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GALERKIN APPROXIMATION WITH QUADRATURE FOR THE SCREEN PROBLEM IN $\mathbf{R}^3$

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ABSTRACT. We study a Galerkin method with quadrature for the single-layer equation on a two-dimensional plate Γ which has the form of a union of rectangles. The trial space consists of piecewise constant functions on a partition of Γ into rectangles, which is assumed to be quasi-uniform. A semi-discrete scheme is obtained by approximating the $L^2(\Gamma)$ inner product in the definition of the Galerkin matrix elements by composite quadrature rules. More precisely, the integral over each rectangular element is replaced by a composite quadrature rule, obtained by subdividing the rectangle into M^2 congruent subrectangles of the same shape as the original, and applying a scaled version of a basic quadrature rule to each subrectangle. The basic quadrature rule is required to have only interior nodes; in this way possible singularities which can be present on the boundary of the rectangles of the partition are not encountered. High precision of the quadrature rules is not necessary. The stability of the semi-discrete scheme is proved, under appropriate conditions, if the subdivision of each rectangle of the partition is fine enough; more precisely, if $M \geq M_0$, with M_0 independent of the partition when the basic quadrature rule is exact for polynomials of degree 1. Error estimates are derived which show that the semi-discrete Galerkin approximations will converge at the same rate as the corresponding Galerkin approximations in some norms.

1. Introduction. This paper is concerned with a semi-discrete Galerkin method for solving the single-layer equation

$$Vu = f \tag{1}$$

for a plane plate. More specifically, the equation is

$$Vu(x) := \frac{1}{4\pi} \int_\Gamma \frac{u(y)}{|x-y|}\, dy, \quad x \in \Gamma, \tag{2}$$

where $\Gamma \subset \mathbf{R}^2$ is a bounded region which is the union of a finite number of rectangles, with all the rectangle edges either parallel or

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perpendicular to each other. The solution of (1) is closely related to the so-called Dirichlet screen problem for the Laplacian in $\mathbf{R}^3\backslash\overline{\Gamma}$ with boundary data f on Γ (see [**8, 21**]). The unknown function u has the physical meaning of the charge density on the plate. It is well known that V is a positive definite operator (see (4)), from which it follows that the solution of (1) is unique.

Our method is a semi-discrete Galerkin method with an approximating subspace S_h of piecewise constant functions on a quasi-uniform partition of Γ into rectangles, where the $L^2(\Gamma)$ inner products in the Galerkin matrix are replaced by quadrature formulas. If we use a 1-point quadrature formula on each rectangle, then our method is equivalent to a collocation method, for which there is as yet no general stability proof (but for the case of a uniform grid and bilinear trial elements see [**6**]). Our aim is to show that nevertheless the method becomes stable if the quadrature formula is sufficiently refined. (The collocation method is of historical interest: James Clerk Maxwell in 1879, see [**16**], used an *ad hoc* improvement of the piecewise-constant collocation method for the case $f \equiv 1$ to estimate the electrical capacitance of a conducting plate.[1])

A corresponding problem in the 2-dimensional case with Γ a closed Lipschitz curve has recently been studied by Sloan and Atkinson [**20**] (and extended to higher order piecewise polynomials by Ainsworth, Grigorieff and Sloan [**1**]), who proved error estimates for the semi-discrete solution $u_h \in S_h$ of the form

$$\|u - u_h\|_{H^t(\Gamma)} \le C\, h_{\max}^{s-t} \|u\|_{H^s(\Gamma)} \tag{3}$$

for s, t satisfying $-1 \le t \le s$, $t < 1/2$, if $u \in H^s(\Gamma)$ with $0 \le s \le 1$. Here $H^t(\Gamma)$ denotes the usual Sobolev space on the curve Γ. The bounds (3) were derived by a perturbation argument from the existence of stable Galerkin approximations $u_h^G \in S_h$. (For the stability of the Galerkin approximation see Costabel [**5**]). The perturbation argument is based on the assumption that the basic quadrature rule is exact for constant functions and has a sufficiently small Peano constant. The latter assumption is closely related to the rule having enough quadrature points.

In higher dimensions the application of quadrature rules to functions in the range $V(S_h)$ is somewhat delicate. Whereas the quadrature error was analyzed in $H^1(\Gamma)$ for the case of Γ a closed Lipschitz curve, this approach is impossible in the higher dimensional setting, because of the well known fact that in $H^1(\mathbf{R}^2)$ point evaluation is not a bounded linear functional. The approach we follow in the present work is to analyze the quadrature rule only in its application to the much more limited set $V(S_h)$.

Functions in $V(S_h)$, with S_h consisting of piecewise-constant functions, in general have singularities on boundaries of the rectangles in the partition of Γ. Singularities in the integrand have been dealt with in the literature in different ways. In the analogous lower-dimensional setting, where Γ is a smooth closed curve and S_h contains piecewise-constant functions on a partition into intervals, Hsiao, Kopp and Wendland [**13, 14**] treat the principal convolutional part of the operator exactly, and the quadrature is applied only to a smooth perturbation of the original kernel obtained by subtracting the principal part. Penzel [**17**], who studies a fully discrete approximation to the Galerkin method for the 3-dimensional screen problem for a square plate, performs first a coordinate transformation to those integrals which contain a singularity to obtain a smooth integrand before applying the (composite) Simpson rule. Wider in scope is the work of Hackbusch and Sauter [**12**] and Sauter [**18**], who handle the singular and nearly singular integrals in the Galerkin method by employing first four-dimensional rotations onto the 4-simplex, and then generalized Duffy transformations to remove singularities and map to the 4-cube. In combination with the so-called panel-clustering technique (see [**11, 10**]) efficient fully discrete schemes are developed based on a computational work of $O(N \log^\alpha N)$ operations for some $\alpha > 0$ for a matrix-vector multiplication with the discretized Galerkin matrix, where N denotes the number of trial functions.

In the present work, in contrast, the focus is entirely on semi-discrete methods, in which the "inner" integrals $V\chi_h$ for $\chi_h \in S_h$ are assumed to be evaluated exactly, whereas we apply quadrature rules to approximate the integral defining the $L^2(\Gamma)$ inner product. Apart from the simpler structure of the resulting discretization, another advantage in avoiding transformations as in [**17**] is that one can use explicit formulas for calculating Vv_h for $v_h \in S_h$. For the case of plane elements and piecewise constants the evaluation of such elements is easy. For more general plane elements analytical results of Maischak [**15**] are available.

Problems with possible singularities lying on the boundary of the rectangles are circumvented in the present paper by using quadrature rules with interior quadrature points only. The rules can differ from rectangle to rectangle. The "richness" of the quadrature rules which we need as a stabilizing device as in [**20, 1**] is obtained by using composite rules based on a further partition of the rectangles defining the global partition of Γ into M^2 congruent subrectangles.

The main results are given in Theorems 7.1 and 7.2. The results are in two parts: first, it is shown that if M is large enough (where M is the number of subdivisions on each element edge in forming the composite rule) then the method is stable; and second, that if the method is stable, then the error estimate

$$\|u - u_h\|_{\tilde{H}^t(\Gamma)} \leq C\, h_{\max}^{s-t} \|f\|_{H^{s+1}(\Gamma)}$$

for $-1 < t \leq s < 0$, $s \geq -1/2$, holds if $f \in H^{s+1}(\Gamma)$ and the partitions are quasi-uniform (for the definition of the function spaces and norms see Section 2). Compared with the result (3) for the planar case, the range of admissible s, t is restricted, reflecting the lower regularity of the solutions u of (1) compared with the case of a closed surface, and also the non-well-posedness of applying quadrature formulas to $H^1(\Gamma)$-functions in the case of a two-dimensional surface piece Γ. In the range of admissible s, t there occurs no degradation of the convergence rate compared with the Galerkin approximation.

Quadrature formulas satisfying the additional assumption of integrating polynomials of degree 1 exactly have, compared with the general case, an improved stabilizing property. For these formulas we prove (see Theorem 7.2) that a value of the multiplicity M^2 of the composite rule guaranteeing stability of the semi-discrete scheme can be chosen independently of the size of $h_{\max}$. In contrast, in the general case covered by Theorem 7.1 our estimates indicate that M might have to increase proportionally to $|\log h_{\max}|$. Note that the requirement of integrating polynomials of degree 1 exactly is easily satisfied: for example, it is satisfied by the basic 1-point rule, if that single point is positioned at the center of the rectangle. Equally, it is satisfied by any rule with inversion symmetry in the center, if the weights sum to 1.

A deficiency in the stability analysis is that we do not know, in either case, how large M needs to be in order to ensure stability. The reason is that the analysis contains constants of unknown size. In particular, it is for this reason that we do not know whether or not the simple mid-point collocation rule (corresponding to $M = 1$) is stable in all cases. What we do know is that if M is large enough then the method is stable; and if the underlying quadrature rule integrates polynomials of degree 1 exactly, then M can be chosen independently of h.

The scope of the present work is restricted to a piecewise constant approximating space on a rectangular mesh. In future work we hope to introduce greater flexibility by extending the method to the case of a triangular mesh.

The Galerkin method is described in Section 2 and our semi-discrete scheme is defined in Section 3. In Section 4 a general formula for the quadrature error is derived which is applied in Section 5 to functions in the range of V when acting on the trial space S_h. The error between the semi-discrete scheme and the Galerkin method is then estimated in Section 6. This is the basic tool used to derive the results in Section 7.

2. The Galerkin scheme. For convenience let us first recall the definition of the spaces $H^s(\Gamma)$ and $\tilde{H}^s(\Gamma)$ for $s \in \mathbf{R}$ (see [**17,19**]). The Hilbert space $H^s(\mathbf{R}^2)$ is the completion of $C_0^\infty(\mathbf{R}^2)$ with respect to the inner product

$$(f,g)_s := \int_{\mathbf{R}^2} \hat{f}(\xi)\,\hat{g}(\xi)(1+|\xi|^2)^s\,d\xi,$$

where '$\hat{\ }$' indicates the Fourier transform, i.e.

$$\hat{f}(\xi) := \int_{\mathbf{R}^2} e^{-i\xi x} f(x)\,dx, \quad \xi \in \mathbf{R}^2.$$

Then $\tilde{H}^s(\Gamma)$ is defined as the closure of the subspace $C_0^\infty(\Gamma)$ in $H^s(\mathbf{R}^2)$. The dual space to $\tilde{H}^s(\Gamma)$ is denoted by $H^{-s}(\Gamma)$, where the norms are defined in the usual way by duality:

$$\|f\|_{H^{-s}(\Gamma)} := \sup_{0\neq g\in C_0^\infty(\Gamma)} \frac{|(f,g)_0|}{\|g\|_{\tilde{H}^s(\Gamma)}}, \quad f \in C^\infty(\overline{\Gamma}),$$

(functions in $L^2(\Gamma)$ are extended by zero outside of Γ and can then be considered as elements in $L^2(\mathbf{R}^2)$). For $|s| < 1/2$ one has $\tilde{H}^s(\Gamma) = H^s(\Gamma)$ algebraically and topologically.

It is known (see [**9**]) that the Fourier transform of the regular distribution $S := 1/|x|$ is given by $\hat{S} = 2\pi/|\xi|$. Thus the operator V is a pseudo-differential operator of order -1, and hence maps $\tilde{H}^{-1/2}(\Gamma)$ into $H^{1/2}(\Gamma)$ continuously. It is also known that V is positive definite in $\tilde{H}^{-1/2}(\Gamma)$ (see [**8**]), in fact, the coercivity inequality

$$(4) \qquad (Vv, v) \geq \frac{1}{8\pi^2}\|v\|^2_{\tilde{H}^{-1/2}(\Gamma)}, \quad v \in \tilde{H}^{-1/2}(\Gamma),$$

holds. Here $(f,g) := (1/(4\pi^2))(f,g)_0$ is the usual $L^2(\Gamma)$ inner product for $f, g \in L^2(\Gamma)$. In the present case of a plate, the forementioned properties can be readily established by Fourier transformation. For example, equation (4) can be seen to hold as follows. If $\varphi \in C_0^\infty(\Gamma)$ then $V\varphi = (1/(4\pi))S * \varphi$, where $S * \varphi$ denotes the convolution of S with φ. Hence, (4) can be directly obtained by Fourier transformation:

$$\begin{aligned}(V\varphi, \varphi) &= \frac{1}{8\pi^2}\int_{\mathbf{R}^2} \frac{1}{|\xi|}|\hat{\varphi}|^2 \, d\xi \\ &\geq \frac{1}{8\pi^2}\int_{\mathbf{R}^2} \frac{1}{(1+|\xi|^2)^{1/2}}|\hat{\varphi}|^2 \, d\xi \, .\end{aligned}$$

It follows that

$$V : \tilde{H}^{-1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma)$$

is bijective and has a continuous inverse. More generally, it is known (see [**19**, Theorem 4.1]) that

$$(5) \qquad V : \tilde{H}^{s}(\Gamma) \longrightarrow H^{s+1}(\Gamma), \quad -1 < s < 0,$$

is continuous and bijective and has a continuous inverse.

We now describe the underlying Galerkin method for solving (1). Let

$$(6) \qquad \Gamma = \bigcup \Gamma_k$$

be a finite partition of Γ into pairwise disjoint rectangles Γ_k. Each $\overline{\Gamma}_k$ can be written in the form

$$(7) \qquad \overline{\Gamma}_k = \{x = (x^{(1)}, x^{(2)}) : x = x_k + h_k \circ y, y \in [0,1]^2\},$$

where x_k is the bottom left corner of Γ_k, $h_k = (h_k^{(1)}, h_k^{(2)})$ is the side length vector of Γ_k in the $x^{(1)}$ and $x^{(2)}$ directions, respectively, and the product $h_k \circ y$ denotes the componentwise product $(h_k^{(1)} y^{(1)}, h_k^{(2)} y^{(2)})$.

As trial space S_h we choose the space of piecewise constant functions on the partition (6). The spaces S_h are subspaces of $\tilde{H}^s(\Gamma)$ for $s < 1/2$. Let $f \in H^{1/2}(\Gamma)$ be given. The Galerkin approximation $u_h^G \in S_h$ is then defined as the solution of

$$(V u_h^G, \chi) = (f, \chi) \quad \forall \chi \in S_h. \tag{8}$$

It follows from (4) that there exists a unique solution u_h^G of (8) and the quasi-optimal error estimate

$$\|u - u_h^G\|_{\tilde{H}^{-1/2}(\Gamma)} \le C \inf_{v_h \in S_h} \|u - v_h\|_{\tilde{H}^{-1/2}(\Gamma)} \tag{9}$$

holds. For example, if the solution of (1) satisfies $u \in H^s(\Gamma)$ for some $s \in (-1/2, 1]$ then it follows from (9) with the aid of the known approximation power of S_h that

$$\|u - u_h^G\|_{\tilde{H}^{-1/2}(\Gamma)} \le C\, h_{\max}^{s+1/2} \|u\|_{H^s(\Gamma)}. \tag{10}$$

The corresponding estimate for the case $s = -1/2$ and $u \in \tilde{H}^{-1/2}(\Gamma)$ is obtained from (9) by taking $v_h = 0$ there. More generally, a duality argument based on (5) (see [**7**]) delivers the estimate

$$\|u - u_h^G\|_{\tilde{H}^t(\Gamma)} \le C\, h_{\max}^{s-t} \|u\|_{H^s(\Gamma)} \tag{11}$$

for $-1 < t \le s \le 1$, $s > -1/2$, $t < 1/2$, if $u \in H^s(\Gamma)$ and the partition is quasi-uniform. One can take $s = -1/2$ in (11) if $H^s(\Gamma)$ is replaced by $\tilde{H}^s(\Gamma)$. (For $-1 < t \le -1/2 \le s \le 1$ the assumption of quasi-uniformity is not needed.) According to (5) we can in general expect, even for smooth data f, only $u \in \tilde{H}^s(\Gamma)$ with $s < 0$, and consequently the error estimate

$$\|u - u_h^G\|_{\tilde{H}^t(\Gamma)} \le C\, h_{\max}^{s-t} \|f\|_{H^{s+1}(\Gamma)}, \tag{12}$$

when $f \in H^{s+1}(\Gamma)$ (and hence $u \in \tilde{H}^s(\Gamma)$) is obtained only for the range of indices $-1 < t \le s < 0$, $s \ge -1/2$.

We rewrite the Galerkin equations (8) as a projection method

$$P_h V u_h^G = P_h f, \tag{13}$$

where P_h is the orthogonal projection in $L^2(\Gamma)$ on S_h. By representing u_h^G in the form

$$u_h^G = \sum_l c_l^G \chi_l, \tag{14}$$

where χ_l denotes the characteristic function of Γ_l, equation (13) can be equivalently written as the linear system

$$\sum_l A_{kl}^G c_l^G = f_k \quad \forall\, k, \tag{15}$$

with

$$A_{kl}^G := (V\chi_l, \chi_k), \qquad f_k := (f, \chi_k) \quad \forall\, k, l. \tag{16}$$

3. The semi-discrete scheme. Our semi-discrete scheme is obtained from (15) by approximating the matrix elements A_{kl}^G by quadrature formulas. Let Q be a finite set of basic quadrature formulas with interior points. By this we mean that each $q \in Q$ is a quadrature formula of the form

$$qf = \sum_{j=1}^{J} w_j f(\xi_j) \sim \int_{[0,1]^2} f(x)\, dx, \tag{17}$$

where $\xi_j \in (0,1)^2$ are the quadrature points and $w_j \in \mathbf{R}$ the corresponding weights. The number $J \in \mathbf{N}$ of quadrature points, and also the weights and quadrature points, may vary for different $q \in Q$. We always assume that constants are integrated exactly, i.e.,

$$\sum_{j=1}^{J} w_j = 1. \tag{18}$$

A simple example is where Q consists of just one rule, e.g., the midpoint rule

$$qf = f(1/2, 1/2), \tag{19}$$

or any other product Gauss rule. Consider now a rectangle $\Gamma(z,h)$ with bottom left corner z and side length vector $h = (h^{(1)}, h^{(2)})$. For $q \in Q$ of the form (17) we then obtain by translation and scaling the quadrature rule

$$q_{\Gamma(z,h)}(f) = |\Gamma(z,h)| \sum_{j=1}^{J} w_j f(z + h \circ \xi_j) \sim \int_{\Gamma(z,h)} f(x)\, dx, \tag{20}$$

where $|\Gamma(z,h)| := h^{(1)} h^{(2)}$.

We are now prepared to define our approximation A_{kl} to A_{kl}^G. It is based on approximating the inner product integral in (16) by a composite quadrature rule. Let $M^2 \in \mathbf{N}$ denote the multiplicity of the composite rule. More precisely, each Γ_k is subdivided into M^2 congruent rectangles Γ_{km}, $m = 1, \dots, M^2$, with side length vector

$$\frac{1}{M}\, h_k. \tag{21}$$

Then, for any $k \leq l$ choose a rule $q \in Q$ and define

$$A_{kl} := \sum_{m=1}^{M^2} q_{\Gamma_{km}}(V\chi_l). \tag{22}$$

For $k > l$ the matrix elements are defined by symmetry (in this way computational work is saved, and at the same time the desirable property of a symmetric coefficient matrix is kept also in the semi-discrete case). Thus for $k > l$ we define

$$A_{kl} = A_{lk} = \sum_{m=1}^{M^2} q_{\Gamma_{lm}}(V\chi_k).$$

The coefficients c_l of the semi-discrete approximate solution

$$u_h = \sum_l c_l \chi_l \tag{23}$$

are then given as the solution (if it exists and is unique) of the linear system

$$\sum_l A_{kl} c_l = f_k \qquad \forall\, k, \tag{24}$$

which is an approximation of (15). These equations can also be equivalently written in an operator form corresponding to (13),

$$V_h u_h = P_h f, \tag{25}$$

where $V_h : S_h \to S_h$ is defined by

$$v_h = \sum_l c_l \chi_l \quad \longmapsto \quad V_h v_h = \sum_k \frac{1}{|\Gamma_k|} \Big(\sum_l A_{kl}\, c_l \Big) \chi_k. \tag{26}$$

The aim of the following sections is to prove, under suitable conditions, the existence of a unique solution u_h of (25) and related error estimates. As a preparation we begin with some nonstandard error estimates for quadrature rules.

4. A quadrature error estimate. In this section $\Gamma(z, h)$ denotes the rectangle with bottom left corner z and side length vector $(h^{(1)}, h^{(2)})$. For example, if $e := (1, 1)$ then $\Gamma(0, e)$ is the unit square. By $\mathring{\Gamma}(z, h)$ we denote the interior of $\Gamma(z, h)$.

Lemma 4.1. *Let $q \in Q$, and let $q_{\Gamma(z,h)}$ denote the corresponding quadrature rule (20). Then, for $f \in L^1(\Gamma(z, h)) \cap C^1(\mathring{\Gamma}(z, h))$ the following error estimate holds:*

$$\begin{aligned} &\left| \int_{\Gamma(z,h)} f(x)\, dx - q_{\Gamma(z,h)} f \right| \\ &\quad \le |\Gamma(z, h)| \sum_{j=1}^{J} |w_j| \int_0^1 \frac{1}{|\Gamma_{\xi_j,\tau}|} \int_{\Gamma_{\xi_j,\tau}} |h \circ Df(z + h \circ s)|_1 \, ds\, d\tau, \end{aligned} \tag{27}$$

where $\Gamma_{\xi_j,\tau} := \Gamma((1 - \tau)\xi_j, \tau e)$ and

$$|h \circ Df|_1 := h^{(1)} |\partial f/\partial x^{(1)}| + h^{(2)} |\partial f/\partial x^{(2)}|. \tag{28}$$

Remark. For $\tau \in (0,1)$ the square $\Gamma_{\xi_j,\tau}$ is contained within the unit square, and contains ξ_j. Since its edge length is τ, the square $\Gamma_{\xi_j,\tau}$ vanishes as $\tau \to 0$, and fills the unit square as $\tau \to 1$.

Proof. Let $E(f)$ denote the quadrature error. After transforming the integral to the unit square through the substitution $x = z + h \circ t$, and taking condition (18) into account, we obtain

$$(29)\quad |E(f)| \le |\Gamma(z,h)| \left| \sum_{j=1}^{J} w_j \int_{(0,1)^2} \Big(f(z+h\circ t) - f(z+h\circ \xi_j) \Big)\, dt \right|.$$

The mean-value theorem in two variables with integral remainder term gives, for $t \in (0,1)^2$,

$$\begin{aligned} &|f(z+h\circ t) - f(z+h\circ\xi_j)| \\ &\qquad = \left| \int_0^1 Df(z+\tau h\circ t + (1-\tau)h\circ\xi_j)\, d\tau [h\circ(t-\xi_j)] \right| \\ &\qquad \le \int_0^1 |h\circ Df(z+\tau h\circ t+(1-\tau)h\circ\xi_j)|_1\, d\tau, \end{aligned}$$

and the estimate (27) follows easily from (29) by replacing t by the new variable $s = \tau t + (1-\tau)\xi_j$, which implies $dt = \tau^{-2}\, ds = |\Gamma_{\xi_j,\tau}|^{-1}\, ds$. □

Corollary 4.2. *For* $f \in C^1(\overline{\Gamma}(z,h))$

(30)

$$\left| \int_{\Gamma(z,h)} f(x)\, dx - q_{\Gamma(z,h)} f \right| \le |\Gamma(z,h)|\, |h|_1 \sum_{j=1}^{J} |w_j| \sup_{x\in\Gamma(z,h)} |Df(x)|_\infty,$$

where $|h|_1 := h^{(1)} + h^{(2)}$ *and* $|Df|_\infty := \max(|\partial f/\partial x^{(1)}|, |\partial f/\partial x^{(2)}|)$.

Proof. The bound (30) follows immediately from (27). □

Lemma 4.3. *Let* $q \in Q$ *and let* $q_{\Gamma(z,h)}$ *be the corresponding quadrature rule* (20). *Assume that* q *integrates polynomials of degree* 1 *exactly. Then, for* $f \in C^2(\overline{\Gamma}(z,h))$,

$$(31)\quad \left| \int_{\Gamma(z,h)} f(x)\,dx - q_{\Gamma(z,h)} f \right| \leq \frac{1}{4} |\Gamma(z,h)|\,|h|_1^2 \sum_{j=1}^{J} |w_j| \sup_{x\in\Gamma(z,h)} |D^2 f(x)|_\infty,$$

where

$$|D^2 f|_\infty := \max\{|\partial^2 f/\partial x^{(i)} \partial x^{(j)}|, i,j = 1,2\}.$$

Proof. We start from the bound (29) and use the Taylor expansion of the integrand with respect to the center of $(0,1)^2$ up to second order terms. Since the first order terms are integrated exactly, we obtain (31) by suitably bounding the error term in the Taylor expansion. □

5. Error in approximating the Galerkin matrix elements. Our main aim in this section is to bound the error in approximating the matrix elements A_{kl}^G from (16) by A_{kl}. We first want to estimate

$$(32)\quad E_{kl} := \int_{\Gamma_k} V\chi_l\,dx - q_{\Gamma_k}(V\chi_l)$$

in the case where Γ_l is in an arbitrary position with respect to Γ_k. (Note that E_{kl} is for $k \leq l$ the error $A_{kl}^G - A_{kl}$ for the case in which $m = 1$.) We will impose a bound ρ for the nonuniformity of the partition, i.e.,

$$(33)\quad \frac{h_{\max}}{h_{\min}} \leq \rho,$$

where

$$h_{\max} := \max\{h_k^{(i)},\ i = 1,2,\ \forall\, k\}, \qquad h_{\min} := \min\{h_k^{(i)},\ i = 1,2,\ \forall\, k\}.$$

A sequence of partitions (6) is said to be quasi-uniform if (33) holds with some fixed ρ for all partitions in the sequence.

Lemma 5.1. *The following estimate holds for all* k, l*:*

$$(34)\quad |E_{kl}| \leq C_1 |\Gamma_k| |h_k|_1,$$

where

$$C_1 := 2\frac{\rho}{\pi}\sum_{j=1}^{J}|w_j|\Big[\min_{j=1,\dots,J}\min_{i=1,2}(\xi_j^{(i)}, 1-\xi_j^{(i)})\Big]^{-1}. \tag{35}$$

Proof. We want to apply the error estimate (27) to the integral in (32), and hence need the first order derivatives of

$$f(x) := 4\pi V\chi_l(x) = \int_{\Gamma_l}\frac{dy}{|x-y|}. \tag{36}$$

Assume first $x \notin \overline{\Gamma}_l$. Then

$$\begin{aligned}\frac{\partial f}{\partial x^{(1)}} &= \int_{\Gamma_l}\frac{\partial}{\partial x^{(1)}}\left(\frac{1}{|x-y|}\right)dy \\ &= -\int_{\Gamma_l}\frac{\partial}{\partial y^{(1)}}\left(\frac{1}{|x-y|}\right)dy \\ &= \int_{x_l^{(2)}}^{x_l^{(2)}+h_l^{(2)}}\left[\frac{1}{|x-x_l^{(1)}e_1-y^{(2)}e_2|}\right. \\ &\qquad\qquad \left. -\frac{1}{|x-(x_l^{(1)}+h_l^{(1)})e_1-y^{(2)}e_2|}\right]dy^{(2)},\end{aligned}$$

where $e_1 := (1,0)$, $e_2 := (0,1)$. The last expression is valid also for $x \in \mathring{\Gamma}_l$, as can be seen by calculating the derivative via the limit of the finite difference quotient. (After an appropriate shift of the integration variable $y^{(1)}$ in one of the resulting terms, the difference reduces to an easily estimated integral over narrow strips at the left and right ends of Γ_l.) A similar formula holds for $\partial f/\partial x^{(2)}$.

We are now going to apply (27) to the integral of f over Γ_k, with f given by (36). It is sufficient to consider the case $x_k = 0$. Assume for the moment that

$$x_l^{(1)} + h_l^{(1)} \le 0, \tag{37}$$

so that Γ_l lies to the left of Γ_k. For any $\tau \in (0,1)$, and any j with $1 \le j \le J$ such that $\xi_j \in (0, \frac{1}{2}]^2$, let

$$T_i := h_k^{(i)}\int_{\Gamma_\tau}\left|\left(\frac{\partial f}{\partial x^{(i)}}\right)(h_k\circ s)\right|ds, \quad i = 1,2,$$

where for brevity we write Γ_τ for $\Gamma_{\xi_j,\tau}$. Note that $h_k \circ \Gamma_\tau \subset \Gamma_k$. In estimating T_1 one has to deal, for example, with the integral

$$\int_{\Gamma_\tau} \left(\int_{x_l^{(2)}}^{x_l^{(2)}+h_l^{(2)}} \frac{dy^{(2)}}{|h_k \circ s - x_l^{(1)} e_1 - y^{(2)} e_2|} \right) ds.$$

This can be viewed as (up to scaling) the potential of a straight uniformly charged line segment with endpoints x_l and $x_l + h_l^{(2)} e_2$ in the Newtonian potential field of a uniformly charged plate $h_k \circ \Gamma_\tau \subset \Gamma_k$. It follows from classical potential theory that the potential will be increased by moving the wire closer to $h_k \circ \Gamma_\tau$ (specifically we imagine moving it to the line $x^{(1)} \equiv 0$ and then adjusting its vertical position so that it finishes in a symmetric position with respect to $h_k \circ \Gamma_\tau$). The potential will be further increased if one takes the length of the line segment to be $\rho h_k^{(1)}$, which cannot be surpassed as a consequence of (33). A similar argument applies to the other term in $\partial f / \partial x^{(1)}$. Thus we have proved

$$T_1 \le 2|h_k|_1 \int_{\Gamma_\tau} \left(\int_I \frac{1}{|h_k \circ s - \sigma e_2|} d\sigma \right) ds,$$

where

$$I := [h_k^{(2)} \tau_m^{(2)} - \tfrac{1}{2} \rho h_k^{(1)}, h_k^{(2)} \tau_m^{(2)} + \tfrac{1}{2} \rho h_k^{(1)}]$$

and $\tau_m^{(i)}$ denotes the midpoint of the interval $I_\tau^{(i)}$, with $I_\tau^{(i)}$ being the projection of Γ_τ on the $x^{(i)}$-axis.

Since $\xi_j \in (0, 1/2]^2$, the same bound trivially holds also in the case that Γ_l lies to the right of Γ_k, i.e.,

$$x_l^{(1)} \ge h_k^{(1)}. \tag{38}$$

For s fixed it is easy to see that

$$\int_I \frac{1}{|h_k \circ s - \sigma e_2|} \, d\sigma \le \int_{-\rho h_k^{(1)}/2}^{\rho h_k^{(1)}/2} \frac{1}{|h_k^{(1)} s^{(1)} e_1 - \sigma e_2|} \, d\sigma,$$

and the last integral is independent of $s^{(2)}$. Consequently, after performing the trivial integration with respect to $s^{(2)}$ we find

$$\begin{aligned} T_1 &\le 2|h_k|_1 \, \tau \int_{-\rho h_k^{(1)}/2}^{\rho h_k^{(1)}/2} \int_{I_\tau^{(1)}} \frac{1}{|h_k^{(1)} s^{(1)} e_1 - \sigma e_2|} \, ds^{(1)} \, d\sigma \\ &= 2|h_k|_1 \, \tau \int_{R_\tau} \frac{d\nu \, d\mu}{\sqrt{\nu^2 + \mu^2}}, \end{aligned}$$

where $R_\tau := \{(\nu,\mu) : \nu \in I_\tau^{(1)},\ |\mu| \le \rho/2\}$. In the Appendix we show that the last integral is bounded by $2\rho\tau/\xi_j^{(1)}$. Thus

$$T_1 \le 4|h_k|_1 \frac{\rho}{\xi_j^{(1)}}\tau^2 \le 4\rho|h_k|_1\tau^2 \left[\min_{i=1,2}(\xi_j^{(i)},\ 1-\xi_j^{(i)})\right]^{-1}.$$

With similar reasoning the same bound as for T_1 holds for T_2. If (37) or (38) does not hold but $k \neq l$ then one has to interchange the role of the variables $x^{(1)}$ and $x^{(2)}$ in the argument given above to see that T_i, $i = 1, 2$, satisfies the same bound. The bound holds also for the three other possible locations in $(0,1)^2$ of the quadrature point ξ_j. Now (27) gives, with $T := T_1 + T_2$,

$$\begin{aligned} |E_{kl}| &\le \frac{1}{4\pi}|\Gamma_k| \sum_{j=1}^{J} |w_j| \int_0^1 \frac{T}{\tau^2}\, d\tau \\ &\le 2\frac{\rho}{\pi}|\Gamma_k|\ |h_k|_1 \sum_{j=1}^{J} |w_j| \left[\min_{i=1,2}(\xi_j^{(i)},\ 1-\xi_j^{(i)})\right]^{-1}, \end{aligned}$$

so that (34) is proved for $k \neq l$. Finally, the bound (34) can be seen to hold also for $k = l$, with no essential change in the argument. □

We now give an alternative error bound to (34) for the case that Γ_k and Γ_l have a positive distance. Then $V\chi_l \in C^2(\overline{\Gamma}_k)$, and we can apply (30) or (31), respectively, to obtain the following estimates.

Lemma 5.2. *Assume* $\overline{\Gamma}_k \cap \overline{\Gamma}_l = \varnothing$. *Then*

$$|E_{kl}| \le C_2\,|\Gamma_k|\,|h_k|_1 \max_{x\in\overline{\Gamma}_k} \int_{\Gamma_l} \frac{dy}{|x-y|^2}, \tag{39}$$

where

$$C_2 := \frac{1}{4\,\pi}\sum_{j=1}^{J} |w_j|. \tag{40}$$

Lemma 5.3. *Assume* $\overline{\Gamma}_k \cap \overline{\Gamma}_l = \varnothing$ *and that the quadrature rules* $q \in Q$ *integrate polynomials of degree* 1 *exactly. Then*

$$|E_{kl}| \leq C_2\, |\Gamma_k|\, |h_k|_1^2 \max_{x \in \overline{\Gamma}_k} \int_{\Gamma_l} \frac{dy}{|x-y|^3}. \tag{41}$$

6. A bound for $P_h V - V_h$. We are now in the position to prove an estimate for the error in approximating the Galerkin operator $P_h V$ by the semi-discrete operator V_h.

Lemma 6.1. *For* $v_h \in S_h$

$$\|(P_h V - V_h) v_h\|_0 \leq \max_k \frac{1}{|\Gamma_k|} \sum_l |A_{kl}^G - A_{kl}| \|v_h\|_0. \tag{42}$$

Proof. For

$$v_h = \sum_l c_l\, \chi_l \in S_h$$

we obtain with (26) and the explicit representation of the orthogonal projection P_h

$$(P_h V - V_h) v_h = \sum_k \frac{1}{|\Gamma_k|} \Big[\sum_l (A_{kl}^G - A_{kl}) c_l \Big] \chi_k.$$

We derive, with the aid of $(\chi_k, \chi_j)_0 = 4\,\pi^2 (\chi_k, \chi_j) = 4\,\pi^2\, |\Gamma_k|\, \delta_{kj}$ and the Cauchy-Schwarz inequality,

$$\begin{aligned}
\|(P_h V - V_h) v_h\|_0^2 &\leq 4\pi^2 \sum_k \frac{1}{|\Gamma_k|} \Big(\sum_l |A_{kl}^G - A_{kl}|^{1/2} |A_{kl}^G - A_{kl}|^{1/2} |c_l| \Big)^2 \\
&\leq 4\pi^2 \sum_k \frac{1}{|\Gamma_k|} \sum_l |A_{kl}^G - A_{kl}| \sum_{l'} |A_{kl'}^G - A_{kl'}| |c_{l'}|^2 \\
&\leq 4\pi^2 \Big(\max_k \frac{1}{|\Gamma_k|} \sum_l |A_{kl}^G - A_{kl}| \Big) \\
&\quad \cdot \sum_{k,l} \frac{1}{|\Gamma_l|} |A_{kl}^G - A_{kl}| |\Gamma_l| |c_l|^2 \\
&\leq \Big(\max_k \frac{1}{|\Gamma_k|} \sum_l |A_{kl}^G - A_{kl}| \Big)^2 \|v_h\|_0^2,
\end{aligned}$$

where in the last step we use the symmetry of A^G and A. □

Theorem 6.2. *Assume that the sequence of partitions* (6) *is quasi-uniform, and that M^2-fold composite quadrature rules of the form* (22) *are used. Then*

$$\|(P_hV - V_h)v_h\|_0 \le C\frac{h_{\max}}{M}\left(\left|\ln\frac{h_{\max}}{M}\right| + 1\right)\|v_h\|_0, \quad v_h \in S_h. \tag{43}$$

Proof. We recall the subdivision of Γ_k into the union of congruent rectangles Γ_{km}, $m = 1, \ldots, M^2$. Correspondingly, we can write

$$\chi_k = \sum_{m=1}^{M^2} \chi_{km}, \tag{44}$$

where χ_{km} denotes the characteristic function of Γ_{km}. We then derive from (42) and the definitions of A^G_{kl} and A_{kl} the estimate

$$\begin{aligned} &\|(P_hV - V_h)v_h\|_0 \\ &\le \max_k \frac{1}{|\Gamma_k|}\sum_l \sum_{m,n=1}^{M^2}\left(\left|\int_{\Gamma_{km}} V\chi_{ln}\,dx - q_{\Gamma_{km}}(V\chi_{ln})\right|\right. \\ &\qquad \left. + \left|\int_{\Gamma_{ln}} V\chi_{km}\,dx \ - q_{\Gamma_{ln}}(V\chi_{km})\right|\right)\|v_h\|_0. \end{aligned} \tag{45}$$

We now want to apply the error estimates (34) and (39). The partition

$$\Gamma = \bigcup_{k,m} \Gamma_{km} \tag{46}$$

is also quasi-uniform with the same nonuniformity bound ρ as in (33). We introduce the notion that Γ_{ln} is close to Γ_{km} if

$$\operatorname{dist}(\Gamma_{ln}, \Gamma_{km}) < \frac{h_{\min}}{2M}. \tag{47}$$

We fix k and m and then split

$$\sum_l \sum_{n=1}^{M^2} \cdot = \sum{}' \cdot + \sum{}'' \cdot, \tag{48}$$

where $\sum'$ contains exactly those terms for which Γ_{ln} is close to Γ_{km}. It can be seen from condition (47) that the close rectangles Γ_{ln} form on each side of Γ_{km} a single layer of not more than $\rho+2$ rectangles, and therefore altogether (including the central rectangle) there are not more than $4\rho+9$ close rectangles. Then it follows from (34), where we take Γ_{km} and Γ_{ln} in place of Γ_k and Γ_l respectively, that

$$\sum{}' \left| \int_{\Gamma_{km}} V\chi_{ln}\, dx - q_{\Gamma_{km}}(V\chi_{ln}) \right| \le (4\rho+9)C_1 \frac{|h_k|_1}{M} |\Gamma_{km}|. \tag{49}$$

For bounding the second sum in (48), i.e., the sum over the terms for which Γ_{ln} is not close to Γ_{km}, we use (39), and derive

$$\begin{aligned} \sum{}'' \left| \int_{\Gamma_{km}} V\chi_{ln}\, dx - q_{\Gamma_{km}}(V\chi_{ln}) \right| \\ \le C_2 \frac{|h_k|_1}{M} |\Gamma_{km}| \sum{}'' \max_{x\in\overline{\Gamma}_{km}} \int_{\Gamma_{ln}} \frac{dy}{|x-y|^2}. \end{aligned} \tag{50}$$

For further discussion of the last integral we assume without loss of generality that Γ_{km} is centered at the origin. With this choice we claim that

$$4\rho|y-x| \ge |y|, \qquad x\in\Gamma_{km}, \quad y\in\Gamma_{ln}, \tag{51}$$

when Γ_{ln} is not close to Γ_{km}. In fact, $|x| \le h_{\max}/(\sqrt{2}M) \le h_{\max}/M$ for $x\in\Gamma_{km}$ and, consequently, if $|y| > 2h_{\max}/M$ then $|y| \ge 2|x|$ and hence $2|y-x| \ge |y|$. Otherwise, taking into account that (47) does not hold,

$$|y-x| \ge \frac{h_{\min}}{2M} \ge \frac{h_{\max}}{2\rho M} \ge \frac{|y|}{4\rho} \quad \text{if } |y| \le \frac{2h_{\max}}{M}.$$

Now it follows with the aid of (51) that

$$\begin{aligned} \sum{}'' \max_{x\in\overline{\Gamma}_{km}} \int_{\Gamma_{ln}} \frac{dy}{|x-y|^2} &\le 16\,\rho^2 \int_{R_h} \frac{dy}{|y|^2} \\ &\le 32\pi\rho^2 \ln\left(\frac{\rho M}{h_{\max}}\ \operatorname{diam}\Gamma\right), \end{aligned} \tag{52}$$

where

$$R_h := \left\{ y : \frac{h_{\min}}{M} \le |y| \le \operatorname{diam}\Gamma \right\}.$$

We use (52) in (50) and then combine with (49) to obtain an estimate for the corresponding sum (48). After summing with respect to m we arrive at

$$\sum_l \sum_{m,n=1}^{M^2} \left| \int_{\Gamma_{km}} V\chi_{ln}\, dx - q_{\Gamma_{km}}(V\chi_{ln}) \right| \leq C|\Gamma_k| \frac{h_{\max}}{M}\left(\left| \ln \frac{h_{\max}}{M} \right| + 1 \right). \tag{53}$$

The remaining part of (45) is bounded in a similar way. We split the sum as in (48). Corresponding to (49) we obtain for k and m fixed

$$\begin{aligned} \sum{}' \left| \int_{\Gamma_{ln}} V\chi_{km}\, dx - q_{\Gamma_{ln}}(V\chi_{km}) \right| &\leq C_1 \sum{}' \left(\frac{|h_l|_1}{M} |\Gamma_{ln}| \right) \\ &\leq (4\rho + 9) C_1 \frac{2\, h_{\max}}{M} \rho^2 |\Gamma_{km}|, \end{aligned} \tag{54}$$

where in the last inequality we used $|\Gamma_{ln}| \leq \rho^2 |\Gamma_{km}|$ for all l and n. In a similar way to (50) and (52) we obtain, again under the assumption that Γ_{km} is centered at the origin, and taking (51) with x and y interchanged into account in the second inequality (but noting that x and y have now exchanged meanings),

$$\begin{aligned} \sum{}'' \left| \int_{\Gamma_{ln}} V\chi_{km}\, dx - q_{\Gamma_{ln}}(V\chi_{km}) \right| \\ &\leq C_2 \sum{}'' \left(\frac{|h_l|_1}{M} |\Gamma_{ln}| \max_{x \in \overline{\Gamma}_{ln}} \int_{\Gamma_{km}} \frac{dy}{|x-y|^2} \right) \\ &\leq C_2 \frac{2h_{\max}}{M} |\Gamma_{km}| \sum{}'' \left(|\Gamma_{ln}| \max_{x \in \overline{\Gamma}_{ln}} \frac{16\,\rho^2}{|x|^2} \right) \\ &\leq C \frac{h_{\max}}{M} |\Gamma_{km}| \sum{}'' \int_{\Gamma_{ln}} \frac{dy}{|y|^2} \\ &\leq C \frac{h_{\max}}{M} |\Gamma_{km}| \left(\left| \ln \frac{h_{\max}}{M} \right| + 1 \right). \end{aligned} \tag{55}$$

In the third inequality we used

$$\begin{aligned} \frac{|y|}{|x|} &\leq 1 + \frac{|y-x|}{|x|} \leq 1 + \sqrt{2}\, \frac{h_{\max}}{M} \frac{1}{|x|} \\ &\leq 1 + 2\sqrt{2}\,\rho, \quad x, y \in \Gamma_{ln}, \end{aligned}$$

where the last step comes from $|x| \geq h_{\min}/2M$, because $x \notin \Gamma_{km}$. Altogether, after summing (54) and (55) with respect to m, the estimate

$$(56)\quad \sum_l \sum_{m,n=1}^{M^2} \left| \int_{\Gamma_{ln}} V\chi_{km}\,dx - q_{\Gamma_{ln}}(V\chi_{km}) \right| \leq C|\Gamma_k| \frac{h_{\max}}{M}\left(\left| \ln \frac{h_{\max}}{M} \right| + 1 \right)$$

is established, which together with (53) proves the assertion. □

Theorem 6.3. *In addition to the assumptions of Theorem* 6.2 *let the quadrature rules* $q \in Q$ *be exact for polynomials of degree* 1. *Then*

$$(57)\qquad \|(P_h V - V_h)v_h\|_0 \leq C\frac{h_{\max}}{M}\|v_h\|_0, \quad v_h \in S_h.$$

Proof. The only change compared to the proof of Theorem 6.2 is to use the estimate (41) in place of (39). The logarithmic factor does not occur in the analogue of (52). □

Remark 6.4. A close inspection of the proof shows that Theorems 6.2 and 6.3 already hold under the assumption that the sequence of partitions is merely locally quasi-uniform, i.e., that there exists $\rho > 0$ such that

$$h_k^{(i)}/h_l^{(j)} \leq \rho, \quad i,j = 1,2, \quad \text{if } \overline{\Gamma}_k \cap \overline{\Gamma}_l \neq \varnothing.$$

7. The main result. Our results on the existence of a solution $u_h \in S_h$ of the semi-discrete method (25) and corresponding error bounds are based on the estimates proved in Section 6 and the quasi-optimality of the Galerkin method.

Theorem 7.1. *Consider a quasi-uniform sequence of partitions of the form* (6). *Then there exists a constant* $\kappa > 0$ *with the following* *property. If the multiplicity M^2 of the composite quadrature rules is large enough to satisfy*

$$\frac{1}{M}\left(\left|\ln\frac{h_{\max}}{M}\right|+1\right)\leq\kappa, \tag{58}$$

then for each $f \in L^2(\Gamma)$ there exists a unique solution $u_h \in S_h$ of (25)*. If $-1 < t \leq s < 0$, $s \geq -1/2$, and $f \in H^{s+1}(\Gamma)$ in* (1)*, then $u \in \tilde{H}^s(\Gamma)$ and*

$$\|u-u_h\|_{\tilde{H}^t(\Gamma)} \leq Ch_{\max}^{s-t}\|u\|_{\tilde{H}^s(\Gamma)} \leq Ch_{\max}^{s-t}\|f\|_{H^{s+1}(\Gamma)}. \tag{59}$$

If $u \in \tilde{H}^0(\Gamma) = L^2(\Gamma)$ and $-1 < t \leq 0$, then

$$\|u-u_h\|_{\tilde{H}^t(\Gamma)} \leq Ch_{\max}^{-t}\|u\|_0. \tag{60}$$

A stronger result holds if the quadrature rules are exact for polynomials of degree 1.

Theorem 7.2. *Consider a quasi-uniform sequence of partitions of the form* (6)*. Assume that the quadrature rules $q \in Q$ are exact for polynomials of degree* 1*. Then there exists an integer $M_0 > 0$, with M_0 independent of $h_{\max}$, such that for all $M \geq M_0$ and for each $f \in L^2(\Gamma)$ there exists a unique solution $u_h \in S_h$ of* (25)*, which satisfies the same error bounds as in Theorem* 7.1.

The proof of Theorems 7.1 and 7.2 will be given after some preparatory lemmas.

Lemma 7.3. *Assume the sequence of partitions to be quasi-uniform. Let $-1/2 \leq s \leq 0$. If the solution u of* (1) *lies in $\tilde{H}^s(\Gamma)$, then the Galerkin approximation u_h^G can be bounded by*

$$\|u_h^G\|_{\tilde{H}^s(\Gamma)} \leq C\|u\|_{\tilde{H}^s(\Gamma)}. \tag{61}$$

If we additionally assume $-1/2 \leq s < 0$ and $f \in H^{s+1}(\Gamma)$ in (1)*, then*

$$\|u_h^G\|_{\tilde{H}^s(\Gamma)} \leq C\|f\|_{H^{s+1}(\Gamma)}. \tag{62}$$

Proof. Choose an optimal simultaneous approximation $v_h \in S_h$ (see [**3, 4**]) such that

$$\|u - v_h\|_{\tilde{H}^{-1/2}(\Gamma)} \leq Ch_{\max}^{s+1/2}\|u\|_{\tilde{H}^s(\Gamma)}$$
$$\|v_h\|_{\tilde{H}^s(\Gamma)} \leq C\|u\|_{\tilde{H}^s(\Gamma)}.$$

Then, with the aid of the inverse inequality (see [**2**]),

$$\|v_h\|_{\tilde{H}^s(\Gamma)} \leq Ch_{\max}^{t-s}\|v_h\|_{\tilde{H}^t(\Gamma)}, \quad v_h \in S_h, \ -1 \leq t \leq s \leq 0, \tag{63}$$

and the quasi-optimality of u_h^G in $\tilde{H}^{-1/2}(\Gamma)$, it follows that, for $-1/2 \leq s \leq 0$,

$$\begin{aligned}\|u_h^G\|_{\tilde{H}^s(\Gamma)} &\leq \|u_h^G - v_h\|_{\tilde{H}^s(\Gamma)} + \|v_h\|_{\tilde{H}^s(\Gamma)}\\ &\leq Ch_{\max}^{-s-1/2}\|u_h^G - v_h\|_{\tilde{H}^{-1/2}(\Gamma)} + \|v_h\|_{\tilde{H}^s(\Gamma)}\\ &\leq C\|u\|_{\tilde{H}^s(\Gamma)}.\end{aligned}$$

For $-1/2 \leq s < 0$ we can use (5) to obtain

$$\|v\|_{\tilde{H}^s(\Gamma)} \leq C\|Vv\|_{H^{s+1}(\Gamma)}, \qquad v \in \tilde{H}^s(\Gamma). \tag{64}$$

Hence, with $Vu = f$, (62) is an immediate consequence of (61). □

The next two lemmas establish a useful stability property in the finite dimensional space S_h, first for the Galerkin operator P_hV, and then for the corresponding operator V_h in our semi-discrete method.

Lemma 7.4. *Assume the sequence of partitions to be quasi-uniform. For* $-1/2 \leq s < 0$ *the following estimate holds*:

$$\|v_h\|_{\tilde{H}^{-s-1}(\Gamma)} \leq Ch_{\max}^s\|P_hVv_h\|_0, \quad v_h \in S_h. \tag{65}$$

Proof. Let $v_h \in S_h$ be given. Then, for all $f \in H^{s+1}(\Gamma)$, since V and P_h are symmetric in $L^2(\Gamma)$, with the aid of (13) and the inverse inequality we obtain

$$\begin{aligned}|(v_h, f)| &= |(P_hVv_h, u_h^G)|\\ &\leq C\|P_hVv_h\|_0\, h_{\max}^s\|u_h^G\|_{\tilde{H}^s(\Gamma)}\\ &\leq Ch_{\max}^s\,\|P_hVv_h\|_0\|f\|_{H^{s+1}(\Gamma)},\end{aligned}$$

where (62) has been used. This proves (65). □

Lemma 7.5. *Let the assumptions of Theorems* 7.1 *and* 7.2 *hold with* κ *small enough or* M_0 *large enough. Then, for* $-1/2 \le s < 0$

$$\|v_h\|_{\tilde{H}^{-s-1}(\Gamma)} \le C h_{\max}^s \|V_h v_h\|_0, \quad v_h \in S_h. \tag{66}$$

Proof. With the aid of (43) and (57), respectively, we obtain from (65)

$$\begin{aligned}\|v_h\|_{\tilde{H}^{-s-1}(\Gamma)} &\le C h_{\max}^s (\|(P_h V - V_h) v_h\|_0 + \|V_h v_h\|_0) \\ &\le C h_{\max}^s (C_h \|v_h\|_0 + \|V_h v_h\|_0), \\ &\le C (h_{\max}^{-1} C_h \|v_h\|_{\tilde{H}^{-s-1}(\Gamma)} + h_{\max}^s \|V_h v_h\|_0),\end{aligned}$$

where

$$C_h := \frac{h_{\max}}{M}\left(\left|\ln \frac{h_{\max}}{M}\right| + 1\right) \quad \text{or} \quad C_h := \frac{h_{\max}}{M} \tag{67}$$

respectively, and we have again invoked the inverse inequality. Under the assumptions of Theorems 7.1 and 7.2 we see that $C_h \le \kappa h_{\max}$ or $C_h \le h_{\max}/M_0$, respectively. Since κ can be chosen small enough or M_0 chosen large enough, the result follows. □

Proof of Theorem 7.1. Since V_h is a map in the finite dimensional space S_h, the existence and uniqueness of u_h for κ sufficiently small both follow from (66). Restrict t to lie in $-1 < t \le -1/2$. We then apply (66), where we take $s = -t-1$ and $v_h = u_h - u_h^G$, to obtain with the aid of (25), (13), Theorem 6.2 and the inverse inequality (63)

$$\begin{aligned}\|u_h - u_h^G\|_{\tilde{H}^t(\Gamma)} &\le C h_{\max}^{-t-1} \|V_h(u_h - u_h^G)\|_0 \\ &= C h_{\max}^{-t-1} \|(P_h V - V_h) u_h^G\|_0 \\ &\le C\kappa h_{\max}^{-t} \|u_h^G\|_0 \\ &\le C\kappa h_{\max}^{s-t} \|u_h^G\|_{\tilde{H}^s(\Gamma)}\end{aligned} \tag{68}$$

if $-1/2 \le s \le 0$. The estimate (68) also holds for $-1/2 < t \le s$, since we can take the result for $t = -1/2$, which we have already proved, and then apply the inverse inequality. The assertions now follow from (68) by taking (62) or (61) into account together with the known accuracy (11) of u_h^G. □

The proof of Theorem 7.2 is similar, but uses Theorem 6.3.

Appendix

Here we obtain an upper bound on a two-dimensional integral arising in Section 5. In less specific notation the integral may be written as

$$\int_R \frac{dx\ dy}{\sqrt{x^2+y^2}},$$

where R is the rectangle

$$R = \{(x,y) : c \le x \le d,\ |y| \le \rho/2\},\ \text{and } c > 0.$$

Letting m and a be defined by

$$m := (c+d)/2, \qquad a := \rho(\pi m)^{-1},$$

we introduce elliptical coordinates r, ϕ,

$$x = r\cos\phi, \qquad y = ar\sin\phi, \qquad r \ge 0, \quad -\pi < \phi \le \pi,$$

and define an elliptical strip

$$S := \{(r\cos\phi,\ ar\sin\phi) : c \le r \le d,\ |\phi| \le \pi/2\}.$$

The elliptical strip and rectangle have the same area, since

$$\begin{aligned} |S| &= \int_S dx\ dy = \int_S ar\,dr\,d\phi \\ &= a\pi \int_c^d r\,dr = a\pi m(d-c) = |R|. \end{aligned}$$

It is clear that

$$(x^2+y^2)^{1/2} \geq \min(1,a^{-1})(a^2x^2+y^2)^{1/2},$$

so that

$$\begin{aligned}\int_R \frac{dx\ dy}{\sqrt{x^2+y^2}} &\leq \max(1,a)\int_R \frac{dx\ dy}{\sqrt{a^2x^2+y^2}}\\ &= \max(1,a)\bigg(\int_{R\cap S}\frac{dx\ dy}{\sqrt{a^2x^2+y^2}}+\int_{R\setminus S}\frac{dx\ dy}{\sqrt{a^2x^2+y^2}}\bigg).\end{aligned}$$

Now with the aid of $(a^2x^2+y^2)^{1/2} = ar$ we have

$$\inf_{R\setminus S}(a^2x^2+y^2)^{1/2} = ad = \sup_{S\setminus R}(a^2x^2+y^2)^{1/2},$$

thus with $|S| = |R|$ it follows that

$$\begin{aligned}\int_R \frac{dx\ dy}{\sqrt{x^2+y^2}} &\leq \max(1,a)\bigg(\int_{R\cap S}\frac{dx\ dy}{\sqrt{a^2x^2+y^2}}+\int_{S\setminus R}\frac{dx\ dy}{\sqrt{a^2x^2+y^2}}\bigg)\\ &= \max(1,a)\int_S \frac{dx\ dy}{\sqrt{a^2x^2+y^2}}\\ &= \max(1,a)\int_S \frac{ar\ dr\ d\phi}{ar}\\ &= \pi\max(1,a)(d-c),\end{aligned}$$

which is the desired result.

In the application in Section 5 we have

$$c = (1-\tau)\xi_j^{(1)} \quad \text{and} \quad d = (1-\tau)\xi_j^{(1)}+\tau,$$

so that

$$d-c=\tau, \qquad m = (1-\tau)\xi_j^{(1)}+\tau/2,$$

from which it follows (on recalling $\xi_j^{(1)} \in (0,1/2)$) that $\xi_j^{(1)} \leq m \leq 1/2$, and hence, since $\rho \geq 1$,

$$a = \frac{\rho}{\pi m} \geq \frac{2}{\pi} \geq \frac{1}{2}.$$

Thus
$$\int_{R_\tau} \frac{d\nu\ d\mu}{\sqrt{\nu^2+\mu^2}} \le \pi 2a\tau = \frac{2\rho\tau}{m} \le \frac{2\rho\tau}{\xi_j^{(1)}}.$$

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ENDNOTES

1. "I am not aware of any method by which the capacity of a square can be found exactly. I have therefore endeavored to find an approximate value by dividing the square into 36 equal squares and calculating the charge of each so as to make the potential at the middle of the square equal to unity."

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