we have

$$\theta_i f_i(t,u) \le h_{M_i,i}(t), \quad \text{a.e.} \quad t \in [0,1], \quad 1 \le i \le n$$

and so

$$\theta_i S_i u(t) \le \int_0^1 g_i(t,s) h_{M_i,i}(s)\,ds, \quad t \in [0,1], \quad 1 \le i \le n. \tag{3.12}$$

Having obtained (3.11) and (3.12), we have shown that $S : D \to D$.

Next, we shall prove that $S : D \to D$ is continuous. Let $\{u^m\}$ be a sequence in D and $u^m \to u$ in B. Then, we find for $t \in [0,1]$ and $1 \le i \le n$,

$$\begin{aligned} |S_i u^m(t) - S_i u(t)| &\le \int_0^1 g_i(t,s)|f_i(s,u^m(s)) - f_i(s,u(s))|\,ds \\ &\le \left(\int_0^1 [g_i(t,s)]^p\,ds\right)^{1/p} \\ &\quad \times \left(\int_0^1 |f_i(s,u^m(s)) - f_i(s,u(s))|^q\,ds\right)^{1/q}. \end{aligned}$$

Since

$$\int_0^1 |f_i(s,u^m(s)) - f_i(s,u(s))|^q \, ds \le 2^q \int_0^1 [h_{M_i,i}(s)]^q \, ds < \infty, \quad 1 \le i \le n,$$

together with (3.3) and (3.4), the Lebesgue dominated convergence theorem gives for each $1 \le i \le n$,

$$\begin{aligned}\sup_{t\in[0,1]} |S_i u^m(t) - S_i u(t)| &\le \left(\sup_{t\in[0,1]} \int_0^1 [g_i(t,s)]^p \, ds \right)^{1/p} \\ &\quad \times \left(\int_0^1 |f_i(s,u^m(s)) - f_i(s,u(s))|^q \, ds \right)^{1/q} \longrightarrow 0\end{aligned}$$

as $m \to \infty$, or $\|Su^m - Su\| \to 0$ as $m \to \infty$. Hence, S is continuous.

Finally, we shall check that $S : D \to D$ is compact. Let $u \in D$. Then, by (3.12) and (3.7) we have

$$\sup_{t\in[0,1]} |S_i u(t)| \le \sup_{t\in[0,1]} \int_0^1 g_i(t,s) h_{M_i,i}(s) \, ds \le M_i, \quad 1 \le i \le n$$

or $\|Su\| \le \max_{1\le i\le n} M_i$. Further, using (3.12) and (3.3) we get for $t, t' \in [0,1]$ and $1 \le i \le n$,

$$\begin{aligned}|S_i u(t) - S_i u(t')| &\le \int_0^1 |g_i(t,s) - g_i(t',s)| h_{M_i,i}(s) \, ds \\ &\le \left(\int_0^1 |g_i^t(s) - g_i^{t'}(s)|^p \, ds \right)^{1/p} \\ &\quad \times \left(\int_0^1 [h_{M_i,i}(s)]^q \, ds \right)^{1/q} \longrightarrow 0\end{aligned}$$

as $t \to t'$. Now Theorem 2.3 guarantees that S is compact.

Hence, we conclude from Theorem 2.1 that S has a fixed point in D. The proof is complete. □

Remark 3.1. In Theorem 3.1, the condition (3.6) can be replaced by the following:

(3.6′)

$$\begin{cases} \text{for any } r_i > 0 \text{ with } \displaystyle\int_0^1 g_i(t,s)\psi_{r_i,i}(s)\,ds \le r_i \text{ for } t \in [0,1], \text{ let} \\ h_{r_i,i}(t) = \sup\left\{ f_i(t,u) : |u_j| \in \left[\displaystyle\int_0^1 g_j(t,s)\psi_{r_j,j}(s)\,ds, rj \right],\ 1 \le j \le n \right\} \\ \text{and assume } h_{r_i,i} \in L^q[0,1]. \end{cases}$$

Remark 3.2. If f_i, $1 \le i \le n$ are nonsingular, i.e., $f_i : [0,1] \times \mathbf{R}^n \to \mathbf{R}$, then we can have a modified Theorem 3.1 with (3.5)–(3.7) replaced by the following conditions:

$$\begin{cases} \text{for any } r_i > 0, \text{ there exists } h_{r_i,i} : [0,1] \to R, \\ h_{r_i,i}(t) \ge 0 \text{ for a.e. } t \in [0,1], \\ h_{r_i,i} \in L^q[0,1] \text{ such that for all } |u_j| \in [0, r_j],\ 1 \le j \le n, \\ 0 \le \theta_i f_i(t,u) \le h_{r_i,i}(t) \text{ for a.e. } t \in [0,1]; \end{cases}$$

there exists $M_i > 0$ such that for $t \in [0,1]$, $M_i \ge \displaystyle\int_0^1 g_i(t,s) h_{M_i,i}(s)\,ds \ge 0$.

Moreover, the conclusion of the modified Theorem 3.1 becomes: system (F) has a constant-sign solution $u \in (C[0,1])^n$ with $\theta_i u_i(t) \ge 0$, $t \in [0,1]$, $1 \le i \le n$.

Theorem 3.2. *Let* $\theta_i \in \{1, -1\}$, $1 \le i \le n$ *be fixed and integers* p, q *be such that* $1 \le p \le q \le \infty$ *and* $1/p + 1/q = 1$. *For each* $1 \le i \le n$, *suppose* (3.2)–(3.5) *hold and the following conditions are satisfied*:

(3.13)

$$\begin{cases} \theta_i f_i(t,u) \le \phi_i(t)[\rho_i(u) + \tau_i(u)] \textit{ for } (t,u) \in [0,1] \times \prod_{j=1}^n [0,\infty)_j, \\ \textit{where } \phi_i : [0,1] \to \mathbf{R},\ \phi_i(t) > 0 \textit{ for a.e. } t \in [0,1], \\ \rho_i, \tau_i : \prod_{j=1}^n (0,\infty)_j \to (0,\infty) \textit{ are continuous}, \\ \textit{if } |u_j| \le |v_j| \textit{ for some } j \in \{1,2,\dots,n\}, \\ \textit{then } \rho_i(u_1,\dots,u_j,\dots,u_n) \ge \rho_i(u_1,\dots,v_j,\dots,u_n) \textit{ and} \\ \tau_i(u_1,\dots,u_j,\dots,u_n) \le \tau_i(u_1,\dots,v_j,\dots,u_n); \end{cases}$$

$$(3.14)\quad \begin{cases} \phi_i \in L^q[0,1], \ \textit{and for any } r_j > 0, \ 1 \le j \le n, \\ \phi_i(t)\rho_i\bigg(\theta_1 \displaystyle\int_0^1 g_1(t,s)\psi_{r_1,1}(s)\,ds, \\ \theta_2 \displaystyle\int_0^1 g_2(t,s)\psi_{r_2,2}(s)\,ds, \dots, \theta_n \displaystyle\int_0^1 g_n(t,s)\psi_{r_n,n}(s)\,ds\bigg) \\ \in L^q[0,1]; \end{cases}$$

(3.15)

$$\begin{cases} \textit{there exists } M_i > 0 \textit{ such that for } t \in [0,1], \\ M_i \ge \displaystyle\int_0^1 g_i(t,s)\phi_i(s)\bigg[\tau_i(\theta_1 M_1, \theta_2 M_2, \dots, \theta_n M_n) \\ + \rho_i\bigg(\theta_1 \displaystyle\int_0^1 g_1(s,x)\psi_{M_1,1}(x)\,dx, \ \theta_2 \displaystyle\int_0^1 g_2(s,x)\psi_{M_2,2}(x)\,dx, \dots, \\ \theta_n \displaystyle\int_0^1 g_n(s,x)\psi_{M_n,n}(x)\,ds\bigg)\bigg]\,dx \\ \ge \displaystyle\int_0^1 g_i(t,s)\psi_{M_i,i}(s)\,ds. \end{cases}$$

Then, (F) has a constant-sign solution $u \in (C[0,1])^n$ with $\theta_i u_i(t) > 0$, almost every $t \in [0,1]$, $1 \le i \le n$.

Proof. We shall show that (3.6) and (3.7) are satisfied; then the conclusion is immediate from Theorem 3.1. In view of (3.13), we obtain for almost every $t \in [0,1]$, $|u_j| \in [\int_0^1 g_j(t,s)\psi_{r_j,j}(s)\,ds, r_j]$, $1 \le j \le n$ and $1 \le i \le n$,

$$(3.16)\quad \theta_i f_i(t,u) \le \\ \phi_i(t)\bigg[\rho_i\bigg(\theta_1 \int_0^1 g_1(t,s)\psi_{r_1,1}(s)\,ds, \ \theta_2 \int_0^1 g_2(t,s)\psi_{r_2,2}(s)\,ds, \dots, \\ \theta_n \int_0^1 g_n(t,s)\psi_{r_n,n}(s)\,ds\bigg) + \tau_i(\theta_1 r_1, \theta_2 r_2, \dots, \theta_n r_n)\bigg] \equiv h_{r_i,i}(t).$$

Observe that we have picked $h_{r_i,i}(t)$ to be the right-hand side of (3.16). Now, (3.6) is fulfilled since (3.14) ensures that $h_{r_i,i} \in L^q[0,1]$. Further, (3.15) implies (3.7). □

As an application of Theorem 3.2, we consider a special case of system (F), viz.,

$$u_i(t) = \int_0^1 g_i(t,s)\theta_i\phi_i(s)[\rho_i(u(s)) + \tau_i(u(s))]\,ds, \quad t \in [0,1], \quad 1 \le i \le n, \tag{3.17}$$

where $\theta_i \in \{1,-1\}$, $1 \le i \le n$ are fixed.

Theorem 3.3. *Let $\theta_i \in \{1,-1\}$, $1 \le i \le n$ be fixed and integers p, q be such that $1 \le p \le q \le \infty$ and $1/p + 1/q = 1$. For each $1 \le i \le n$, suppose* (3.2) *and* (3.3) *hold and the following conditions are satisfied*:

$$\begin{cases} \phi_i : [0,1] \to \mathbf{R},\ \phi_i(t) > 0 \text{ for a.e. } t \in [0,1], \\ \rho_i,\ \tau_i : \prod_{j=1}^n (0,\infty)_j \to (0,\infty) \text{ are continuous}, \\ \text{if } |u_j| \le |v_j| \text{ for some } j \in \{1,2,\dots,n\}, \\ \text{then } \rho_i(u_1,\dots,u_j,\dots,u_n) \ge \rho_i(u_1,\dots,v_j,\dots,u_n) \text{ and} \\ \tau_i(u_1,\dots,u_j,\dots,u_n) \le \tau_i(u_1,\dots,v_j,\dots,u_n); \end{cases} \tag{3.18}$$

(3.19)

$$\begin{cases} \phi_i \in L^q[0,1], \text{ and for any } r_j > 0,\ 1 \le j \le n, \\ \phi_i(t)\rho_i\bigg(\theta_1\rho_1(\theta_1 r_1, \theta_2 r_2, \dots, \theta_n r_n)\displaystyle\int_0^1 g_1(t,s)\phi_1(s)\,ds, \\ \theta_2\rho_2(\theta_1 r_1, \theta_2 r_2, \dots, \theta_n r_n)\displaystyle\int_0^1 g_2(t,s)\phi_2(s)\,ds, \dots, \\ \theta_n\rho_n(\theta_1 r_1, \theta_2 r_2, \dots, \theta_n r_n)\displaystyle\int_0^1 g_n(t,s)\phi_n(s)\,ds\bigg) \in L^q[0,1]; \end{cases}$$

$$\begin{cases} \text{there exists } M_i > 0 \text{ such that for } t \in [0,1], \\ M_i \ge \displaystyle\int_0^1 g_i(t,s)\phi_i(s)\bigg[\tau_i(\theta_1 M_1, \theta_2 M_2, \dots, \theta_n M_n) \\ +\rho_i\bigg(\theta_1\rho_1(\theta_1 M_1, \theta_2 M_2, \dots, \theta_n M_n)\displaystyle\int_0^1 g_1(s,x)\phi_1(x)\,dx, \\ \theta_2\rho_2(\theta_1 M_1, \theta_2 M_2, \dots, \theta_n M_n)\displaystyle\int_0^1 g_2(s,x)\phi_2(x)\,dx, \dots, \\ \theta_n\rho_n(\theta_1 M_1, \theta_2 M_2, \dots, \theta_n M_n)\displaystyle\int_0^1 g_n(s,x)\phi_n(x)\,dx\bigg)\bigg]\,ds \\ \ge \rho_i(\theta_1 M_1, \theta_2 M_2, \dots, \theta_n M_n)\displaystyle\int_0^1 g_i(t,s)\phi_i(s)\,ds. \end{cases} \tag{3.20}$$

Then (3.17) *has a constant-sign solution* $u \in (C[0,1])^n$ *with* $\theta_i u_i(t) > 0$*, almost every* $t \in [0,1]$*,* $1 \le i \le n$*.*

Proof. Taking $\psi_{r_i,i}(t) = \phi_i(t)\rho_i(\theta_1 r_1, \theta_2 r_2, \dots, \theta_n r_n)$, the conclusion follows immediately from Theorem 3.2. □

Example 3.1. Consider (3.17) where, for each $1 \le i \le n$,

$$\begin{gathered}\theta_i = 1, \quad \rho_i(u) = |u_i|^{-\alpha_i}, \qquad \tau_i(u) = A_i|u_i|^{\beta_i} + B_i, \\ 0 < \alpha_i < 1, \quad 0 \le \beta_i < 1, \quad A_i, B_i \ge 0,\end{gathered} \tag{3.21}$$

$$g_i \text{ fulfills (3.2) and (3.3),} \qquad \phi_i \text{ satisfies (3.18) and (3.19).} \tag{3.22}$$

Then, (3.20) reduces to

(3.23)

$$\begin{aligned} M_i &\ge \int_0^1 g_i(t,s)\phi_i(s)\left[A_i M_i^{\beta_i} + B_i + M_i^{\alpha_i^2}\left(\int_0^1 g_i(s,x)\phi_i(x)\,dx\right)^{-\alpha_i}\right] ds \\ &\ge M_i^{-\alpha_i}\int_0^1 g_i(t,s)\phi_i(s)\,ds, \quad 1 \le i \le n, \end{aligned}$$

which is satisfied for large M_i. Thus, by Theorem 3.3 the system (3.17) with (3.21) and (3.22) has a constant-sign solution $u \in (C[0,1])^n$ with $\theta_i u_i(t) > 0$, almost everywhere $t \in [0,1]$, $1 \le i \le n$.

System (V). Next, we shall investigate the system of Volterra integral equations (V), where the nonlinearities f_i, $1 \le i \le n$, may be singular at $u_j = 0$, $j \in \{1,2,\dots,n\}$, and may also be singular in the independent variable at some set $\Omega \subset [0,T]$ with measure zero. Let the Banach space $B = \{u \mid u \in (C[0,T])^n\}$ be equipped with the norm $\|u\| = \max_{1\le i\le n} \sup_{t\in[0,T]} |u_i(t)|$.

Theorem 3.4. *Let* $\theta_i \in \{1,-1\}$*,* $1 \le i \le n$ *be fixed, and let integers* p, q *be such that* $1 \le p \le q \le \infty$ *and* $1/p + 1/q = 1$*. For each* $1 \le i \le n$*, suppose the following conditions are satisfied*:

$$\begin{cases} \textit{for all } t \in [0,T],\ g_i^t(s) \equiv g_i(t,s) \ge 0 \textit{ for a.e. } s \in [0,t] \textit{ and} \\ \textit{for a.e. } t \in [0,T],\ g_i^t(s) > 0 \textit{ for a.e. } s \in [0,t]; \end{cases} \tag{3.24}$$

(3.25)

$$g_i^t(s) \in L^p[0,t] \quad \textit{for all} \quad t \in [0,T] \quad \textit{and} \quad \sup_{t\in[0,T]} \int_0^t [g_i^t(s)]^p\,ds < \infty;$$

$$\textit{for any} \quad t,\, t' \in [0,T],$$

(3.26)
$$\int_0^{\min\{t,t'\}} |g_i^t(s) - g_i^{t'}(s)|^p\,ds \longrightarrow 0 \textit{ as } t \to t';$$

(3.27)
$$\begin{cases} f_i : [0,T] \times (\mathbf{R} \setminus \{0\})^n \to \mathbf{R} \textit{ with} \\ t \to f_i(t,u) \textit{ measurable for all } u \in (\mathbf{R} \setminus \{0\})^n \\ \textit{and } u \to f_i(t,u) \textit{ continuous for a.e. } t \in (0,T); \end{cases}$$

(3.28)

$$\begin{cases} \textit{for any } r_i > 0, \textit{ there exists } \psi_{r_i,i} : [0,T] \to R,\ \psi_{r_i,i}(t) > 0 \\ \textit{for a.e. } t \in [0,T],\ \psi_{r_i,i} \in L^q[0,T] \textit{ s.t. } \forall\, |u_j| \in (0, r_j],\ 1 \le j \le n, \\ \theta_i f_i(t,u) \ge \psi_{r_i,i}(t) \textit{ for a.e. } t \in [0,T]; \end{cases}$$

(3.29)

$$\begin{cases} \textit{for any } r_i > 0 \textit{ with } \displaystyle\int_0^t g_i(t,s)\psi_{r_i,i}(s)\,ds \le r_i \textit{ for } t \in [0,T], \\ \exists\, h_{r_i,i} : [0,T] \to \mathbf{R},\ h_{r_i,i}(t) \ge 0 \textit{ for a.e. } t \in [0,T],\ h_{r_i,i} \in L^q[0,T] \\ \textit{s.t. for a.e. } t \in [0,T] \textit{ and all } |u_j| \in \left[\displaystyle\int_0^t g_j(t,s)\psi_{r_j,j}(s)\,ds, r_j\right], \\ 1 \le j \le n,\ \theta_i f_i(t,u) \le h_{r_i,i}(t); \end{cases}$$

(3.30)
$$\begin{cases} \textit{there exists } M_i > 0 \textit{ such that for } t \in [0,T], \\ M_i \ge \displaystyle\int_0^t g_i(t,s)h_{M_i,i}(s)\,ds \ge \int_0^t g_i(t,s)\psi_{M_i,i}(s)\,ds. \end{cases}$$

Then, (V) has a constant-sign solution $u \in (C[0,T])^n$ with $\theta_i u_i(t) > 0$, almost every $t \in [0,T]$, $1 \le i \le n$.

Proof. Define a closed convex subset of $B = (C[0,T])^n$ as

$$D = \left\{ u \in B \;\Big|\; \int_0^t g_i(t,s)h_{M_i,i}(s)\,ds \ge \theta_i u_i(t) \ge \int_0^t g_i(t,s)\psi_{M_i,i}(s)\,ds \right.$$
$$\left. \text{for } t \in [0,T],\ 1 \le i \le n \right\}.$$

Let the operator $S : D \to B$ be defined by

$$Su(t) = (S_1u(t), S_2u(t), \dots, S_nu(t)), \quad t \in [0,T], \tag{3.31}$$

where

$$S_iu(t) = \int_0^t g_i(t,s) f_i(s,u(s))\, ds, \quad t \in [0,T], \quad 1 \le i \le n. \tag{3.32}$$

Clearly, a fixed point of S obtained in D will be a constant-sign solution of the system (V).

Following a similar argument as in the proof of Theorem 3.1, we can show that S maps D into D.

Next, we shall prove that $S : D \to D$ is continuous. Let $\{u^m\}$ be a sequence in D and $u^m \to u$ in B. Then, we have for $t \in [0,T]$ and $1 \le i \le n$,

$$\begin{aligned} &|S_iu^m(t) - S_iu(t)| \\ &\quad \le \int_0^t g_i(t,s)|f_i(s,u^m(s)) - f_i(s,u(s))|\, ds \\ &\quad \le \left(\int_0^t [g_i(t,s)]^p\, ds \right)^{1/p} \left(\int_0^T |f_i(s,u^m(s)) - f_i(s,u(s))|^q\, ds \right)^{1/q}. \end{aligned}$$

Noting that

$$\int_0^T |f_i(s,u^m(s)) - f_i(s,u(s))|^q\, ds \le 2^q \int_0^T [h_{M_i,i}(s)]^q\, ds < \infty, \quad 1 \le i \le n$$

and also (3.25) and (3.27), the Lebesgue dominated convergence theorem yields for each $1 \le i \le n$,

$$\begin{aligned} &\sup_{t\in[0,T]} |S_iu^m(t) - S_iu(t)| \\ &\le \left(\sup_{t\in[0,T]} \int_0^t [g_i(t,s)]^p\, ds \right)^{1/p} \left(\int_0^T |f_i(s,u^m(s)) - f_i(s,u(s))|^q\, ds \right)^{1/q} \\ &\qquad \longrightarrow 0 \end{aligned}$$

as $m \to \infty$, or $\|Su^m - Su\| \to 0$ as $m \to \infty$. Hence, S is continuous.

Finally, we shall show that $S : D \to D$ is compact. Let $u \in D$. Then, by (3.29) and (3.30) we have

$$\sup_{t\in[0,T]} |S_i u(t)| \le \sup_{t\in[0,T]} \int_0^t g_i(t,s) h_{M_i,i}(s)\, ds \le M_i, \quad 1 \le i \le n$$

or $\|Su\| \le \max_{1\le i\le n} M_i$. Further, in view of (3.25), (3.26) and (3.29), we get for $t, t' \in [0,T]$, with $t' < t$ and $1 \le i \le n$,

$$\begin{aligned}
&|S_i u(t) - S_i u(t')| \\
&\quad \le \int_0^{t'} |g_i(t,s) - g_i(t',s)| f_i(s,u(s))\, ds + \int_{t'}^t g_i(t,s) f_i(s,u(s))\, ds \\
&\quad \le \int_0^{t'} |g_i(t,s) - g_i(t',s)| h_{M_i,i}(s)\, ds + \int_{t'}^t g_i(t,s) h_{M_i,i}(s)\, ds \\
&\quad \le \left(\int_0^{t'} |g_i^t(s) - g_i^{t'}(s)|^p\, ds \right)^{1/p} \left(\int_0^T [h_{M_i,i}(s)]^q\, ds \right)^{1/q} \\
&\qquad + \left(\sup_{t\in[0,T]} \int_0^t [g_i^t(s)]^p\, ds \right)^{1/p} \left(\int_{t'}^t [h_{M_i,i}(s)]^q\, ds \right)^{1/q} \longrightarrow 0
\end{aligned}$$

as $t \to t'$. A similar argument also holds for $t' > t$. Now Theorem 2.3 guarantees that S is compact.

It now follows from Theorem 2.1 that S has a fixed point in D. The proof is complete. □

Remark 3.3. In Theorem 3.4, the condition (3.29) can be replaced by the following:
(3.29)′

$$\begin{cases}
\text{for any } r_i > 0 \text{ with } \displaystyle\int_0^t g_i(t,s)\psi_{r_i,i}(s)\, ds \le r_i \text{ for } t \in [0,T], \text{ let} \\
h_{r_i,i}(t) = \sup\left\{ f_i(t,u) : |u_j| \in \left[\displaystyle\int_0^t g_j(t,s)\psi_{r_j,j}(s)\, ds, r_j \right],\ 1 \le j \le n \right\} \\
\text{and assume } h_{r_i,i} \in L^q[0,T].
\end{cases}$$

Remark 3.4. In Theorem 3.4, the condition (3.26) can be replaced by the following: for any $t, t' \in [0,T]$,
(3.26)′
$$\int_0^{\min\{t,t'\}} |g_i^t(s) - g_i^{t'}(s)|^p\, ds + \int_{\min\{t,t'\}}^{\max\{t,t'\}} |g_i^{\max\{t,t'\}}(s)|^p\, ds \longrightarrow 0 \quad \text{as} \quad t \to t'.$$

Note that (3.26)′ implies $\sup_{t\in[0,T]} \int_0^t [g_i^t(s)]^p\, ds < \infty$ in (3.25).

Remark 3.5. If f_i, $1 \le i \le n$ are nonsingular, i.e., $f_i : [0,T]\times\mathbf{R}^n \to \mathbf{R}$, then we can have a modified Theorem 3.4 with (3.28)–(3.30) replaced by the following conditions:

$$\begin{cases} \text{for any } r_i > 0, \text{ there exists } h_{r_i,i} : [0,T] \to R,\ h_{r_i,i}(t) \ge 0, \\ \text{for a.e. } t \in [0,T],\ h_{r_i,i} \in L^q[0,T] \text{ s.t. } \forall\, |u_j| \in [0,r_j],\ 1 \le j \le n, \\ 0 \le \theta_i f_i(t,u) \le h_{r_i,i}(t) \text{ for a.e. } t \in [0,T]; \end{cases}$$

there exists $M_i > 0$ such that for $t \in [0,T]$, $M_i \ge \int_0^t g_i(t,s) h_{M_i,i}(s)\, ds \ge 0$.

Moreover, the conclusion of the modified Theorem 3.4 becomes: system (V) has a constant-sign solution $u \in (C[0,T])^n$ with $\theta_i u_i(t) \ge 0$, $t \in [0,T]$, $1 \le i \le n$.

Theorem 3.5. *Let $\theta_i \in \{1,-1\}$, $1 \le i \le n$ be fixed, and let integers p, q be such that $1 \le p \le q \le \infty$ and $1/p+1/q = 1$. For each $1 \le i \le n$, suppose* (3.24)–(3.28) *hold and the following conditions are satisfied*:

(3.33)
$$\begin{cases} \theta_i f_i(t,u) \le \phi_i(t)[\rho_i(u) + \tau_i(u)] \textit{for } (t,u) \in [0,T] \times \prod_{j=1}^n [0,\infty)_j, \\ \textit{where } \phi^i : [0,T] \to \mathbf{R},\ \phi_i(t) > 0 \textit{ for a.e. } t \in [0,T], \\ \rho_i,\ \tau_i : \prod_{j=1}^n (0,\infty)_j \to (0,\infty) \textit{ are continuous, if } |u_j| \le |v_j| \\ \textit{for some } j \in \{1,2,\dots,n\}, \\ \textit{then } \rho_i(u_1,\dots,u_j,\dots,u_n) \ge \rho_i(u_1,\dots,v_j,\dots,u_n) \textit{ and} \\ \tau_i(u_1,\dots,u_j,\dots,u_n) \le \tau_i(u_1,\dots,v_j,\dots,u_n); \end{cases}$$

(3.34)
$$\begin{cases} \phi_i \in L^q[0,T], \textit{ and for any } r_j > 0,\ 1 \le j \le n, \\ \phi_i(t)\rho_i\bigg(\theta_1 \int_0^t g_1(t,s)\psi_{r_1,1}(s)\,ds,\ \theta_2 \int_0^t g_2(t,s)\psi_{r_2,2}(s)\,ds, \dots , \\ \theta_n \int_0^t g_n(t,s)\psi_{r_n,n}(s)\,ds\bigg) \in L^q[0,T]; \end{cases}$$

(3.35)
$$\begin{cases} \textit{there exists } M_i > 0 \textit{ such that for } t \in [0,T], \\ M_i \ge \int_0^t g_i(t,s)\phi_i(s)\bigg[\tau_i(\theta_1 M_1, \theta_2 M_2, \dots , \theta_n M_n) \\ + \rho_i\bigg(\theta_1 \int_0^s g_1(s,x)\psi_{M_1,1}(x)\,dx,\ \theta_2 \int_0^s g_2(s,x)\psi_{M_2,2}(x)\,dx, \dots , \\ \theta_n \int_0^s g_n(s,x)\psi_{M_n,n}(x)\,dx\bigg)\bigg]\,ds \ge \int_0^t g_i(t,s)\psi_{M_i,i}(s)\,ds. \end{cases}$$

Then, (V) has a constant-sign solution $u \in (C[0,T])^n$ with $\theta_i u_i(t) > 0$, almost every $t \in [0,T]$, $1 \le i \le n$.

Proof. For each $1 \le i \le n$, let

$$h_{r_i,i}(t) = \phi_i(t)\bigg[\tau_i(\theta_1 r_1, \theta_2 r_2, \dots , \theta_n r_n) + \rho_i\bigg(\theta_1 \int_0^t g_1(t,s)\psi_{r_1,1}(s)\,ds, \\ \theta_2 \int_0^t g_2(t,s)\psi_{r_2,2}(s)\,ds, \dots , \theta_n \int_0^t g_n(t,s)\psi_{r_n,n}(s)\,ds\bigg)\bigg].$$

Then, using a similar argument as in the proof of Theorem 3.2, we can show that (3.29) and (3.30) are satisfied, and so the conclusion is immediate from Theorem 3.4. □

As an application of Theorem 3.5, we consider a special case of system (V), viz.,

$$u_i(t) = \int_0^t g_i(t,s)\theta_i\phi_i(s)[\rho_i(u(s)) + \tau_i(u(s))]\,ds, \qquad t \in [0,T],\ 1 \le i \le n \tag{3.36}$$

where $\theta_i \in \{1, -1\}$, $1 \le i \le n$ are fixed.

The following result is immediate from Theorem 3.5. The proof is similar to that of Theorem 3.3.

Theorem 3.6. *Let* $\theta_i \in \{1, -1\}$, $1 \le i \le n$ *be fixed, and let integers* p, q *be such that* $1 \le p \le q \le \infty$ *and* $1/p + 1/q = 1$. *For each* $1 \le i \le n$, *suppose* (3.24)–(3.26) *hold and the following conditions are satisfied*:

$$(3.37)\quad \begin{cases} \phi_i : [0,T] \to \mathbf{R},\ \phi_i(t) > 0 \text{ for a.e. } t \in [0,T], \\ \rho_i,\ \tau_i : \prod_{j=1}^{n} (0,\infty)_j \to (0,\infty) \text{ are continuous}, \\ \text{if } |u_j| \le |v_j| \text{ for some } j \in \{1,2,\ldots,n\}, \\ \text{then } \rho_i(u_1,\ldots,u_j,\ldots,u_n) \ge \rho_i(u_1,\ldots,v_j,\ldots,u_n) \text{ and} \\ \tau_i(u_1,\ldots,u_j,\ldots,u_n) \le \tau_i(u_1,\ldots,v_j,\ldots,u_n); \end{cases}$$

$$(3.38)\quad \begin{cases} \phi_i \in L^q[0,T], \text{ and for any } r_j > 0,\ 1 \le j \le n, \\ \phi_i(t)\rho_i(\theta_1\rho_1(\theta_1 r_1, \theta_2 r_2, \ldots, \theta_n r_n) \displaystyle\int_0^t g_1(t,s)\phi_1(s)\,ds, \\ \theta_2\rho_2(\theta_1 r_1, \theta_2 r_2, \ldots, \theta_n r_n) \displaystyle\int_0^t g_2(t,s)\phi_2(s)\,ds, \ldots, \\ \theta_n\rho_n(\theta_1 r_1, \theta_2 r_2, \ldots, \theta_n r_n) \displaystyle\int_0^t g_n(t,s)\phi_n(s)\,ds) \in L^q[0,T]; \end{cases}$$

$$(3.39)\quad \begin{cases} \text{there exists } M_i > 0 \text{ such that for } t \in [0,T], \\ M_i \ge \displaystyle\int_0^t g_i(t,s)\phi_i(s)\Big[\tau_i(\theta_1 M_1, \theta_2 M_2, \ldots, \theta_n M_n) \\ +\rho_i\Big(\theta_1\rho_1(\theta_1 M_1, \theta_2 M_2, \ldots, \theta_n M_n) \displaystyle\int_0^s g_1(s,x)\phi_1(x)\,dx, \\ \theta_2\rho_2(\theta_1 M_1, \theta_2 M_2, \ldots, \theta_n M_n) \displaystyle\int_0^s g_2(s,x)\phi_2(x)\,dx, \ldots, \\ \theta_n\rho_n(\theta_1 M_1, \theta_2 M_2, \ldots, \theta_n M_n) \displaystyle\int_0^s g_n(s,x)\phi_n(x)\,dx\Big)\Big]\,ds \\ \ge \rho_i(\theta_1 M_1, \theta_2 M_2, \ldots, \theta_n M_n) \displaystyle\int_0^t g_i(t,s)\phi_i(s)\,ds. \end{cases}$$

Then, (3.36) *has a constant-sign solution* $u \in (C[0,T])^n$ *with* $\theta_i u_i(t) > 0$, *almost every* $t \in [0,1]$, $1 \le i \le n$.

System $(F)_\infty$. We shall now study the system of Fredholm integral equations $(F)_\infty$, where the nonlinearities f_i, $1 \le i \le n$ may be singular at $u_j = 0$, $j \in \{1, 2, \dots, n\}$ and may also be singular in the independent variable at some set $\Omega \subset [0,\infty)$ with measure zero. Let the Banach space $B = \{u \mid u \in (BC[0,\infty))^n\}$ be equipped with the norm $\|u\| = \max_{1\le i\le n} \sup_{t\in[0,\infty)} |u_i(t)|$. Note that $BC[0,\infty)$ is the space of bounded continuous functions on $[0,\infty)$. Let $C_l[0,\infty)$ be defined as in (2.1). We are interested to obtain a solution of $(F)_\infty$ in $(C_l[0,\infty))^n$.

Theorem 3.7. *Let $\theta_i \in \{1,-1\}$, $1 \le i \le n$ be fixed, and let integers p, q be such that $1 \le p \le q \le \infty$ and $1/p+1/q = 1$. For each $1 \le i \le n$, suppose the following conditions are satisfied*:

$$\text{(3.40)} \quad \begin{cases} g_i^t(s) \equiv g_i(t,s) \ge 0 \text{ for all } t \in [0,\infty), \text{a.e. } s \in [0,\infty) \text{ and} \\ g_i^t(s) > 0 \text{ for a.e. } t \in [0,\infty), \text{ a.e. } s \in [0,\infty); \end{cases}$$

$$\text{(3.41)} \quad \begin{cases} g_i^t(s) \in L^p[0,\infty) \text{ for all } t \in [0,\infty) \text{ and} \\ \text{the map } t \to g_i^t \text{ is continuous from } [0,\infty) \text{ to } L^p[0,\infty); \end{cases}$$

(3.42)

$$\begin{cases} \text{there exists } \tilde{g}_i \in L^p[0,1) \text{ s.t. } g_i^t \to \tilde{g}_i \text{ in } L^p[0,\infty) \text{ as } t \to \infty, \\ \text{i.e., } \lim_{t\to\infty} \|g_i^t - \tilde{g}_i\|_p = 0; \end{cases}$$

(3.43)

$$\begin{cases} f_i : [0,\infty) \times (\mathbf{R} \setminus \{0\})^n \to \mathbf{R} \text{ with } t \to f_i(t,u) \text{ measurable} \\ \forall\, u \in (\mathbf{R} \setminus \{0\})^n \text{ and } u \to f_i(t,u) \text{ continuous for a.e. } t \in (0,\infty); \end{cases}$$

$$\text{(3.44)} \quad \begin{cases} \text{for any } r_i > 0, \text{ there exists } \psi_{r_i,i} : [0,\infty) \to R, \\ \psi_{r_i,i}(t) > 0 \text{ for a.e. } t \in [0,\infty), \ \psi_{r_i,i} \in L^q[0,\infty) \\ \text{such that for all } |u_j| \in (0, r_j], \ 1 \le j \le n, \\ \theta_i f_i(t,u) \ge \psi_{r_i,i}(t) \text{ for a.e. } t \in [0,\infty); \end{cases}$$

(3.45)

$$\begin{cases} \textit{for any } r_i > 0 \textit{ with } \displaystyle\int_0^\infty g_i(t,s)\psi_{r_i,i}(s)\,ds \le r_i \textit{ for } t \in [0,\infty), \\ \exists h_{r_i,i} : [0,\infty) \to \mathbf{R},\ h_{r_i,i}(t) \ge 0 \textit{ for a.e. } t \in [0,\infty), \\ h_{r_i,i} \in L^q[0,\infty) \textit{ s.t. } \forall\, |u_j| \in \left[\displaystyle\int_0^\infty g_j(t,s)\psi_{r_j,j}(s)\,ds, r_j \right],\ 1 \le j \le n, \\ \theta_i f_i(t,u) \le h_{r_i,i}(t) \textit{ for a.e. } t \in [0,\infty); \end{cases}$$

$$\text{(3.46)} \quad \begin{cases} \textit{there exists } M_i > 0 \textit{ such that for } t \in [0,\infty), \\ M_i \ge \displaystyle\int_0^\infty g_i(t,s)h_{M_i,i}(s)\,ds \ge \int_0^\infty g_i(t,s)\psi_{M_i,i}(s)\,ds. \end{cases}$$

Then, $(F)_\infty$ *has a constant-sign solution* $u \in (C_l[0,\infty))^n$ *with* $\theta_i u_i(t) > 0$*, almost every* $t \in [0,\infty)$*,* $1 \le i \le n$*.*

Proof. To begin, we define

$$D = \left\{ u \in (C_l[0,\infty))^n \,\Big|\, \int_0^\infty g_i(t,s)h_{M_i,i}(s)\,ds \ge \theta_i u_i(t) \right.$$
$$\left. \ge \int_0^\infty g_i(t,s)\psi_{M_i,i}(s)\,ds \text{ for } t \in [0,\infty),\ 1 \le i \le n \right\}.$$

Clearly, D is a closed subset of $(C_l[0,\infty))^n$ as $(C_l[0,\infty))^n$ is a closed subspace of $(BC[0,\infty))^n$. Let the operator $S : D \to (BC[0,\infty))^n$ be defined by

$$\text{(3.47)} \qquad Su(t) = (S_1u(t), S_2u(t), \ldots, S_nu(t)), \quad t \in [0,\infty)$$

where

$$\text{(3.48)} \quad S_iu(t) = \int_0^\infty g_i(t,s)f_i(s,u(s))\,ds, \quad t \in [0,\infty), \quad 1 \le i \le n.$$

It is clear that a fixed point of the operator S is a solution of system $(F)_\infty$. Indeed, a fixed point of S obtained in D will be a *constant-sign solution* of system $(F)_\infty$.

First we shall show that S maps D into D. Let $u \in D$. Using a similar argument as in the proof of Theorem 3.1, we obtain

$$\psi_{M_i,i}(t) \le \theta_i f_i(t,u) \le h_{M_i,i}(t), \quad \text{a.e.} \quad t \in [0,\infty), \quad 1 \le i \le n,$$

and so

$$\int_0^\infty g_i(t,s)\psi_{M_i,i}(s)\,ds \le \theta_i S_i u(t) \le \int_0^\infty g_i(t,s) h_{M_i,i}(s)\,ds, \quad t \in [0,\infty), \quad 1 \le i \le n. \tag{3.49}$$

It also follows from (3.49) and (3.46) that

$$|S_i u(t)| \le \int_0^\infty g_i(t,s) h_{M_i,i}(s)\,ds \le M_i, \quad t \in [0,\infty), \quad 1 \le i \le n, \tag{3.50}$$

i.e., $S_i u$, $1 \le i \le n$ are bounded. Moreover, $S_i u \in C[0,\infty)$, $1 \le i \le n$ since if $t, t' \in [0,\infty)$, then (3.41) and (3.45) provide

$$\begin{aligned} &|S_i u(t) - S_i u(t')| \\ &\quad \le \int_0^\infty |g_i(t,s) - g_i(t',s)| h_{M_i,i}(s)\,ds \\ &\quad \le \left(\int_0^\infty |g_i^t(s) - g_i^{t'}(s)|^p\,ds\right)^{1/p} \left(\int_0^\infty [h_{M_i,i}(s)]^q\,ds\right)^{1/q} \longrightarrow 0 \end{aligned} \tag{3.51}$$

as $t \to t'$. It remains to show that $\lim_{t\to\infty} S_i u(t)$, $1 \le i \le n$ exist. Applying (3.42), we get for $1 \le i \le n$,

$$\begin{aligned} &\int_0^\infty |[g_i^t(s) - \tilde{g}_i(s)] f_i(s,u(s))|\,ds \\ &\quad \le \int_0^\infty |g_i^t(s) - \tilde{g}_i(s)| h_{M_i,i}(s)\,ds \\ &\quad \le \left(\int_0^\infty |g_i^t(s) - \tilde{g}_i(s)|^p\,ds\right)^{1/p} \left(\int_0^\infty [h_{M_i,i}(s)]^q\,ds\right)^{1/q} \longrightarrow 0 \end{aligned}$$

as $t \to \infty$. Hence, it follows that

$$
\begin{aligned}
\lim_{t\to\infty} S_i u(t) &= \lim_{t\to\infty} \int_0^\infty g_i^t(s) f_i(s,u(s))\,ds \\
&= \int_0^\infty \tilde{g}_i(s) f_i(s,u(s))\,ds, \quad 1 \le i \le n.
\end{aligned} \tag{3.52}
$$

This completes the proof of $S : D \to D$. $\square$

Next, using a similar argument as in the proof of Theorem 3.1, we see that $S : D \to D$ is continuous.

Finally, we shall show that $S : D \to D$ is compact. Let $u \in D$. Then, clearly, from (3.50)

$$
\sup_{t\in[0,\infty)} |S_i u(t)| \le \sup_{t\in[0,\infty)} \int_0^\infty g_i(t,s) h_{M_i,i}(s)\,ds \le M_i, \quad 1 \le i \le n, \tag{3.53}
$$

or $\|Su\| \le \max_{1\le i\le n} M_i$. Further, we have (3.51) as $t \to t'$. Also, for each $1 \le i \le n$, from (3.52) it follows that, given $\varepsilon_i > 0$, there exists $T_i > 0$ such that $|S_i u(t) - S_i u(\infty)| < \varepsilon_i$ for any $t \ge T_i$. Now, Theorem 2.4 guarantees that S is compact.

Hence, it follows from Theorem 2.1 that S has a fixed point in D. This completes the proof. $\square$

Remark 3.6. In Theorem 3.7, the condition (3.45) can be replaced by the following:

(3.45)′

$$
\begin{cases}
\text{for any } r_i > 0 \text{ with } \displaystyle\int_0^\infty g_i(t,s)\psi_{r_i,i}(s)\,ds \le r_i \text{ for } t \in [0,\infty), \text{ let} \\
h_{r_i,i}(t) = \sup\{f_i(t,u) : |u_j| \in \left[\displaystyle\int_0^\infty g_j(t,s)\psi_{r_j,j}(s)\,ds, r_j\right], \\
1 \le j \le n\} \text{ and assume } h_{r_i,i} \in L^q[0,\infty).
\end{cases}
$$

Remark 3.7. If f_i, $1 \le i \le n$ are nonsingular, i.e., $f_i : [0,\infty) \times \mathbf{R}^n \to \mathbf{R}$, then we can have a variant of Theorem 3.7 with (3.44)–(3.46) replaced by the following conditions:

$$\begin{cases} \text{for any } r_i > 0, \text{ there exists } h_{r_i,i} : [0,\infty) \to R, \\ h_{r_i,i}(t) \geq 0 \text{ for a.e. } t \in [0,\infty), \\ h_{r_i,i} \in L^q[0,\infty) \text{ such that for all } |u_j| \in [0, r_j],\ 1 \leq j \leq n, \\ 0 \leq \theta_i f_i(t,u) \leq h_{r_i,i}(t) \text{ for a.e. } t \in [0,\infty); \end{cases}$$

there exists $M_i > 0$ such that for $t \in [0,\infty)$, $M_i \geq \int_0^\infty g_i(t,s) h_{M_i,i}(s)\,ds \geq 0$.

Moreover, the conclusion of the modified Theorem 3.7 becomes: system $(F)_\infty$ has a constant-sign solution $u \in (C_l[0,\infty))^n$ with $\theta_i u_i(t) \geq 0$, $t \in [0,\infty)$, $1 \leq i \leq n$.

Using a similar argument as in the proof of Theorem 3.2, we obtain the following result.

Theorem 3.8. *Let $\theta_i \in \{1,-1\}$, $1 \leq i \leq n$ be fixed and integers p, q be such that $1 \leq p \leq q \leq \infty$ and $1/p + 1/q = 1$. For each $1 \leq i \leq n$, suppose* (3.40)–(3.44) *hold and the following conditions are satisfied*:

$$(3.54)\quad \begin{cases} \theta_i f_i(t,u) \leq \phi_i(t)[\rho_i(u) + \tau_i(u)] \\ \textit{for } (t,u) \in [0,\infty) \times \prod_{j=1}^n [0,\infty)_j, \textit{ where} \\ \phi_i : [0,\infty) \to \mathbf{R},\ \phi_i(t) > 0 \textit{ for a.e. } t \in [0,\infty), \\ \rho_i,\ \tau_i : \prod_{j=1}^n (0,\infty)_j \to (0,\infty) \textit{ are continuous}, \\ \textit{if } |u_j| \leq |v_j| \textit{ for some } j \in \{1,2,\ldots,n\}, \\ \textit{then } \rho_i(u_1,\ldots,u_j,\ldots,u_n) \geq \rho_i(u_1,\ldots,v_j,\ldots,u_n) \textit{ and} \\ \tau_i(u_1,\ldots,u_j,\ldots,u_n) \leq \tau_i(u_1,\ldots,v_j,\ldots,u_n); \end{cases}$$

(3.55)

$$\begin{cases} \phi_i \in L^q[0,\infty), \textit{ and for any } r_j > 0,\ 1 \leq j \leq n, \\ \phi_i(t)\rho_i\Big(\theta_1 \int_0^\infty g_1(t,s)\psi_{r_1,1}(s)\,ds,\ \theta_2 \int_0^\infty g_2(t,s)\psi_{r_2,2}(s)\,ds, \ldots, \\ \theta_n \int_0^\infty g_n(t,s)\psi_{r_n,n}(s)\,ds\Big) \in L^q[0,\infty); \end{cases}$$

(3.56)

$$\begin{cases} \textit{there exists } M_i > 0 \textit{ such that for } t \in [0,\infty), \\ M_i \ge \int_0^\infty g_i(t,s)\phi_i(s)\Big[\tau_i(\theta_1 M_1, \theta_2 M_2, \dots, \theta_n M_n) \\ + \rho_i\Big(\theta_1 \int_0^\infty g_1(s,x)\psi_{M_1,1}(x)\,dx,\ \theta_2 \int_0^\infty g_2(s,x)\psi_{M_2,2}(x)\,dx, \dots, \\ \theta_n \int_0^\infty g_n(s,x)\psi_{M_n,n}(x)\,ds\Big)\Big]\,dx \\ \ge \int_0^\infty g_i(t,s)\psi_{M_i,i}(s)\,ds. \end{cases}$$

Then, $(F)_\infty$ *has a constant-sign solution* $u \in (C_l[0,\infty))^n$ *with* $\theta_i u_i(t) > 0$*, almost every* $t \in [0,\infty)$*,* $1 \le i \le n$*.*

As an application of Theorem 3.8, we consider a special case of system $(F)_\infty$, viz.,

$$u_i(t) = \int_0^\infty g_i(t,s)\theta_i\phi_i(s)[\rho_i(u(s)) + \tau_i(u(s))]\,ds, \quad t \in [0,\infty), \quad 1 \le i \le n, \tag{3.57}$$

where $\theta_i \in \{1,-1\}$, $1 \le i \le n$ are fixed. A similar argument as in the proof of Theorem 3.3 yields the following result.

Theorem 3.9. *Let* $\theta_i \in \{1,-1\}$*,* $1 \le i \le n$ *be fixed and integers* p, q *be such that* $1 \le p \le q \le \infty$ *and* $1/p + 1/q = 1$*. For each* $1 \le i \le n$*, suppose* (3.40)–(3.42) *hold and the following conditions are satisfied*:

$$\begin{cases} \phi_i : [0,\infty) \to \mathbf{R},\ \phi_i(t) > 0 \textit{ for a.e. } t \in [0,\infty), \\ \rho_i,\ \tau_i : \prod_{j=1}^n (0,\infty)_j \to (0,\infty) \textit{are continuous}, \\ \textit{if } |u_j| \le |v_j| \textit{ for some } j \in \{1,2,\dots,n\}, \\ \textit{then } \rho_i(u_1,\dots,u_j,\dots,u_n) \ge \rho_i(u_1,\dots,v_j,\dots,u_n) \textit{ and} \\ \tau_i(u_1,\dots,u_j,\dots,u_n) \le \tau_i(u_1,\dots,v_j,\dots,u_n); \end{cases} \tag{3.58}$$

(3.59)
$$\begin{cases} \phi_i \in L^q[0,\infty), \ \text{and for any } r_j > 0, \ 1 \le j \le n, \\ \phi_i(t)\rho_i\biggl(\theta_1\rho_1(\theta_1 r_1, \theta_2 r_2, \dots, \theta_n r_n)\displaystyle\int_0^\infty g_1(t,s)\phi_1(s)\,ds, \\ \theta_2\rho_2(\theta_1 r_1, \theta_2 r_2, \dots, \theta_n r_n)\displaystyle\int_0^\infty g_2(t,s)\phi_2(s)\,ds, \dots, \\ \theta_n\rho_n(\theta_1 r_1, \theta_2 r_2, \dots, \theta_n r_n)\displaystyle\int_0^\infty g_n(t,s)\phi_n(s)\,ds\biggr) \in L^q[0,\infty); \end{cases}$$

$$(3.60)\quad \begin{cases} \text{there exists } M_i > 0 \text{ such that for } t \in [0,\infty), \\ M_i \ge \displaystyle\int_0^\infty g_i(t,s)\phi_i(s)\biggl[\tau_i(\theta_1 M_1, \theta_2 M_2, \dots, \theta_n M_n) \\ +\rho_i(\theta_1\rho_1(\theta_1 M_1, \theta_2 M_2, \dots, \theta_n M_n)\displaystyle\int_0^\infty g_1(s,x)\phi_1(x)\,dx, \\ \theta_2\rho_2(\theta_1 M_1, \theta_2 M_2, \dots, \theta_n M_n)\displaystyle\int_0^\infty g_2(s,x)\phi_2(x)\,dx, \dots, \\ \theta_n\rho_n(\theta_1 M_1, \theta_2 M_2, \dots, \theta_n M_n)\displaystyle\int_0^\infty g_n(s,x)\phi_n(x)\,dx)\biggr]\,ds \\ \ge \rho_i(\theta_1 M_1, \theta_2 M_2, \dots, \theta_n M_n)\displaystyle\int_0^\infty g_i(t,s)\phi_i(s)\,ds. \end{cases}$$

Then, (3.57) *has a constant-sign solution* $u \in (C_l[0,\infty))^n$ *with* $\theta_i u_i(t) > 0$*, almost every* $t \in [0,\infty)$*,* $1 \le i \le n$*.*

In Theorems 3.7–3.9, we require solutions of $(F)_\infty$ to lie in $(C_l[0,\infty))^n$. We shall now seek solutions of $(F)_\infty$ in $(C[0,\infty))^n$. Since $C[0,\infty)$ is a Fréchet space, we shall apply the Schauder-Tychonoff fixed point theorem (Theorem 2.2) instead of the Schauder fixed point theorem (Theorem 2.1).

Theorem 3.10. *Let* $\theta_i \in \{1,-1\}$*,* $1 \le i \le n$ *be fixed, and let integers* p, q *be such that* $1 \le p \le q \le \infty$ *and* $1/p + 1/q = 1$*. For each* $1 \le i \le n$*, suppose* (3.40), (3.41) *and* (3.43)–(3.46) *are satisfied. Then,* $(F)_\infty$ *has a constant-sign solution* $u \in (BC[0,\infty))^n$ *with* $\theta_i u_i(t) > 0$*, almost every* $t \in [0,1)$*,* $1 \le i \le n$*.*

Proof. To begin, we define

$$D = \Big\{ u \in (C[0,\infty))^n \mid u \in (BC[0,\infty))^n \text{ and } \int_0^\infty g_i(t,s) h_{M_i,i}(s)\,ds \ge \theta_i u_i(t) \ge \int_0^\infty g_i(t,s)\psi_{M_i,i}(s)\,ds \quad \text{for } t \in [0,\infty),\ 1 \le i \le n \Big\}.$$

Clearly, D is a closed (Note (3.46)) convex subset of the Fréchet space $(C[0,\infty))^n$. Let the operator $S : D \to (C[0,\infty))^n$ be defined by (3.47) and (3.48). As seen from (3.49)–(3.51), we have $S : D \to D$.

Next, $S : D \to D$ is compact since we have (3.53) for $u \in D$ which gives $\|Su\| \le \max_{1\le i\le n} M_i$, and we already have (3.51) as $t \to t'$.

Finally, we shall show that $S : D \to D$ is continuous. Let $\{u^m\}$ be a sequence in D and $u^m \to u$ in $(C[0,\infty))^n$, i.e., $u_i^m \to u_i$ in $C[0,\infty)$, $1 \le i \le n$. Then, for each $1 \le i \le n$, $u_i^m \to u_i$ in $C[0,k]$ for each $k \in \mathbf{Z}^+$, and u_i^m converges pointwise to u_i on $[0,\infty)$. Fix $k \in \mathbf{Z}^+$. Using a similar argument as in the proof of Theorem 3.1, we see that for each $1 \le i \le n$, $S_i u^m(t) \to S_i u(t)$ for each $t \in [0,\infty)$, and $S_i u^m \to S_i u$ in $C[0,k]$. Since this is true for each $k \in \mathbf{Z}^+$, it follows that $S_i u^m \to S_i u$ in $C[0,\infty)$. Hence, $S : D \to D$ is continuous.

We now conclude from Theorem 2.2 that S has a fixed point in D. □

Remark 3.8. Remarks 3.6 and 3.7 (with $(C_l[0,\infty))^n$ replaced by $(BC[0,\infty))^n$) also hold for Theorem 3.10.

A similar argument as in Theorems 3.8 and 3.9 give the following results.

Theorem 3.11. *Let $\theta_i \in \{1,-1\}$, $1 \le i \le n$ be fixed, and let integers p,q be such that $1 \le p \le q \le \infty$ and $1/p + 1/q = 1$. For each $1 \le i \le n$, suppose* (3.40), (3.41), (3.43), (3.44) *and* (3.54)–(3.56) *hold. Then, $(F)_\infty$ has a constant-sign solution $u \in (BC[0,\infty))^n$ with $\theta_i u_i(t) > 0$, almost every $t \in [0,\infty)$, $1 \le i \le n$.*

Theorem 3.12. *Let $\theta_i \in \{1, -1\}$, $1 \le i \le n$ be fixed, and let integers p, q be such that $1 \le p \le q \le \infty$ and $1/p + 1/q = 1$. For each $1 \le i \le n$, suppose* (3.40), (3.41) *and* (3.58)–(3.60) *hold. Then,* (3.57) *has a constant-sign solution $u \in (BC[0,\infty))^n$ with $\theta_i u_i(t) > 0$, almost every $t \in [0,\infty)$, $1 \le i \le n$.*

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