

DISCRETE COLLOCATION FOR A FIRST KIND  
CAUCHY SINGULAR INTEGRAL EQUATION  
WITH WEAKLY SINGULAR SOLUTION

HEINRICH N. MÜLTHEI AND CLAUS SCHNEIDER

ABSTRACT. A fully discrete scheme for the numerical solution of a first kind Cauchy singular integral equation is analyzed. The underlying mesh may be graded in order to approximate weakly singular solutions (functions behaving like  $|t|^\alpha$ ,  $\alpha > 0$ ,  $t \in [-1, 1]$ ) as well as smooth ones on a uniform grid. Order of convergence results are established in sup-norms and weighted  $l_2$ -norms. They exactly reflect the outcome of the numerical computations. For the stability analyses it is shown that the row differencing imbedded in the method just yields a Moore-Penrose inverse in an unconventional way.

**1. Introduction.** This paper concerns a fully discrete numerical method for the solution of Cauchy singular integral equations on a smooth closed curve. The method was introduced in [4, 11] and is closely connected to the midpoint collocation studied in [3]. These papers provided a first step towards the analysis of nonuniform meshes which are useful for nonsmooth solutions. Obviously, adaptive mesh refinement will produce such grids for rapidly varying or weakly singular solutions. But the theory did not yet cover completely these cases and did not reflect comprehensively the order of convergence achieved with the methods in practice. Here we will take a second step by giving optimal error estimates even for weakly singular solutions approximated on an appropriately graded mesh. We study in detail the equation

$$(1.1) \quad \mathcal{H}u(t) = f(t), \quad t \in [0, 2\pi],$$

with the Hilbert transform on the unit circle (cf. [6], e.g.) which is defined for Hölder continuous functions  $u \in C^\alpha[0, 2\pi]$ ,  $\alpha > 0$ , as

$$\mathcal{H}u(t) := \frac{1}{2\pi} \int_0^{2\pi} u(s) \cot \frac{t-s}{2} ds, \quad t \in [0, 2\pi].$$

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Equation (1.1) is solvable for  $f \in C^\alpha[0, 2\pi]$  with  $\int_0^{2\pi} f(t) dt = 0$ . For uniqueness of a naturally  $2\pi$ -periodic solution  $u$  a side condition is required (see [6] or [7]). Here we will consider the point side condition  $u(0) = 0$ . The integral side condition  $\int_0^{2\pi} u(t) dt = 0$  may be analyzed similarly. For the connections of the model problem (1.1) with classical boundary value problems of potential theory, with boundary integral equations, and with the more realistic situation of a region bounded by a curve  $\Gamma$  which is not the unit circle, we refer to [1], [7], and the instructive introduction of [3]. These papers also include a lot of references connected with the theory and practice of Cauchy singular integral equations.

**2. The numerical method.** As constants vanish under  $\mathcal{H}$ , we will consider only the following regularized form of equation (1.1):

$$\mathcal{H}u(t) = \frac{1}{2\pi} \int_0^{2\pi} \{u(s) - u(t)\} \cot \frac{t-s}{2} ds = f(t),$$

$$t \in [0, 2\pi].$$

Then it will be easier to study the remainder of a quadrature applied to the integral. But the approximation of a Cauchy principal value can only be hidden for the moment and will inevitably appear later. For the discretization we define a sequence of grids by the knots

$$(2.2) \quad x_0^n = 0 < x_1^n < \dots < x_n^n = 2\pi, \quad n \in \mathbf{N}.$$

Then  $h_i^n := x_i^n - x_{i-1}^n$ ,  $i = 1(1)n$ ,  $h_0^n := h_n^n$ ,  $h_{n+1}^n := h_1^n$ , are the local stepsizes. Furthermore, we need intermediate points

$$t_i^n := (1 - \lambda_i^n)x_{i-1}^n + \lambda_i^n x_i^n, \quad 0 < \lambda_i^n < 1, \quad i = 1(1)n.$$

It is convenient to define also  $t_0^n := t_n^n - 2\pi$  and  $t_{n+1}^n := t_1^n + 2\pi$ . With the weights of the trapezoidal rule on the mesh

$$\omega_i^n := \frac{1}{2}(h_i^n + h_{i+1}^n), \quad i = 0(1)n,$$

we define  $\mathcal{H}_n$  for functions  $u_n$  which are  $2\pi$ -periodic, continuous at  $x_1^n, \dots, x_{n-1}^n$ , and piecewise linear on the grid, i.e., we are working in

an  $n$ -dimensional subspace of  $C^\alpha[0, 2\pi]$ :

$$\mathcal{H}_n u_n(t) := \frac{1}{2\pi} \sum_{j=1}^n \omega_j^n \{u_n(x_j^n) - u_n(t)\} \cot \frac{t - x_j^n}{2},$$

$$t \in [0, 2\pi].$$

Note that  $\mathcal{H}_n u_n(x_j^n)$  could be defined as a one-sided limit. Hence, the function has  $n - 1$  jump-discontinuities in the worst case. Actually, we have replaced  $\mathcal{H}u$  by its trapezoidal approximation and the function  $u$  by a piecewise linear interpolant. Then collocation at the points  $t_i^n$  leads to the linear system of equations  $\mathcal{H}_n u_n(t_i^n) = f(t_i^n)$ ,  $i = 1(1)n$ , which can be stated explicitly without any integrals involved. This explains why we call this method *discrete collocation*. Thus, we have to analyze the system

$$\begin{aligned} \mathcal{H}_n u_n(t_i^n) &= \frac{1}{2\pi} \sum_{j=1}^n \omega_j^n \cot \frac{t_i^n - x_j^n}{2} \\ &\quad \cdot (u_j^n - (1 - \lambda_i^n)u_{i-1}^n - \lambda_i^n u_i^n) \\ &= f_i^n := f(t_i^n), \quad i = 1(1)n, \end{aligned}$$

where  $u_i^n := u_n(x_i^n)$ ,  $i = 0(1)n$ , and therefore  $u_0^n = u_n^n$ . With the side conditions  $u(0) = u(2\pi) = 0$ , however, this gives  $n$  equations for only  $n - 1$  unknowns  $\tilde{u}^n := (u_1^n, \dots, u_{n-1}^n)^T$ :

$$(2.3) \quad B_{n,n-1} \tilde{u}^n = f^n,$$

with

$$\begin{aligned} B_{n,n-1} &:= B_{n,n-1}^0 - S_{n,n-1}^0, \\ B_{n,n-1}^0 &:= \frac{1}{2\pi} \left( \omega_j^n \cot \frac{t_i^n - x_j^n}{2} \right)_{i=1(1)n, j=1(1)n-1}, \\ S_{n,n-1}^0 &:= \begin{pmatrix} \lambda_1^n S_1^n & 0 & \cdot & \cdot & \cdot & 0 \\ (1 - \lambda_2^n) S_2^n & \lambda_2^n S_2^n & 0 & \cdot & \cdot & 0 \\ & & \cdot & \cdot & \cdot & \\ 0 & \cdot & \cdot & 0 & (1 - \lambda_{n-1}^n) S_{n-1} & \lambda_{n-1}^n S_{n-1}^n \\ 0 & \cdot & \cdot & \cdot & 0 & (1 - \lambda_n^n) S_n^n \end{pmatrix}, \end{aligned}$$

$$(2.4) \quad S_i^n := \frac{1}{2\pi} \sum_{j=1}^n \omega_j^n \cot \frac{t_i^n - x_j^n}{2}, \quad i = 1(1)n.$$

Obviously the system (2.3) could be solved with a Moore-Penrose inverse of  $B_{n,n-1}$ . But we use the much simpler procedure of taking row differences. This means was introduced by Dyn and Levin [5] for regular systems. It was successfully applied and analyzed in detail by Chandler [3] in the case of a nonsquare system. Formally the method may be described by the matrix

$$R_{n-1,n} := \begin{pmatrix} -1 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & -1 & 1 & 0 & \cdot & 0 \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ 0 & \cdot & \cdot & 0 & -1 & 1 \end{pmatrix} \in \mathbf{R}^{n-1,n},$$

finally leading to the equations considered in [4] and [11]:

$$(2.5) \quad \begin{aligned} A_{n-1,n-1} \tilde{u}^n &= R_{n-1,n} f^n, \\ A_{n-1,n-1} &:= R_{n-1,n} B_{n,n-1}, \end{aligned}$$

where

$$\begin{aligned} A_{n-1,n-1} &= A_{n-1,n-1}^0 + S_{n-1,n-1}, \\ A_{n-1,n-1}^0 &= (a_{ij}^0)_{i,j=1(1)n-1}, \\ a_{ij}^0 &:= \frac{1}{2\pi} \omega_j^n \left( \cot \frac{t_{i+1}^n - x_j^n}{2} - \cot \frac{t_i^n - x_j^n}{2} \right), \\ & \quad i, j = 0(1)n. \end{aligned}$$

The entries  $a_{ij}^0$  which do not belong to  $A_{n-1,n-1}^0$  will be needed later or they are used in the proofs of results cited in the next section.  $S_{n-1,n-1}$  is a tri-diagonal matrix with

$$\begin{aligned} (S_{n-1,n-1})_{ii} &:= \lambda_i^n S_i^n - (1 - \lambda_{i+1}^n) S_{i+1}^n, \quad i = 1(1)n - 1, \\ (S_{n-1,n-1})_{i,i+1} &:= -\lambda_{i+1}^n S_{i+1}^n, \quad i = 1(1)n - 2, \\ (S_{n-1,n-1})_{i,i-1} &:= (1 - \lambda_i^n) S_i^n, \quad i = 2(1)n - 1. \end{aligned}$$

In [2, 3, 4, 11] the stability of  $A_{n-1,n-1}^{-1}$  was studied in detail. Here we will analyze the stability of  $A_{n-1,n-1}^{-1} R_{n-1,n}$ . Furthermore, we will

show that this matrix actually is or is close to a Moore-Penrose inverse of  $B_{n,n-1}$  explaining the connection between a solution of (2.3) and the solution of (2.5). But at first let us collect some properties of  $A_{n-1,n-1}$  which will prove row differencing a numerically stable means to solve (2.3).

**3. Known results.** The following assumptions and results are given in detail in [11]. But their origins are [2, 3] and a lot of discussions with G. Chandler during the second author's visit at the University of Queensland.

At first we impose the following restrictions on the mesh and the  $t_i^n$ :

$$(A_1) \text{ there exists a } \mu_1 \geq 1: \mu_1^{-1} \leq h_{i+1}^n/h_i^n \leq \mu_1, \quad i = 1(1)n - 1,$$

$$(A_2) \text{ there exists a } \mu_2 > 0: h_{\max}^n := \max\{h_i^n, i = 1(1)n\} \leq \mu_2/n,$$

$$(A_3) \text{ there exists a } \mu_3 \in (0, 1/2]: \mu_3 \leq (t_i^n - x_{i-1}^n)/(x_i^n - x_{i-1}^n) \leq 1 - \mu_3.$$

For a uniform mesh, i.e.,  $h_i^n = h_{\max}^n$  for all  $i \leq n$ , with midpoints  $t_i^n$  these conditions are trivially satisfied. For a graded mesh centered at  $\pi$ , e.g., usually the following mesh points are used:

$$x_j^n := \pi\{1 - (1 - 2j/n)^q\}, \quad j = 0(1)[n/2],$$

$$(3.6) \quad x_j^n := \pi\{1 + (1 - 2(n - j)/n)^q\}, \quad j = [n/2] + 1(1)n,$$

$q \geq 1$  the grading exponent.

Then  $(A_1)$ – $(A_2)$  are still valid as a short calculation shows (cf. [11]). Order of consistency results for smooth functions are preserved by such meshes for functions  $u$  being weakly singular (at  $\pi$  in the example) in the following sense:

$$(3.7) \quad \begin{aligned} &1. \quad u \in C_{2\pi}^3([0, \pi) \cup (\pi, 2\pi]), \\ &\quad \text{i.e., } u \text{ and its derivatives are } 2\pi\text{-periodic,} \\ &2. \quad u \in C^\alpha[0, 2\pi], \\ &3. \quad \exists C_i > 0: |u^{(i)}(t)| \leq C_i |t - \pi|^{\alpha-i}, \\ &\quad \quad \quad i = 1, 2, 3, \quad \forall t \in [0, 2\pi], \end{aligned}$$

i.e.,  $u$  is  $2\pi$ -periodic and of **type**  $(\alpha, 3, \{\pi\})$ , a notation introduced by Rice in [8]. The mesh restrictions yield

**Lemma 1.** *Let  $(A_1)$ ,  $(A_2)$  and  $(A_3)$  hold. Then*

1.  $A_{n-1, n-1}^0 = (a_{ij}^0)$  is an  $M$ -matrix,
2. there exists a  $\gamma = \gamma(\mu_1, \mu_2, \mu_3) > 0$  such that  $a_{ii}^0 \geq 2\gamma$ ,  $i = 0(1)n$ ,  $a_{i, i+1}^0 \leq -\gamma$ ,  $i = 0(1)n - 1$ ,  $a_{i, i-1}^0 \leq -\gamma$ ,  $i = 1(1)n$ .

Elementary arguments show the first assertion. The second one mainly follows from the fact that  $h(\cot h\alpha - \cot h\beta) \geq 1/\alpha - 1/\beta$  for  $0 < \alpha < \beta$  and  $h\beta < \pi$ . In order to maintain the  $M$ -property even for  $A_{n-1, n-1}$  another assumption on the intermediate points is necessary:

$(A_4)$  there exists  $\gamma' \in (0, \gamma]$ :  $a_{i, i-1} = a_{i, i-1}^0 + (1 - \lambda_i^n)S_i^n < -\gamma'$ ,  $i = 1(1)n$ , and  $a_{i, i+1} = a_{i, i+1}^0 - \lambda_{i+1}^n S_{i+1}^n < -\gamma'$ ,  $i = 0(1)n - 1$ .

*Remarks.* (i)  $(A_4)$  obviously ensures that the sub- and super-diagonal entries of the matrix  $A_{n-1, n-1}$  stay negative.

(ii) Furthermore, the  $S_i^n$ ,  $i = 1(1)n$ , remains uniformly bounded with respect to  $i$  and  $n$ , which will be an important property regarding consistency.

Note that the quantities  $S_i^n$  are defined as approximations of the Cauchy principal values  $\int_0^{2\pi} \cot((t_i^n - t)/2) dt = 0$  by the trapezoidal rule. The following result from [11] shows that we can find points for which  $(A_4)$  applies.

**Lemma 2.** *For any mesh (2.2) there exists one and only one  $t_i^n \in (x_{i-1}^n, x_i^n)$  such that  $S_i^n = 0$ . This holds for  $i = 1(1)n$ .*

The proof consists of simply studying the sign behavior of the  $S_i^n$  as functions of  $t_i^n$ . Such  $t_i^n$  solving  $S_i^n = 0$  will be called *optimal* in the remainder of this paper. They may be computed approximately by some Newton steps which would need an additional effort. Unfortunately, it is not yet clear if the optimal points satisfy  $(A_3)$ . Fortunately, we are able to derive convergence results for these points using another approach which does not need  $(A_3)$  and its implications given in this section. On the other hand, the midpoints obviously satisfy  $(A_3)$ —if  $(A_4)$  holds is still open. But the numerical results imply that the midpoints

perform well. Hence, the theory is not yet sufficiently sophisticated. Anyway  $(A_4)$  finally leads to

**Lemma 3.** *The mesh conditions  $(A_1)$ – $(A_4)$  imply that  $A_{n-1,n-1}$  is a strictly diagonally dominant  $M$ -matrix, i.e.,  $A_{n-1,n-1}$  is regular and has an inverse with nonnegative entries. Therefore,  $B_{n,n-1}$  has full column rank.*

Furthermore,  $A_{n-1,n+1} := (a_{i0}|A_{n-1,n-1}|a_{in})$  with  $a_{i0}, a_{in}$  given in  $(A_4)$  fulfills a discrete maximum principle (cf. [3]) providing the main tool for the stability analysis given in that paper. With the second differences matrix  $T_{n-1,n-1} := R_{n-1,n}R_{n-1,n}^T$  another stability result due to G. Chandler was proved in [2]:

**Theorem.** *Let  $E_{n-1,n-1} := A_{n-1,n-1} - \gamma'T_{n-1,n-1}$ ,  $\gamma'$  from  $(A_4)$ . Under the assumptions  $(A_1)$ – $(A_4)$  holds*

$$v^T E_{n-1,n-1} v \geq 0, \quad v \in \mathbf{R}^{n-1}.$$

Hence

$$\begin{aligned} \gamma'v^T T_{n-1,n-1} v &= \gamma' \sum_{i=1}^n (v_i - v_{i-1})^2 \\ &\leq v^T A_{n-1,n-1} v, \\ v \in \mathbf{R}^{n-1} \quad &\text{with } v_0 := v_n := 0. \end{aligned}$$

And the last inequality was used to derive error estimates in [4, 11]. Unfortunately, all these results from [3, 4, 11] were not entirely satisfactory because they proved convergence of order  $\log n/n$  whereas numerical evidence shows  $\log n/n^2$ . Moreover, the analysis in [3] required the solution  $u$  to have a uniformly bounded first derivative, an assumption not needed in reality and not needed in our framework. But the properties of  $A_{n-1,n-1}$  collected in this section show that this matrix is well suited for numerically solving equations (2.5) directly or iteratively.

In the next sections we will restore the lost  $1/n$  for optimal  $t_i^n$  by studying  $A_{n-1,n-1}^{-1}R_{n-1,n}$  which seemed to be quite untractable at first sight compared with the nice  $A_{n-1,n-1}$ .

**4. The Moore-Penrose inverse of  $B_{n,n-1}$ .** In this section we will show that  $A_{n-1,n-1}^{-1}R_{n,n-1}$  is just the Moore-Penrose inverse of  $B_{n,n-1}$  with respect to an appropriate scalar product if the collocation points  $t_i^n$  are chosen optimally. In the other cases this matrix yields ‘nearly’ a pseudo-inverse.

**Theorem.** *Let  $B \in \mathbf{R}^{n,n-1}$ ,  $\text{rank}(B) = n - 1$ ,  $X \in \mathbf{R}^{n-1,n}$ ,  $\text{rank}(X) = n - 1$ ,  $\text{kernel}(B^*) = \text{span}\{\eta\}$ ,  $\text{kernel}(X) = \text{span}\{\rho\}$ , and  $\eta^*\rho = 1$ . Then*

1.  $XB$  is regular,
2.  $(XB)^{-1}X = B^\dagger(I - \rho\eta^*)$ .

*Remarks.* (i) Note that we have not considered a special scalar product but that only the notion of  $*$  and  $^\dagger$  depend on it.

(ii) If we assume the regularity of  $XB$  instead of the normalization  $\eta^*\rho = 1$ , then it is easy to see that such a normalization is possible.

*Proof.* 1. The normalization of  $\eta^*\rho$  implies that  $\rho \notin \text{range}(B)$  which is the orthogonal complement of  $\text{span}\{\eta\}$ . Hence,  $\text{range}(B) \cap \text{kernel}(X) = \{0\}$ . As  $B$  has independent columns its kernel is trivial finally yielding the regularity of  $XB$ .

2.  $I - BB^\dagger$  is the orthogonal projection on  $\text{kernel}(B^\dagger) = \text{kernel}(B^*) = \text{span}\{\eta\}$ . Therefore we have  $I - BB^\dagger = \eta\eta^\dagger = \eta\eta^*/\eta^*\eta$ . This implies  $BB^\dagger(I - \rho\eta^*) = I - \rho\eta^*$ , since  $\eta^*\rho = 1$ . Furthermore,  $X\rho = 0$  implies that  $X(I - \rho\eta^*) = X$  holds too. So we get  $X = X(I - \rho\eta^*) = XBB^\dagger(I - \rho\eta^*)$  which completes the proof.  $\square$

Two easy consequences of this theorem are

1. If  $\text{kernel}(B^*) = \text{kernel}(X)$ , then  $(XB)^{-1}X = B^\dagger$ .
2. If the equation  $Bu = f$  is solvable and  $XB$  regular, then  $(XB)^{-1}Xf = B^\dagger f$  because the assumptions imply that  $\eta^*f = 0$ . Hence  $XB$  provides in this case just the solution  $u$ .

Clearly the theorem could be generalized in order to cover higher rank deficiency. Then the vectors  $\eta, \rho$  have to be replaced by orthogonal matrices  $H$  and  $P$ .



In the case of a matrix  $B$  with full column rank, taking  $X = B^T$  is the obvious choice but not always the cheapest or most stable one. Also  $X = (I|0)Q^T$  ( $Q$  being the orthogonal factor in the  $QR$ -decomposition of  $B$ ) is easy to understand within the theorem.

In our problem we have  $X = R_{n-1,n}$  and thus  $\rho = (1, 1, \dots, 1)^T =: e$ . Furthermore, we know that  $A_{n-1,n-1} = R_{n-1,n}B_{n,n-1}$  is a regular matrix under the mesh restrictions  $(A_1)$ – $(A_4)$ . Thus the theorem shows that for a solvable equation  $B_{n,n-1}\tilde{u}^n = f^n$  the row-differencing amazingly yields the solution of minimal length. Otherwise it provides an approximation of  $B_{n,n-1}^\dagger f^n$ , the minimizer of  $\|B_{n,n-1}\tilde{u}^n - f^n\|$ . Therefore, row-differencing was an excellent idea of G. Chandler's. But this insight does not yet facilitate the stability analysis which has still to come. On the other hand, there are recent results (cf. [9]) about  $B_{n,n-1}^\dagger$  for the case of *optimally chosen* collocation points. In order to apply them, we have to exhibit the connection between  $A_{n-1,n-1}$  and  $B_{n,n-1}^\dagger$  in that situation.

**Lemma 4.** *If the  $t_i^n$  are optimal, then  $A_{n-1,n-1}$  is regular, without any mesh restrictions, and*

$$A_{n-1,n-1}^{-1}R_{n-1,n} = B_{n,n-1}^\dagger,$$

where the pseudo-inverse is formed with respect to the scalar product induced by the vector  $p^n$  with components

$$(4.8) \quad p_i^n := \left( \sum_{\mu=1}^n \frac{\omega_\mu^n}{2\pi} / \sin^2 \left( \frac{t_i^n - x_\mu^n}{2} \right) \right)^{-1} > 0, \quad i = 1(1)n.$$

*Proof.* The theorem can be applied because the kernel of  $R_{n-1,n}$  is spanned by  $\rho = e$ , and it was shown in [9] that  $(p^n)^T \rho = 1$  and  $\text{kernel}(B_{n,n-1}^T) = \text{span}\{p^n\}$ . With the right scalar product being induced now by this  $p^n > 0$ , we have  $\eta = e \in \text{kernel}(B_{n,n-1}^*)$  and  $\eta^* \rho = 1$ . Then the theorem tells us that  $A_{n-1,n-1}$  is regular and that  $A_{n-1,n-1}^{-1}R_{n-1,n} = B_{n,n-1}^\dagger$  because now again  $B_{n,n-1}^\dagger e = 0$ .  $\square$

*Remark.* The lemma shows that row-differencing with optimal  $t_i^n$  finally solves the approximation problem  $\min_{\tilde{u}^n \in \mathbf{R}^{n-1}} \|B_{n,n-1}\tilde{u}^n - f^n\|$  in the  $p^n$ -norm.

Let us collect results from [9] which will be needed in the sequel. They hold under the single restriction of optimal collocation points.

$$B_{n,n-1}^\dagger = \widetilde{W}_{n-1}^{-1} \left( I_{n-1} + \frac{2\pi}{\omega_n^n} \widetilde{W}_{n-1} e e^T \right) B_{n,n-1}^T P_n,$$

where

$$P_n = \text{diag}(p_1^n, \dots, p_n^n) \quad \text{and} \quad \widetilde{W}_{n-1} = \text{diag} \left( \frac{1}{2\pi} \omega_i^n \right)_{i=1(1)n-1}.$$

Just as the optimal knots  $t_i^n$  guarantee that the trapezoidal approximations  $S_i^n$  of the Cauchy principal value

$$\frac{1}{2\pi} \int_0^{2\pi} \cot \frac{t-x}{2} dx$$

are exact at those points, so do the  $p_i^n$  make sure that

$$\frac{1}{2\pi} \int_0^{2\pi} \cot \frac{t-x}{2} dt$$

is approximated exactly at least at the knots  $x_0^n, \dots, x_n^n$  by the quadrature with weights  $p_i^n$  and knots  $t_i^n$ .

Note that for uniform meshes we have  $p_i^n = (1/(2\pi))\omega_i^n = 1/n$  leading to simpler expressions:

$$B_{n,n-1}^\dagger = B_{n,n-1}^T + e(B_{n,n-1}e)^T = B_{n,n-1}^T - \frac{1}{n} e \left( \cot \frac{t_j^n}{2} \right)_{j=1(1)n}^T.$$

Finally, the following form of the Moore-Penrose inverse is more suitable for the stability analysis in the sup-norm:

(4.9)

$$\begin{aligned} B_{n,n-1}^\dagger &= \left( \left( \cot \frac{t_j^n - x_i^n}{2} - \cot \frac{t_j^n}{2} \right) p_j^n \right)_{i=1(1)n-1, j=1(1)n} \\ &= \text{diag} \left( \sin \frac{x_i^n}{2} \right) \left( 1 / \sin \frac{t_j^n - x_i^n}{2} \right) \cdot \text{diag} \left( p_j^n / \sin \frac{t_j^n}{2} \right). \end{aligned}$$

In the case of a uniform mesh with midpoint-collocation it is known (cf. [9]) that  $p^n = e/n$ , i.e.,  $A_{n-1,n}^{-1}R_{n-1,n}$  is just the usual  $l_2$  Moore-Penrose inverse.

These results obviously imply that we could compute an approximation for  $u$  by directly applying  $B_{n,n-1}^\dagger$  to  $f^n$ . This needs only the optimal  $t_i^n$  and a matrix vector multiplication. Anyway, it seems to be useful to consider both,  $A_{n-1,n-1}^{-1}R_{n-1,n} = B_{n,n-1}^\dagger$  and  $A_{n-1,n-1}$  for the analysis of more realistic situations where  $\mathcal{H}$  is compactly perturbed. But then it is not yet quite clear which one should be used for actually computing an approximate solution.

**5. Stability.** In this section we discuss the stability of the method in a weighted  $l_2$ -norm and in the sup-norm as well.

5.1. *Weighted  $l_2$  estimates.* If  $\varepsilon = B_{n,n-1}^\dagger \tau$ , then it has been shown for optimal  $t_i^n$  in [9] that

$$(5.10) \quad \|\varepsilon\|_{\tilde{W}_n} \leq \sqrt{\frac{2\pi}{\omega_n^n}} \|\tau\|_{P_n},$$

and that the amplification factor is  $\sqrt{n}$  in the equidistant case. Furthermore, it can be seen that an amplification appears only in the direction of the singular vector belonging to the singular value  $\sqrt{2\pi/\omega_n^n}$  whereas all the other singular values are one. Hence, as the numerical results will show, an amplification of the errors is hard to see; and it can happen only if the error is mainly lying in the ‘bad’ direction. In general, not much is known about  $\|\tau\|_{P_n}$ . Therefore, some knowledge about  $p^n$  is useful.

**Lemma 5.** *For  $i = 1(1)n$ ,  $n > 1$ , we have*

$$0 < p_i^n < 2\pi \sin^2\left(\frac{h_i^n}{2}\right) / h_i^n = \frac{\pi}{2} h_i^n + \mathcal{O}((h_i^n)^3).$$

*Proof.* Using (4.8) yields

$$\begin{aligned} \frac{1}{p_i^n} &= \frac{1}{4\pi} \sum_{\mu=1}^n h_\mu^n \left( 1 / \sin^2 \left( \frac{x_{\mu-1}^n - t_i^n}{2} \right) + 1 / \sin^2 \left( \frac{x_\mu^n - t_i^n}{2} \right) \right) \\ &\geq \frac{1}{4\pi} h_i^n \left( 1 / \sin^2 \left( \frac{x_{i-1}^n - t_i^n}{2} \right) + 1 / \sin^2 \left( \frac{x_i^n - t_i^n}{2} \right) \right) \\ &> \frac{1}{2\pi} h_i^n / \sin^2 \left( \frac{h_i^n}{2} \right), \end{aligned}$$

because  $x_i^n - t_i^n < h_i^n$  and  $t_i^n - x_{i-1}^n < h_i^n$ .  $\square$

With this result, (5.10) implies

**Theorem.**

$$(5.11) \quad \|\varepsilon\|_{\tilde{W}_n} \leq \sqrt{\frac{2\pi}{\omega_n}} \|\tau\|_{\Xi_n} (C + \mathcal{O}(h_{\max}^n)),$$

where  $\Xi_n := \text{diag}(h_1^n, \dots, h_n^n)$ , and  $C$  is a constant not depending on  $n$ .

In the next step we will derive a sup-norm estimate.

5.2. *Sup-norm estimates.* In this subsection we still assume the  $t_i^n$  being optimal. Let

$$\tau_j^n := \min\{x_j^n - t_j^n, t_j^n - x_{j-1}^n\}, \quad j = 1(1)n.$$

Obviously,  $0 < \tau_j^n \leq h_j^n$  and  $\tau_j^n = \pi/n = h_{\max}^n/2$  on a uniform mesh with midpoint collocation. Then we have, using (4.9) and scaling with these  $\tau_j^n$ ,

$$B_{n,n-1}^\dagger = \tilde{B}_{n-1,n} Q_{n,n},$$

where

$$\tilde{B}_{n-1,n} := \left( 2\tau_j^n \left( \cot \frac{t_j^n - x_i^n}{2} - \cot \frac{t_j^n}{2} \right) \right)_{i=1(1)n-1, j=1(1)n}$$

and

$$Q_{n,n} := \text{diag} (p_j^n / (2\tau_j^n))_{j=1(1)n}.$$

This factorization trivially implies

$$\|B_{n,n-1}^\dagger\|_\infty \leq \|\tilde{B}_{n-1,n}\|_\infty \|Q_{n,n}\|_\infty.$$

The following lemma shows that  $\|Q_{n,n}\|_\infty$  is uniformly bounded.

**Lemma 6.** *For  $j = 1(1)n$ ,  $n > 1$ , the following holds:*

$$p_j^n / (2\tau_j^n) < \frac{\pi}{2} (1 + \mathcal{O}((h_j^n)^2));$$

hence,

$$\|Q_{n,n}\|_\infty < \frac{\pi}{2} (1 + \mathcal{O}((h_{\max}^n)^2)).$$

*Proof.* The second inequality in the proof of Lemma 5 immediately implies

$$\begin{aligned} p_j^n / (2\tau_j^n) &< \frac{2\pi}{\tau_j^n h_j^n} \left( 1 / \sin^2 \left( \frac{x_{j-1}^n - t_j^n}{2} \right) + 1 / \sin^2 \left( \frac{x_j^n - t_j^n}{2} \right) \right)^{-1} \\ &< \frac{2\pi}{\tau_j^n h_j^n} \min \left\{ \sin^2 \left( \frac{x_{j-1}^n - t_j^n}{2} \right), \sin^2 \left( \frac{x_j^n - t_j^n}{2} \right) \right\} \\ &\leq \frac{2\pi}{\tau_j^n h_j^n} \sin^2 \left( \frac{\tau_j^n}{2} \right) \\ &\leq 2\pi \left( \sin \left( \frac{\tau_j^n}{2} \right) / \tau_j^n \right) \left( \sin \left( \frac{h_j^n}{2} \right) / h_j^n \right). \quad \square \end{aligned}$$

The estimate for  $\|\tilde{B}_{n-1,n}\|_\infty$  will be given in the next proof, finally yielding

**Theorem.** *For optimally chosen points  $t_i^n$ , the following holds:*

$$(5.12) \quad \|B_{n,n-1}^\dagger\|_\infty = \mathcal{O}(|\log h_{\min}^n|).$$

If  $h_{\min}^n = \mathcal{O}(n^{-s})$  for some  $s > 0$  (for a uniform mesh we have  $s = 1$ , for the graded mesh (3.6)  $s = q$ ), then

$$(5.13) \quad \|B_{n,n-1}^\dagger\|_\infty = \mathcal{O}(\log n).$$

*Proof.* It remains to bound properly  $\tilde{B}_{n-1,n}$ . It is no restriction to consider only knots  $x_i^n \leq \pi$ . Define

$$g_i^n(t) := \begin{cases} \cot((x_i^n - t)/2) + \cot(t/2), & 0 < t < x_i^n, \\ \cot((t - x_i^n)/2) - \cot(t/2), & x_i^n < t < 2\pi. \end{cases}$$

Then

$$(|\tilde{B}_{n-1,n}|e)_i = \sum_{j=1}^n 2\tau_j^n g_i^n(t_j^n).$$

The  $g_i^n$  are convex functions. Hence we have the inequalities

$$2\tau_j^n g_i^n(t_j^n) \leq \int_{t_j^n - \tau_j^n}^{t_j^n + \tau_j^n} g_i^n(t) dt.$$

But the bounds do not help much close to the poles at 0,  $x_i^n$  and  $2\pi$ . Therefore we extract the corresponding terms and replace the others by the integrals over an eventually enlarged region. Hence we achieve for sufficiently large  $n$  the estimate

$$(|\tilde{B}_{n-1,n}|e)_i \leq 2r_i^n + \mathfrak{I}_i,$$

where

$$r_i^n := \tau_1^n g_i^n(t_1^n) + \tau_i^n g_i^n(t_i^n) + \tau_{i+1}^n g_i^n(t_{i+1}^n) + \tau_n^n g_i^n(t_n^n),$$

$$\mathfrak{I}_i := \int_{t_2^n - \tau_2^n}^{t_{i-1}^n + \tau_{i-1}^n} g_i^n(t) dt + \int_{t_{i+2}^n - \tau_{i+2}^n}^{t_{n-1}^n + \tau_{n-1}^n} g_i^n(t) dt.$$

Formulas for the integrals are readily available yielding

$$\begin{aligned} \frac{\mathfrak{J}_i}{2} &= \log \left[ \sin \frac{x_i^n - t_2^n + \tau_2^n}{2} \sin \frac{t_{i-1}^n + \tau_{i-1}^n}{2} \right. \\ &\quad \cdot \left. \sin \frac{t_{i+2}^n - \tau_{i+2}^n}{2} \sin \frac{t_{n-1}^n + \tau_{n-1}^n - x_i^n}{2} \right] \\ &\quad - \log \left[ \sin \frac{t_2^n - \tau_2^n}{2} \sin \frac{x_i^n - t_{i-1}^n - \tau_{i-1}^n}{2} \right. \\ &\quad \cdot \left. \sin \frac{t_{i+2}^n - \tau_{i+2}^n - x_i^n}{2} \sin \frac{t_{n-1}^n + \tau_{n-1}^n}{2} \right] \\ &< -\log \left[ \sin \frac{t_2^n - \tau_2^n}{2} \sin \frac{x_i^n - t_{i-1}^n - \tau_{i-1}^n}{2} \right. \\ &\quad \cdot \left. \sin \frac{t_{i+2}^n - \tau_{i+2}^n - x_i^n}{2} \sin \frac{t_{n-1}^n + \tau_{n-1}^n}{2} \right] \\ &< -\log \left[ \sin \frac{h_1^n}{2} \sin \frac{h_i^n}{2} \sin \frac{h_{i+1}^n}{2} \sin \frac{h_n^n}{2} \right]. \end{aligned}$$

This result immediately implies that

$$\mathfrak{J}_i < -2 \log \left[ \sin \frac{h_1^n}{2} \sin^2 \left( \frac{h_{\min}^n}{2} \right) \sin \frac{h_n^n}{2} \right] = \mathcal{O}(-\log h_{\min}^n),$$

where  $h_{\min}^n := \min\{h_1^n, \dots, h_n^n\}$ . Note that, for  $i < 3$  and  $i > n - 3$  one of the integrals in the definitions of  $\mathfrak{J}_i$  should not be present. The resulting modifications in the estimates are easily done. But we still have to show that the  $r_i^n$  are uniformly bounded with respect to  $i$  and  $n$ . It holds, e.g.,

$$\begin{aligned} \tau_1^n g_i^n(t_1^n) &= \tau_1^n \left( \cot \frac{x_i^n - t_1^n}{2} + \cot \frac{t_1^n}{2} \right) \\ &\leq 2\tau_1^n \cot \frac{\tau_1^n}{2} \leq \text{const.}, \\ \tau_i^n g_i^n(t_i^n) &= \tau_i^n \left( \cot \frac{x_i^n - t_i^n}{2} + \cot \frac{t_i^n}{2} \right) \\ &\leq 2\tau_i^n \cot \frac{\tau_i^n}{2} \leq \text{const.} \end{aligned}$$

In the same way the remaining terms in  $r_i^n$  may be handled, thus completing the proof.  $\square$

*Remark.* It can be shown for a uniform mesh with similar techniques that

$$\begin{aligned} \|B_{n,n-1}^\dagger\|_\infty &= \frac{2\pi}{n} \sin \frac{x_m^n}{2} \sum_{j=1}^n 1 / \left( \sin \frac{t_j^n}{2} \left| \sin \frac{x_m^n - t_j^n}{2} \right| \right) \\ &= \mathcal{O} \left( 4 \log \left[ \sin \frac{t_m^n}{2} \sin \frac{t_{m+1}^n}{2} / \sin^2(t_1^n) \right] \right) \\ &= \mathcal{O}(\log n), \end{aligned}$$

where  $m := \lfloor n/2 \rfloor$ . To see this,  $(|\tilde{B}_{n-1,n}|e)_i = (|B_{n,n-1}^\dagger|e)_i$  has to be understood also as a trapezoidal rule, after excluding some terms once again; then convexity implies that the  $\mathfrak{J}_i$  provide a lower bound which is of the same order as the upper one.

Note that all of these results hold without any mesh restrictions. But in order to achieve convergence we need the consistency results which hold for smooth functions already on a uniform mesh whereas the weakly singular solutions need the graded mesh.

**6. Consistency for weakly singular solutions.** For smooth solutions consistency is no problem at all, and the order 2 may be derived easily. For solutions  $u \in C^3[0, 2\pi]$  the same result has been proved in [11] assuming that the mesh conditions  $(A_1)$ – $(A_4)$  hold. At first glance it seems to be quite clear that the appropriate grading of the mesh should yield consistency of order two also for weakly singular solutions. But for actually proving this result we need some unpleasant estimates and technical details. Therefore we will state the theorem giving the order of consistency in a first subsection. Then some technical lemmas will be collected preparing the proof which will be completed in the last subsection.

### 6.1. The consistency theorem.

**Theorem.** *Let  $u$  be weakly singular in the sense of (3.7) and the mesh graded according to (3.6). If the assumptions  $(A_3)$  and  $(A_4)$  hold*



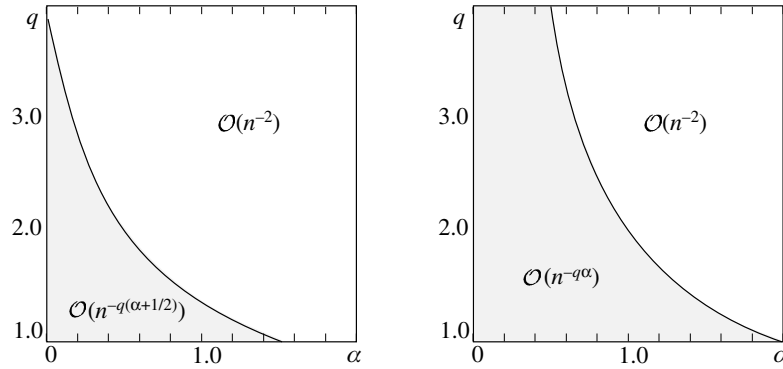


FIGURE 1.

for the collocation points  $t_i^n$  or if the  $t_i^n$  are optimally chosen, then

$$\sqrt{\sum_{i=1}^n h_i^n (\mathcal{H}u(t_i^n) - \mathcal{H}_n u(t_i^n))^2} = \begin{cases} \mathcal{O}(1/n^{q\alpha+q/2}), & 1 \leq q < 2/(\alpha + 1/2), \\ \mathcal{O}(\sqrt{\log n}/n^2), & q = 2/(\alpha + 1/2), \\ \mathcal{O}(1/n^2), & q > 2/(\alpha + 1/2), \end{cases}$$

and

$$\max_{i=1}^n |\mathcal{H}u(t_i^n) - \mathcal{H}_n u(t_i^n)| = \begin{cases} \mathcal{O}(1/n^{q\alpha}), & 1 \leq q < 2/\alpha, \\ \mathcal{O}(\log n/n^2), & q = 2/\alpha, \\ \mathcal{O}(1/n^2), & q > 2/\alpha. \end{cases}$$

Here  $\mathcal{H}_n u(t_i^n)$  means that  $\mathcal{H}_n$  is applied to a piecewise linear function which coincides with  $u$  at the mesh points  $x_j^n$  and is then evaluated at the collocation points  $t_i^n$ .

Figure 1 displays the regions of different orders for the two different norms considered in the theorem. The logarithmic terms appear on the boundary curves.

*Remark.* For solutions being  $2\pi$ -periodic and of **type**  $(\alpha, r + 1, \{\pi\})$ ,  $r > 2$ , it may be interesting to study quadrature rules yielding an order

$r$ . Then the stability results do still hold, and studying consistency will obviously become more complicated, but only in the notations and technical details, finally leading to an  $n^{-r}$  for  $q > r/(\alpha + 1/2)$ , etc. Nonoptimal points  $t_i^n$  then clearly require also higher order interpolation.

The main problems for the proof are introduced by the behavior of  $u$  at  $\pi$  and by the fact that we need some knowledge about the location of the optimal collocation points relative to the mesh points.

6.2. *Technical lemmas.* We start with two lemmas about the location of the optimal  $t_i^n$  replacing the assumption  $(A_3)$ .

**Lemma 7.** *If the mesh (2.2) is symmetrical with respect to  $\pi$ , then the optimal knots  $t_i^n$  are also symmetrical with respect to  $\pi$ . If  $n = 2m + 1$ , then  $t_{m+1}^n = \pi$  and*

$$S(t) := \sum_{j=1}^n \omega_j^n \cot \frac{t - x_j^n}{2} = \omega_n^n \cot \frac{t}{2} + \sin t \sum_{j=1}^m \omega_j^n / \left( \sin \frac{t - x_j^n}{2} \sin \frac{t + x_j^n}{2} \right).$$

*Proof.* Symmetry of the mesh, the identity  $S(t) = -S(2\pi - t)$ , and  $t_i^n \in (x_{i-1}^n, x_i^n)$  prove the first assertions. Furthermore, this symmetry and some manipulations with trigonometric identities yield the representation of  $S$  showing once more explicitly that  $t_{m+1}^n = \pi$ .  $\square$

Therefore, assumption  $(A_3)$  holds at least for  $i = m + 1$  and  $n = 2m + 1$ :

$$t_{m+1}^n - x_m^n = x_{m+1}^n - t_{m+1}^n = h_{m+1}^n/2.$$

A similar estimate can be deduced for  $i = m$ .

**Lemma 8.** *If  $n = 2m + 1$  and the  $t_m^n, t_{m+1}^n, t_{m+2}^n$  are optimal on*

a graded mesh (3.6), then there exists a constant  $C \in (0, 1)$  such that  $x_m^n - t_m^n \geq Ch_m^n$  and  $t_{m+2}^n - x_{m+1}^n \geq Ch_{m+2}^n$  for all  $m \in \mathbf{N}$ .

*Proof.* Fix  $m$  and assume that  $t_m^n \geq x_m^n - h_m^n/2$  (otherwise  $C = 1/2$  is good enough). As  $S(t_m^n) = 0$ , we have

$$\begin{aligned} 1 / \sin \frac{x_m^n - t_m^n}{2} &= \sin \frac{x_m^n + t_m^n}{2} / \omega_m^n \left[ \omega_n^n \cot \frac{t_m^n}{2} / \sin t_m^n \right. \\ &\quad \left. + \sum_{j=1}^{m-1} \omega_j^n / \left( \sin \frac{t_m^n - x_j^n}{2} \sin \frac{t_m^n + x_j^n}{2} \right) \right]. \end{aligned}$$

Note that all terms in the brackets are positive. Now we will derive bounds for the terms appearing on the righthand side of the equation. For sufficiently large  $m$  the following holds:

$$\begin{aligned} \sin \frac{x_m^n + t_m^n}{2} &= \sin \frac{\pi - x_m^n + \pi - t_m^n}{2} \\ &= \sin \frac{h_{m+1}^n + x_m^n - t_m^n}{2} \\ &< \sin \frac{h_{m+1}^n + h_m^n}{2}; \end{aligned}$$

hence,

$$\sin \frac{x_m^n + t_m^n}{2} / \omega_m^n \leq \frac{2}{h_{m+1}^n + h_m^n} \sin \frac{h_{m+1}^n + h_m^n}{2} \leq 1 + \varepsilon_0.$$

Furthermore, for  $m$  large enough, the following holds:

$$\begin{aligned} \cot \frac{t_m^n}{2} / \sin t_m^n &= \cos \frac{t_m^n}{2} / \left( \sin t_m^n \sin \frac{t_m^n}{2} \right) \\ &= \sin \frac{\pi - t_m^n}{2} / \left( \sin(\pi - t_m^n) \sin \frac{t_m^n}{2} \right) \\ &\leq \frac{1}{2}(1 + \varepsilon_1). \end{aligned}$$

By the assumption on  $t_m^n$ , we have for  $j \leq m - 1$

$$\sin \frac{t_m^n - x_j^n}{2} \geq \sin \frac{x_{m-1}^n + h_m^n/2 - x_j^n}{2}$$

and

$$\sin \frac{t_m^n + x_j^n}{2} \geq \sin \frac{x_m^n + x_j^n}{2}$$

as far as  $(t_m^n + x_1^n)/2 > \pi/2$  which is true for large  $n$ . Collecting the results we obtain

$$\begin{aligned} & 1 / \sin \frac{x_m^n - t_m^n}{2} \\ & < (1 + \varepsilon_0) \left[ \frac{1 + \varepsilon_1}{2} \omega_n^n + \sum_{j=1}^{m-1} \omega_j^n / \left( \sin \frac{x_{m-1}^n - x_j^n + h_m^n/2}{2} \sin \frac{x_m^n + x_j^n}{2} \right) \right]. \end{aligned}$$

Note that the bound does not depend on  $t_m^n$ . With the estimate

$$\begin{aligned} & \frac{1}{x_m^n - t_m^n} / \left( \sin \frac{x_m^n - t_m^n}{2} \frac{1}{x_m^n - t_m^n} \right) \\ & = \frac{2}{x_m^n - t_m^n} \frac{1}{1 + \mathcal{O}(x_m^n - t_m^n)} \\ & > \frac{2 - \varepsilon_2}{x_m^n - t_m^n} \quad \text{with } 0 < \varepsilon_2 \ll 1, m \text{ sufficiently large,} \end{aligned}$$

and the inequality  $\sin x \geq 2x/\pi$  for  $0 \leq x \leq \pi/2$ , we finally achieve

$$\frac{1}{x_m^n - t_m^n} < c_1 h_1^n + c_2 \sum_{j=1}^{m-1} \frac{h_{j+1}^n}{(x_{m-1}^n - x_j^n + h_m^n/2)(\pi - x_j^n + h_m^n/2)},$$

where  $(A_1)$  was applied. For the sum, let us call it  $\sigma_m$ , the following holds:

$$\begin{aligned} \sigma_m & \leq \frac{2q}{\pi} n^q \sum_{j=1}^{m-1} \frac{(n-2j)^{q-1}}{((n-2j)^q - (3^q + 1)/2)((n-2j)^q + 1)} \\ & < \frac{2q}{\pi} n^q \sum_{j=1}^{m-1} \frac{1}{(n-2j)((n-2j)^q - (3^q + 1)/2)} \\ & = \frac{2q}{\pi} n^q \sum_{j=1}^{m-1} \frac{1}{(2j+1)((2j+1)^q - (3^q + 1)/2)} \\ & < c_3 n^q, \end{aligned}$$

because the sum is convergent for  $q \geq 1$ . Hence, we arrive at the estimate

$$\frac{1}{x_m^n - t_m^n} < c_1 h_1^n + c_4 n^q \leq c_5 n^q = 1/(c_6 h_m^n)$$

with the definition of  $h_m^n$ . Thus  $x_m^n - t_m^n > c_6 h_m^n$  where necessarily  $0 < c_6 < 1$ . Symmetry implies the result for  $t_{m+2}^n$ .  $\square$

The next lemma will help to bound sharply the influence of a singular  $u'''$ .

**Lemma 9.** *If  $2 > \alpha > 0$ , then there exists a  $C = C(\alpha)$  such that*

$$(6.14) \quad \left| \int_x^t (t-\tau)^2 |\pi-\tau|^{\alpha-3} d\tau / (t-x)^3 \right| \\ \leq C |\pi-x|^{\alpha-2} \begin{cases} |\pi-x|^{-1}, & x \leq t \leq \pi \text{ or } \pi \leq t \leq x, \\ |\pi-t|^{-1}, & t \leq x < \pi \text{ or } \pi < x \leq t. \end{cases}$$

*Proof.* It follows from [10, p. 20] that

$$\left| \int_x^t (t-\tau)^2 |\pi-\tau|^{\alpha-3} d\tau / (t-x)^3 \right| \\ = \frac{1}{3} \begin{cases} |\pi-x|^{\alpha-3} {}_2F_1(3-\alpha, 1, 4, (t-x)/(\pi-x)), & x \leq t \leq \pi \\ & \text{or } \pi \leq t \leq x, \\ |\pi-t|^{\alpha-3} {}_2F_1(3-\alpha, 3, 4, (x-t)/(\pi-t)), & t \leq x < \pi \\ & \text{or } \pi < x \leq t, \end{cases}$$

where  ${}_2F_1$  denotes the hypergeometric function which is well defined with the given parameters for  $|t-x|/|\pi-x| \leq 1$  and  $|x-t|/|\pi-t| < 1$ , respectively. For  $\alpha < 2$ , the following transformation (cf. [10, p. 10]) shows the behavior of the second  ${}_2F_1$ -function:

$${}_2F_1\left(3-\alpha, 3, 4, \frac{x-t}{\pi-t}\right) = \left(\frac{\pi-x}{\pi-t}\right)^{\alpha-2} {}_2F_1\left(1+\alpha, 1, 4, \frac{x-t}{\pi-t}\right).$$

Now the  ${}_2F_1$ -function on the righthand side is uniformly bounded with respect to  $x$  and  $t$  as far as  $|x-t|/|\pi-t| \leq 1$ .  $\square$

*Remark.* In the confluent case  $\alpha = 2$ , there appears a logarithmic singularity in  ${}_2F_1(1, 3, 4, (x-t)/(\pi-t))$  as  $(x-t)/(\pi-t) \rightarrow 1$ .

With these results we are prepared to prove the consistency results of the theorem.

6.3. *Consistency proof.* We have to study

$$\mathcal{H}u(t_i^n) - \mathcal{H}_n u(t_i^n) = Q^n(t_i^n) + I^n(t_i^n)$$

with the remainder of the quadrature

$$Q^n(t_i^n) := \frac{1}{2\pi} \left( \int_0^{2\pi} \frac{u(x) - u(t_i^n)}{\tan((t_i^n - x)/2)} dx - \sum_{j=1}^n \omega_j^n \frac{u(x_j^n) - u(t_i^n)}{\tan((t_i^n - x_j^n)/2)} \right)$$

and the error of the interpolation weighted with the  $S_i^n$  (cf. (2.4)):

$$I^n(t_i^n) := (u_n(t_i^n) - u(t_i^n))S_i^n, \quad i = 1(1)n.$$

$u_n$  denotes the piecewise linear interpolant of  $u$  at the grid points  $x_j^n$ ,  $j = 0(1)n$ . Obviously, all the  $I^n(t_i^n)$  vanish for the optimal  $t_i^n$ —another advantage of these collocation points. For nonoptimal knots the  $S_i^n$  do not generally vanish, although the assumption  $(A_4)$  ensures that they are uniformly bounded. Therefore, the interpolation error appears and has to be studied.

**Lemma 10.** *If  $u$  is of type  $(\alpha, 3, \{\pi\})$  and  $u_n$  its piecewise linear interpolant at the knots  $x_j^n$ ,  $j = 0(1)n$ , of a graded mesh with grading exponent  $q$ , then*

$$\sqrt{\sum_{i=1}^n h_i^n (u(t_i^n) - u_n(t_i^n))^2} = \begin{cases} \mathcal{O}(1/n^{q\alpha+q/2}), & 1 \leq q < 2/(\alpha + 1/2), \\ \mathcal{O}(\sqrt{\log n}/n^2), & q = 2/(\alpha + 1/2), \\ \mathcal{O}(1/n^2), & q > 2/(\alpha + 1/2), \end{cases}$$

and

$$\max_{i=1}^n |u(t_i^n) - u_n(t_i^n)| = \begin{cases} \mathcal{O}(1/n^{q\alpha}), & 1 \leq q \leq 2/\alpha, \\ \mathcal{O}(1/n^2), & q \geq 2/\alpha. \end{cases}$$

*Proof.*

$$\begin{aligned} \sum_{i=1}^n h_i^n (u(t_i^n) - u_n(t_i^n))^2 &\leq D_1 (h_{m+1}^n)^{2\alpha+1} \\ &\quad + D_2 \sum_{i=1}^m (h_{i+1}^n)^5 (\pi - x_i^n)^{2\alpha-4} \\ &\leq \frac{D_3}{n^{2q\alpha+q}} \left( 1 + \sum_{i=1}^m (n-2i)^{2q\alpha+q-5} \right), \end{aligned}$$

if  $n = 2m+1$ . Minor modifications are necessary for even  $n$ . This result is slightly better than the rough estimate given in [8] for the  $L_2$ -norm. For the sup-norm there is [8] just the right reference.  $\square$

Hence the remainder of the quadrature alone yields all the difficulties. Unfortunately, the proof for its bounds given in [11] is not quite complete. Therefore we still have to derive consistency of order two. But we reproduce some of the useful results from [11]. In order to avoid a lot of notational inconveniences, we consider only the case of odd  $n$ . Now we have to look closer at the remainder  $Q^n(t_i^n)$  of the quadrature. Define

$$\begin{aligned} \Phi(t, x) &:= (u(x) - u(t)) \cot \frac{t-x}{2}, \\ x, t &\in [0, \pi) \cup (\pi, 2\pi], \end{aligned}$$

for a weakly singular  $u$ . Then

$$Q^n(t_i^n) = \sum_{j=1}^{2m+1} e_{ji}^n = \frac{1}{12} \sum_{\substack{j=1 \\ j \neq m+1}}^{2m+1} (h_j^n)^3 \Phi_{xx}(t_i^n, \zeta_j^n) + e_{m+1,i}^n,$$

with  $\zeta_j^n \in [x_{j-1}^n, x_j^n]$  and

$$\begin{aligned} e_{ji}^n &:= \int_{x_{j-1}^n}^{x_j^n} \Phi(t_i^n, x) dx - \frac{h_j^n}{2} [\Phi(t_i^n, x_{j-1}^n) + \Phi(t_i^n, x_j^n)], \\ &\quad i, j = 1(1)n. \end{aligned}$$

At first we bound the term  $e_{m+1,i}^n$ .

**Lemma 11.** *If the mesh and the knots  $t_i^n$  are symmetrical with respect to  $\pi$  and if  $n = 2m + 1$ , then there exists a  $C = C(\alpha)$  such that*

$$|e_{m+1,i}^n| \leq C(h_{m+1}^n)^{\alpha+1} \left( \frac{1}{|x_m^n - t_i^n|} + \frac{1}{|x_{m+1}^n - t_i^n|} \right), \quad i = 1(1)n.$$

*Proof.*

$$\begin{aligned} e_{m+1,i}^n &= \int_{x_m^n}^{x_{m+1}^n} \left[ \Phi(t_i^n, x) - \frac{1}{2}(\Phi(t_i^n, x_m^n) + \Phi(t_i^n, x_{m+1}^n)) \right] dx \\ &=: \mathfrak{J}_{1i} + \mathfrak{J}_{2i}, \end{aligned}$$

where

$$\begin{aligned} \mathfrak{J}_{1i} &:= \int_{x_m^n}^{\pi} [\Phi(t_i^n, x) - \Phi(t_i^n, x_m^n)] dx, \\ \mathfrak{J}_{2i} &:= e_{m+1,i}^n - \mathfrak{J}_{1i}. \end{aligned}$$

We will only show how to bound  $\mathfrak{J}_{1i}$ . Then it will be clear how to bound the other term.

$$\begin{aligned} \mathfrak{J}_{1i} &= \int_{x_m^n}^{\pi} \left[ (u(x) - u(t_i^n)) \left( \cot \frac{t_i^n - x}{2} - \cot \frac{t_i^n - x_m^n}{2} \right) \right. \\ &\quad \left. - \cot \frac{t_i^n - x_m^n}{2} (u(x_m^n) - u(x)) \right] dx \\ &= \left[ \int_{x_m^n}^{\pi} (u(x) - u(t_i^n)) \sin \frac{x - x_m^n}{2} / \sin \frac{t_i^n - x}{2} dx \right. \\ &\quad \left. - \cos \frac{t_i^n - x_m^n}{2} \int_{x_m^n}^{\pi} (u(x_m^n) - u(x)) dx \right] / \sin \frac{t_i^n - x_m^n}{2}. \end{aligned}$$

If  $x \in [x_m^n, \pi]$ , then  $x - t_i^n < \pi$  obviously holds for  $i = 1(1)m$ . Furthermore,  $t_i^n - x \leq t_{n-1}^n - x_m^n = 2\pi - t_2^n - x_m^n = (1/2)h_{m+1}^n + \pi - t_2^n < \pi$  for  $i = m + 1(1)n - 1$ . Only  $t_n^n - x$  could be slightly larger than  $\pi$ , namely,  $t_n^n - x \leq 2\pi - x_m^n = (1/2)h_{m+1}^n + \pi$ . Then, using  $\sin z \geq (2/\pi)z$  for  $z \in [0, \pi/2]$ , we have

$$1 / \left| \sin \frac{t_i^n - x}{2} \right| \leq \pi / |t_i^n - x|, \quad i = 1(1)n - 1.$$



And, for  $i = n \geq n_0(\varepsilon_0)$ , such an equality holds too if  $\pi$  is replaced by  $\pi + \varepsilon_0$ ,  $0 < \varepsilon_0 \ll 1$ . Hence,

$$\begin{aligned} \mathfrak{J}_{1i} &\leq C_1 \left[ \int_{x_m^n}^{\pi} (x - x_m^n) |x - t_i^n|^{\alpha-1} dx + \int_{x_m^n}^{\pi} (x - x_m^n)^{\alpha} dx \right] / |t_i^n - x_m^n| \\ &\leq C_1 \left[ \frac{1}{\alpha} (\pi - x_m^n) \left| |\pi - t_i^n|^{\alpha} - |x_m^n - t_i^n|^{\alpha} \right| + \frac{1}{\alpha+1} (\pi - x_m^n)^{\alpha+1} \right] / |t_i^n - x_m^n| \\ &\leq \frac{C_2}{|t_i^n - x_m^n|} (\pi - x_m^n)^{\alpha+1} \\ &\leq \frac{C_3}{|t_i^n - x_m^n|} (h_{m+1}^n)^{\alpha+1}. \end{aligned}$$

Similarly the following bound is achieved

$$\mathfrak{J}_{2i} \leq C_4 (h_{m+1}^n)^{\alpha+1} (1/|t_i^n - x_m^n| + 1/|t_i^n - x_{m+1}^n|). \quad \square$$

*Remark.* If the knots  $t_i^n$  are even optimally chosen or satisfy  $(A_3)$ , then this lemma implies

$$\begin{aligned} (6.15) \quad |e_{m+1,i}^n| &\leq C \begin{cases} (h_{m+1}^n)^{\alpha+1} / |\pi - x_i^n|, & i < m+1, \\ (h_{m+1}^n)^{\alpha}, & i = m+1, \\ (h_{m+1}^n)^{\alpha+1} / |\pi - x_{i-1}^n|, & i > m+1, \end{cases} \\ &\leq \frac{C^*}{n^{q\alpha}} \begin{cases} (n - 2i)^{-q}, & i < m+1, \\ 1, & i = m+1, \\ (2(i - 1) - n)^{-q}, & i > m+1, \end{cases} \\ &=: \beta_i^n(q, \alpha), \end{aligned}$$

a bound which will be used later. Hence, for a graded mesh, the following holds.

$$|e_{m+1,i}^n| \leq C (h_{m+1}^n)^{\alpha} = \mathcal{O}(1/n^{q\alpha}), \quad i = 1(1)n.$$

*Proof.* If  $i < m$ , then

$$\begin{aligned} x_m^n - t_i^n &\geq x_m^n - x_i^n = \pi - x_i^n - h_{m+1}^n/2 \\ &\geq (\pi - x_i^n) \left(1 - \frac{1}{2} \frac{h_{m+1}^n}{\pi - x_{m-1}^n}\right) \\ &= (\pi - x_i^n) \left(1 - \frac{1}{2} \frac{h_{m+1}^n}{h_m^n + h_{m+1}^n/2}\right) \\ &\geq (\pi - x_i^n) \left(1 - \frac{1}{1 + 2/\mu_1}\right) > 0 \end{aligned}$$

with  $\mu_1$  from (A<sub>1</sub>). If  $i = m$ , then Lemma 8 or (A<sub>3</sub>) implies that

$$x_m^n - t_m^n \geq Ch_m^n = C(\pi - x_m^n)2h_m^n/h_{m+1}^n \geq \frac{C}{\mu_1}(\pi - x_m^n).$$

The remaining cases are shown similarly.  $\square$

Now we have to study  $\Phi_{xx}$ . If  $x \neq \pi$  and  $x \neq t$ , then

$$\begin{aligned} \Phi_{xx}(t, x) &= \frac{-1}{2} \left[ \cos \frac{t-x}{2} (u(t) - u(x)) - 2u'(x) \sin \frac{t-x}{2} \right. \\ &\quad \left. - 2u''(x) \cos \frac{t-x}{2} \sin^2 \left(\frac{t-x}{2}\right) \right] / \sin^3 \left(\frac{t-x}{2}\right). \end{aligned}$$

For smooth  $u$ , it is quite easy to derive estimates for  $\Phi_{xx}$  which becomes a slightly more delicate matter in the weakly singular case. Symmetry allows us to assume that  $0 \leq x < \pi$  or even  $x \leq x_m^n = \pi - h_{m+1}^n/2$  because we will treat the interval  $[x_m^n, x_{m+1}^n]$  separately. Nevertheless,  $t$  may vary all over  $[0, 2\pi]$ . But if  $t \in (\pi, 2\pi]$ , then the periodicity of  $u$  yields  $u(t) = u(t^*)$  where  $t^* := t - 2\pi \in (-\pi, 0]$ . And the identity  $\Phi_{xx}(t, x) = \Phi_{xx}(t^*, x)$  follows. Hence,  $u(t) - u(x)$  or  $u(t^*) - u(x)$ , respectively, can be expanded because the singularity at  $\pi$  is avoided. Consider the case  $0 \leq x, t < \pi$  at first. If  $|x - t| \geq \varepsilon > 0$  for some  $\varepsilon \ll 1$ ,  $\varepsilon$  fixed, then

$$\begin{aligned} |\Phi_{xx}(t, x)| &\leq C_0(|u'(x)| + |u''(x)|) \\ &\leq C_1(|\pi - x|^{\alpha-1} + |\pi - x|^{\alpha-2}), \end{aligned}$$

where  $C_0, C_1$  depend on  $\|u\|_\infty$  and  $\varepsilon$ . If  $|x - t| < \varepsilon$ , then expanding  $\Phi_{xx}$  yields

$$|\Phi_{xx}(t, x)| \leq C_2 \left( |u'(x)| + |u''(x)| + \left| \int_x^t (t-\tau)^2 u'''(\tau) d\tau \right| / |t-x|^3 \right).$$

Continuity arguments assure that all the inequalities even hold for  $t = \pi$ . The same reasoning holds if  $t > \pi$ . But in this case we have to replace  $t$  by  $t^*$ . Therefore, the following bound is achieved with (6.14) for  $0 \leq t \leq \pi$ ,

$$\begin{aligned} |\Phi_{xx}(t, x)| &\leq C_3 \left\{ (\pi - x)^{\alpha-1} + (\pi - x)^{\alpha-2} \right. \\ &\quad \left. + \left| \int_x^t (t-\tau)^2 (\pi - \tau)^{\alpha-3} d\tau \right| / |t-x|^3 \right\} \\ &\leq C_4 (\pi - x)^{\alpha-2} \begin{cases} (\pi - x)^{-1}, & 0 \leq x \leq t \leq \pi, \\ (\pi - t)^{-1}, & 0 \leq t \leq x < \pi. \end{cases} \end{aligned}$$

If  $\pi < t \leq 2\pi$ , then we have, with  $t^* := t - 2\pi \in (-\pi, 0]$  and (6.14),

$$\begin{aligned} |\Phi_{xx}(t, x)| &= |\Phi_{xx}(t^*, x)| \\ &\leq C_5 \left\{ (\pi - x)^{\alpha-1} + (\pi - x)^{\alpha-2} \right. \\ &\quad \left. + \int_{t^*}^0 (\tau - t^*)^2 (\pi + \tau)^{\alpha-3} d\tau / |t^* - x|^3 \right. \\ &\quad \left. + \int_0^x (\tau - t^*)^2 (\pi - \tau)^{\alpha-3} d\tau / |t^* - x|^3 \right\} \\ &\leq C_6 \left\{ (\pi - x)^{\alpha-2} + \int_{-x}^{-t^*} (-t^* - \tau)^2 (\pi - \tau)^{\alpha-3} d\tau / |t^* - x|^3 \right. \\ &\quad \left. - \int_x^{t^*} (t^* - \tau)^2 (\pi - \tau)^{\alpha-3} d\tau / |t^* - x|^3 \right\} \\ &\leq C_7 \{ (\pi - x)^{\alpha-2} + (\pi + x)^{\alpha-3} + (\pi - x)^{\alpha-2} (3\pi - t)^{-1} \} \\ &\leq C_8 (\pi - x)^{\alpha-2}. \end{aligned}$$

Hence, for such  $x$  and  $t$ ,  $\Phi_{xx}$  behaves like the *second* derivative of  $u$ . Now we are in a position to bound

$$\frac{1}{12} \sum_{\substack{j=1 \\ j \neq m+1}}^{2m+1} (h_j^n)^3 \Phi_{xx}(t_i^n, \zeta_j^n) = \sum_{\substack{j=1 \\ j \neq m+1}}^{2m+1} e_{ji}^n, \quad i = 1(1)n.$$

Symmetry allows us to consider only  $|\sum_{j=1}^m e_{ji}^n|$ . But then we have to study the cases  $i \leq m$ ,  $i = m + 1$  and  $i \geq m + 2$  separately. Let  $i \leq m$ . Without loss of generality, we assume that  $t_i^n \leq \zeta_i^n$ . Otherwise we would have to discuss the sums from 1 to  $i$  and from  $i + 1$  to  $m$  with the same final result. Therefore, we get

$$\left| \sum_{j=1}^m e_{ji}^n \right| \leq C_9 \sum_{j=1}^{i-1} (h_j^n)^3 (\pi - \zeta_j^n)^{\alpha-3} + \frac{C_9}{|\pi - t_i^n|} \sum_{j=i}^m (h_j^n)^3 (\pi - \zeta_j^n)^{\alpha-2}.$$

With  $h_j^n \leq \mu_1 h_{j+1}^n$ , it follows

$$\begin{aligned} \left| \sum_{j=1}^m e_{ji}^n \right| &\leq C_{10} \sum_{j=1}^{i-1} (h_{j+1}^n)^3 (\pi - x_j^n)^{\alpha-3} \\ &\quad + \frac{C_{10}}{|\pi - t_i^n|} \sum_{j=i}^m (h_{j+1}^n)^3 (\pi - x_j^n)^{\alpha-2} \\ &\leq \frac{C_{11}}{n^{q\alpha}} \sum_{j=1}^{i-1} (n - 2j)^{q\alpha-3} \\ &\quad + \frac{C_{11}}{n^{q\alpha}(n - 2i)^q} \sum_{j=i}^m (n - 2j)^{q\alpha+q-3} \\ &= \frac{C_{11}}{n^2} \left[ \frac{1}{n} \sum_{j=1}^{i-1} \left( \frac{n-2j}{n} \right)^{q\alpha-3} \right] \\ &\quad + \frac{C_{11}}{n^2} \left( \frac{n}{n-2i} \right)^q \left[ \frac{1}{n} \sum_{j=i}^m \left( \frac{n-2j}{n} \right)^{q\alpha+q-3} \right]. \end{aligned}$$

The quadrature rules in the brackets are bounded by the integrals over convex or concave, monotone functions. Hence we can keep on estimating, for  $i = 1(1)m$ , by

$$\frac{C_{12}}{n^{q\alpha}} \begin{cases} (n - 2i)^{-q} & \text{if } q\alpha + q < 2, \\ (n - 2i)^{-q} \log((n - 2i)e) & \text{if } q\alpha + q = 2, \\ (n - 2i)^{q\alpha-2} & \text{if } q\alpha + q > 2 > q\alpha, \\ |\log((n - 2i)/n)| & \text{if } 2 = q\alpha, \\ n^{q\alpha-2} & \text{if } 2 < q\alpha \end{cases} \\ =: b_i^n(q, \alpha).$$

If  $i = m + 1$ , then we have  $\zeta_j^n < t_{m+1}^n = \pi$ , and thus

$$\begin{aligned} \left| \sum_{j=1}^m e_{j,m+1}^n \right| &\leq C_{13} \sum_{j=1}^m (h_{j+1}^n)^3 (\pi - x_j^n)^{\alpha-3} \\ &\leq \frac{C_{14}}{n^{q\alpha}} \sum_{j=1}^m (n-2j)^{q\alpha-3} \\ &\leq C_{15} \begin{cases} n^{-q\alpha} & \text{if } q\alpha < 2, \\ \log n/n^2 & \text{if } q\alpha = 2, \\ n^{-2} & \text{if } q\alpha > 2 \end{cases} \\ &=: b_{m+1}^n(q, \alpha). \end{aligned}$$

If  $i > m + 1$ , then  $t_i^n > \pi > x_j^n$  for  $j = 1(1)m$ . Thus,  $|\Phi_{xx}(t_i^n, x_j^n)| \leq C_8(\pi - x_j^n)^{\alpha-2}$  yielding

$$\begin{aligned} \left| \sum_{j=1}^m e_{ji}^n \right| &\leq C_{16} \sum_{j=1}^m (h_{j+1}^n)^3 (\pi - x_j^n)^{\alpha-2} \\ &\leq \frac{C_{17}}{n^2} \left[ \frac{1}{n} \sum_{j=1}^m \left( \frac{n-2j}{n} \right)^{q\alpha+q-3} \right] \\ &\leq C_{18} \begin{cases} n^{-q\alpha-q} & \text{if } q\alpha + q < 2, \\ \log n/n^2 & \text{if } q\alpha + q = 2, \\ n^{-2} & \text{if } q\alpha + q > 2 \end{cases} \\ &=: b_i^n(q, \alpha). \end{aligned}$$

These bounds, (6.15), and symmetry finally lead to an estimate for  $i = 1(1)n$ :

$$\begin{aligned} \left| \sum_{j=1}^n e_{ji}^n \right| &\leq \left| \sum_{j=1}^m e_{ji}^n \right| + |e_{m+1,i}^n| + \left| \sum_{j=m+2}^n e_{ji}^n \right| \\ &\leq b_i^n(q, \alpha) + \beta_i^n(q, \alpha) + b_{n-i+1}^n(q, \alpha) =: \mathbf{b}_i^n(q, \alpha). \end{aligned}$$

Comparing all the bounds immediately yields that

$$\beta_i^n(q, \alpha) \leq \text{const.} (b_i^n(q, \alpha) + b_{n-i+1}^n(q, \alpha))$$

and, secondly, that  $b_i^n(q, \alpha) + b_{n-i+1}^n(q, \alpha)$  is dominated by the terms depending on  $i$  if  $i \neq m+1$  and  $q\alpha \leq 2$ . Therefore, we have

$$\mathbf{b}_i^n(q, \alpha) \leq C_{19} \begin{cases} b_i^n(q, \alpha) & \text{if } i < m+1, \\ b_{n-i+1}^n(q, \alpha) & \text{if } i > m+1, \end{cases}$$

and

$$\mathbf{b}_{m+1}^n(q, \alpha) \leq C_{19} \begin{cases} 1/n^{q\alpha} & \text{if } q\alpha < 2, \\ \log n/n^2 & \text{if } q\alpha = 2, \\ 1/n^2 & \text{if } q\alpha > 2. \end{cases}$$

Thus, we have derived the order of consistency in the sup-norm. For the weighted sum of the squared bounds, it follows

$$\begin{aligned} & \sqrt{\sum_{i=1}^n h_i^n \left( \sum_{j=1}^n e_{ji}^n \right)^2} \\ & \leq C_{20} \sqrt{2 \sum_{i=1}^m h_{i+1}^n (\mathbf{b}_i^n(q, \alpha))^2 + h_{m+1}^n (\mathbf{b}_{m+1}^n(q, \alpha))^2} \\ & \leq C_{21} \sqrt{\frac{1}{n} \sum_{i=1}^m \left(1 - \frac{2i}{n}\right)^{q-1} (\mathbf{b}_i^n(q, \alpha))^2 + (1/n^q) (\mathbf{b}_{m+1}^n(q, \alpha))^2} \\ & \leq C_{22} \begin{cases} n^{-q(\alpha+1/2)} & \text{if } q(\alpha+1/2) < 2, \\ n^{-2} \sqrt{\log n} & \text{if } q(\alpha+1/2) = 2, \\ n^{-2} & \text{if } q(\alpha+1/2) > 2. \end{cases} \end{aligned}$$

The last inequality is obtained again by comparing quadrature rules with appropriate integrals. As an example, we will only show the case

$q\alpha + q > 2 > q\alpha$ . Then

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^m \left(1 - \frac{2i}{n}\right)^{q-1} (\mathbf{b}_i^n(q, \alpha))^2 \\ & \leq \frac{C_{19}^2}{n} \sum_{i=1}^m \left(1 - \frac{2i}{n}\right)^{q-1} (n-2i)^{2q\alpha-4} / n^{2q\alpha} \\ & = \frac{C_{19}^2}{n^4} \left[ \frac{1}{n} \sum_{i=1}^m \left(1 - \frac{2i}{n}\right)^{2q\alpha+q-5} \right] \\ & \leq \text{const.} \begin{cases} n^{-2q(\alpha+1/2)}, & q(\alpha+1/2) < 2, \\ n^{-4} \log n, & q(\alpha+1/2) = 2, \\ n^{-4}, & q(\alpha+1/2) > 2. \end{cases} \end{aligned}$$

Hence, the order of consistency is established for the weighted norm too. Actually, all of these results hold also for nonoptimal knots as far as the mesh restrictions  $(A_1)$ – $(A_3)$  and  $(A_4)$ , which bounds the weights of the interpolation error, apply. Therefore, the consistency result is completely proved.

**7. Convergence.** Here we simply have to collect and to combine results from the previous sections.

Let  $u$  be the solution of the continuous equation,  $\tilde{u}^n$  the solution of the discrete collocation with optimal  $t_i^n$ . Then

$$(u(x_i^n) - \tilde{u}_i^n)_{i=1(1)n-1} = B_{n,n-1}^\dagger (\mathcal{H}_n u(t_i^n) - \mathcal{H}u(t_i^n))_{i=1(1)n}.$$

Here we have used  $\mathcal{H}u = f$  and  $u(2\pi) = u(x_n^n) = 0$ . Now the consistency theorem and the stability results (5.11)–(5.13) yield, as usual, convergence and the order of convergence:

**Theorem.** *Let  $u$  be  $2\pi$ -periodic, of type  $(\alpha, 3, \{\pi\})$ , and a solution of the integral equation  $\mathcal{H}u = f$  with the side conditions  $u(0) = 0$ . Let  $\tilde{u}^n$  be the solution of the discrete collocation with optimal  $t_i^n$  on a mesh (3.6) with grading exponent  $q \geq 1$ . If  $q > 2/(\alpha + 1/2)$ , then*

$$\|u_r - \tilde{u}^n\|_{\tilde{W}_n} \leq \sqrt{\frac{2\pi}{\omega_n^n}} n^{-2} (C + \mathcal{O}(h_{\max}^n)) = \sqrt{n} \mathcal{O}\left(\frac{1}{n^2}\right),$$

$u_r$  being the restriction of  $u$  to the gridpoints  $x_i^n$ . If  $q > 2/\alpha$ , then

$$\|u_r - \tilde{u}^n\|_\infty = \mathcal{O}((h_{\max}^n)^2 |\log h_{\min}^n|) = \mathcal{O}\left(\frac{\log n}{n^2}\right).$$

With the piecewise linear polynomial  $u_n$  defined by its values  $\tilde{u}_i^n$  at the knots  $x_i^n$ , this result implies with an interpolation argument that

$$\sup\{|u(t) - u_n(t)|, 0 \leq t \leq 2\pi\} = \mathcal{O}\left(\frac{\log n}{n^2}\right).$$

In the theorem we have separated the  $\sqrt{n}$  term indicating that it only appears in the worst case.

*Remark.* If the grading exponent is too small, then the order drops down according to the consistency results. The error bounds hold too on a uniform grid with midpoint collocation for  $u \in C^3[0, 2\pi]$ . Clearly, for a smooth and  $2\pi$  periodic solution we then even have an exponential decay of the remainder as always when applying the trapezium rule to such functions.

*Remark.* A similar analysis for the integral side condition  $\int_0^{2\pi} u(t) dt = 0$  (discretized by the trapezoidal rule, too) instead of  $u(0) = u(2\pi) = 0$ , leads to an analogous error estimate but without the  $\sqrt{n}$  for the weighted norm (cf. [9]).

If the  $t_i^n$  are not optimal, but if  $(A_3)$  and  $(A_4)$  hold, then the consistency results are still valid. In order to study the influence of the collocation points on the stability, we will compare the matrix  $A_{n-1, n-1}$  of the optimal points with the matrix belonging to the  $t_i^n$ , namely  $A(\tau)_{n-1, n-1}$  where  $\tau := \{t_1^n, \dots, t_n^n\}$ . If  $\tilde{u}^n$  is the solution of the system  $A(\tau)_{n-1, n-1} \tilde{u}^n = R_{n-1, n} f^n$ , then

$$\begin{aligned} (u(x_i^n) - \tilde{u}_i^n)_{i=1(1)n-1} &= A(\tau)_{n-1, n-1}^{-1} A_{n-1, n-1} A_{n-1, n-1}^{-1} R_{n-1, n} \\ &\quad \cdot (B(\tau)_{n, n-1} u(x_i^n)_{i=1(1)n-1} - f^n) \\ &= (A(\tau)_{n-1, n-1}^{-1} A_{n-1, n-1}) B_{n, n-1}^\dagger (\mathcal{H}_n u(t_i^n) \\ &\quad - \mathcal{H} u(t_i^n))_{i=1(1)n}. \end{aligned}$$



Therefore the convergence theorem remains true for  $t_i^n$  satisfying  $(A_3)$  and  $(A_4)$ , and furthermore

$$(7.16) \quad \|A(\tau)_{n-1,n-1}^{-1}A_{n-1,n-1}\| = \mathcal{O}(1).$$

For knots  $t_i^n$  not satisfying (7.16) we still have consistency (assuming the conditions  $(A_3)$  and  $(A_4)$ ) but our stability results do not apply. However, all numerical computations indicate that the estimates given in the previous theorem are valid at the easily available midpoints  $t_i^n$  even for graded meshes. Unfortunately, neither (7.16) nor  $(A_4)$  have been proven for them up to now. Hence, there remains an open problem.

**8. Numerical results.** The numerical results given in [3] and [11] clearly displayed a better order of convergence than the theoretical results of these papers, thus motivating our research. Now these results perfectly fit into the theory, as far as the optimal points are concerned. Therefore, it suffices to show only some additional numbers in order to ascertain this statement.

Our computations were performed with the test functions

$$(8.17) \quad u(s) = \mathbf{R}\{(e^{is} + 1)^\alpha\} - 2^\alpha, \quad s \in [0, 2\pi] \text{ and } \alpha > 0.$$

Obviously, they are  $2\pi$ -periodic, of **type**  $(\alpha, \infty, \{\pi\})$ , and satisfy the point side condition  $u(0) = 0$ . If  $\alpha = 1/2$ , e.g., then

$$u(s) = \sqrt{2 \cdot |\cos(s/2)|} \cdot \max \left\{ \cos \left( \frac{s}{4} \right), \sin \left( \frac{s}{4} \right) \right\} - \sqrt{2}, \quad s \in [0, 2\pi].$$

The righthand side  $f$  of the equation  $\mathcal{H}u = f$  is given by

$$f(t) = \mathfrak{I}\{(e^{it} + 1)^\alpha\}, \quad t \in [0, 2\pi].$$

We computed the approximations  $\tilde{u}^n$  and the norms of the absolute errors  $\|u_r - \tilde{u}^n\|_{\tilde{W}_n}$ ,  $\|u_r - \tilde{u}^n\|_\infty$  for some  $n$  and for different grading exponents  $q$ . Here  $u_r$  denotes again the restriction of  $u$  to the mesh. In

TABLE 1. Trapezoidal rule.

Function (8.17), $\alpha = 1/2$							
Mesh graded with $q$							
$t_i^n = \text{midpoints}, i = 1(1)n$							
$q$	$n$	$\ u_r - \tilde{u}^n\ _{\tilde{W}_n}$	$\mathcal{O}_{\tilde{W}_n}^{CE}$	$\mathcal{O}_{\tilde{W}_n}^{CY}$	$\ u_r - \tilde{u}^n\ _\infty$	$\mathcal{O}_\infty^{CE}$	$\mathcal{O}_\infty^{CY}$
1	4	$1.8_{10} - 1$			$3.5_{10} - 1$		
	8	$8.7_{10} - 2$	1.02	0.90	$2.5_{10} - 1$	0.52	0.47
	16	$4.3_{10} - 2$	1.01	0.96	$1.7_{10} - 1$	0.51	0.50
	32	$2.1_{10} - 2$	1.01	0.98	$1.2_{10} - 1$	0.51	0.50
	64	$1.1_{10} - 2$	1.00	0.99	$8.6_{10} - 2$	0.50	0.50
2	4	$1.3_{10} - 1$			$3.4_{10} - 1$		
	8	$3.9_{10} - 2$	*2.00	*1.99	$1.9_{10} - 1$	0.84	0.94
	16	$1.1_{10} - 2$	*2.04	*2.03	$9.9_{10} - 2$	0.94	1.00
	32	$3.0_{10} - 3$	*2.03	*2.02	$5.1_{10} - 2$	0.97	1.00
	64	$8.1_{10} - 4$	*2.02	*2.02	$2.5_{10} - 2$	0.99	1.00
4	4	$1.8_{10} - 1$			$4.5_{10} - 1$		
	8	$6.0_{10} - 2$	1.55	1.72	$2.5_{10} - 1$	*1.43	1.94
	16	$1.6_{10} - 2$	1.86	2.01	$1.0_{10} - 1$	*1.74	2.00
	32	$4.0_{10} - 3$	2.03	2.07	$3.4_{10} - 2$	*1.88	2.00
	64	$9.6_{10} - 4$	2.08	2.07	$1.1_{10} - 2$	*1.93	2.00
6	4	$2.5_{10} - 1$			$5.0_{10} - 1$		
	8	$9.8_{10} - 2$	1.37	1.37	$3.4_{10} - 1$	0.57	2.72
	16	$3.0_{10} - 2$	1.69	1.90	$1.4_{10} - 1$	1.29	2.20
	32	$7.9_{10} - 3$	1.94	2.03	$4.2_{10} - 2$	1.71	2.08
	64	$1.9_{10} - 3$	2.06	2.08	$1.1_{10} - 2$	1.90	2.05

order to estimate the order of convergence and the order of consistency, respectively, we use the quantities

$$\mathcal{O}_*^{CE} := \frac{\log \|u_r - \tilde{u}^{n_1}\|_* - \log \|u_r - \tilde{u}^{n_2}\|_*}{\log(n_2) - \log(n_1)}, \quad n_1 < n_2,$$

and the analogously defined  $\mathcal{O}_*^{CY}$  for the two norms studied in this paper. Unfortunately, these quantities are not very significant if the error behaves like  $(\log n)^\beta/n^\gamma$ . In order to estimate the  $\gamma$  it is quite convenient to modify them by subtracting  $\beta \log(\log n_1/\log n_2)/\log(n_2/n_1)$ . An \* indicates such modifications in the tables. They appear for the

TABLE 2. Simpson's rule with the inversion formula (4.9).

Function (8.17), $\alpha = 1/2$					
Mesh graded with $q$					
$t_i^n =$ optimal points, $i = 1(1)n$					
$q$	$n$	$\ u_r - \tilde{u}^n\ _{\tilde{W}_n}$	$\mathcal{O}_{\tilde{W}_n}^{CE}$	$\ u_r - \tilde{u}^n\ _\infty$	$\mathcal{O}_\infty^{CE}$
2	8	$2.7_{10} - 2$		$1.3_{10} - 1$	
	16	$6.7_{10} - 3$	2.00	$6.5_{10} - 2$	1.00
	32	$1.7_{10} - 3$	2.00	$3.3_{10} - 2$	1.00
	64	$4.2_{10} - 4$	2.00	$1.6_{10} - 2$	1.00
	128	$1.0_{10} - 4$	2.00	$8.2_{10} - 3$	1.00
4	8	$2.3_{10} - 2$		$1.5_{10} - 1$	
	16	$2.7_{10} - 3$	*3.32	$4.5_{10} - 2$	1.71
	32	$2.3_{10} - 4$	*3.68	$1.2_{10} - 2$	1.93
	64	$1.8_{10} - 5$	*3.82	$3.0_{10} - 3$	1.98
	128	$1.3_{10} - 6$	*3.89	$7.5_{10} - 4$	2.00
8	8	$7.4_{10} - 2$		$2.5_{10} - 1$	
	16	$1.3_{10} - 2$	2.50	$1.0_{10} - 1$	*1.71
	32	$1.1_{10} - 3$	3.56	$1.5_{10} - 2$	*3.06
	64	$7.6_{10} - 5$	3.86	$1.6_{10} - 3$	*3.53
	128	$4.9_{10} - 6$	3.95	$1.5_{10} - 4$	*3.63

weighted norm with  $\beta = 1/2$  if  $q(\alpha + 1/2) = 2$  and for the sup-norm with  $\beta = 1$  if  $q\alpha = 2$ .

Table 1 gives all these numbers for the midpoints  $t_i^n$  and the weights  $\omega_j^n$  of the trapezoidal rule. It is really amazing how exactly the different orders show up already at relatively small  $n$ . Also, the modified  $\mathcal{O}_*^{CY}$  or  $\mathcal{O}_*^{CE}$  are quite close to the proven exponent  $\gamma = 2$ . Furthermore, the convergence results show that the  $\sqrt{n}$  does not appear in the weighted  $l_2$ -norm and that the  $\log n$  of the sup-norm can only be seen in the case  $q\alpha = 2$ . But then consistency seems not to be spoiled by another  $\log n$ —at least in our specially structured examples. Computations with the optimal points  $t_i^n$  yield nearly the same results. Also, the explicit inversion formulas behave quite similarly. In order to apply them, we need the optimal points. But one Newton step, starting with the midpoints, is sufficient to achieve good results. It can be seen in all cases that the absolute errors begin to increase if the grading is

stronger than necessary for an optimal order. Hence, overgrading is quite useless.

In Table 2 we want to illustrate that Simpson's rule, for example, really leads to an improvement and to order-4 results. Here the optimal points were approximated with three Newton steps and then we applied the explicit inversion formulas with Simpson's weights  $\omega_j^n$ . Notice that the midpoints are no longer optimal on a uniform mesh. The results with the \* seem to indicate some more logarithms. But that leads to another open problem.

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JOHANNES GUTENBERG-UNIVERSITÄT, FACHBEREICH MATHEMATIK, SAARSTRASSE  
21, D-55099 MAINZ, GERMANY  
*E-mail address:* MUELTHEI@mat.mathematik.Uni-Mainz.DE

JOHANNES GUTENBERG-UNIVERSITÄT, FACHBEREICH MATHEMATIK, SAARSTRASSE  
21, D-55099 MAINZ, GERMANY  
*E-mail address:* Schneider@mat.mathematik.Uni-Mainz.DE