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SINGULAR PERTURBATIONS IN A NONLINEAR VISCOELASTICITY

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ABSTRACT. A nonlinear equation in viscoelasticity of the form

$$\rho u^\rho_{tt}(t,x) = \phi(u^\rho_x(t,x))_x + \int_{-\infty}^t F(t-s)\phi(u^\rho_x(s,x))_x\,ds + \rho g(t,x) + f(x), \qquad t \geq 0, \quad x \in [0,1], \tag{0.1}$$

$$u^\rho(t,0) = u^\rho(t,1) = 0, \qquad t \geq 0, \tag{0.2}$$

$$u^\rho(s,x) = v^\rho(s,x), \qquad s \leq 0, \quad x \in [0,1], \tag{0.3}$$

(where ϕ is nonlinear) is studied when the density ρ of the material goes to zero. It will be shown that when $\rho \downarrow 0$, solutions u^ρ of the dynamical system (0.1)–(0.3) approach the unique solution w (which is independent of t) of the steady state obtained from (0.1)–(0.3) with $\rho = 0$. Moreover, the rate of convergence in ρ is obtained to be $\|u^\rho - w\|_{L^2} \leq K\sqrt{\rho}$ and $\|u^\rho_x - w_x\|_{L^2} \leq K\sqrt{\rho}$ for some constant K independent of ρ.

1. Introduction.

Let us begin with the following quasi-static approximation studied in MacCamy [**11**],

$$u_{tt}(t) = -A(0)g(u(t)) - \int_0^t A'(t-s)g(u(s))\,ds + F(t), \tag{1.1}$$

and

$$0 = -A(0)g(w(t)) - \int_0^t A'(t-s)g(w(s))\,ds + F(t). \tag{1.2}$$

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Here $A(t)$ is a bounded and linear operator and g is a nonlinear and unbounded operator in a Hilbert space. It is shown in [**11**] that if $F(t)$ approaches a constant vector $F(\infty)$ as $t \to \infty$, then, under appropriate conditions, one has

$$g(u(t)) \longrightarrow A(\infty)^{-1}F(\infty) \quad \text{weakly in } H, \quad \text{as } t \longrightarrow \infty, \tag{1.3}$$

$$g(w(t)) \longrightarrow A(\infty)^{-1}F(\infty) \quad \text{in } H, \quad \text{as } t \longrightarrow \infty, \tag{1.4}$$

where u and w are solutions of (1.1) and (1.2), respectively. This result motivates the procedure of using the quasi-static approximation in viscoelasticity, which drops the "acceleration" term u_{tt} when t is large. That is, use w to approximate u.

Now, let us look at the following nonlinear equation in viscoelasticity,

$$\begin{aligned} \rho u^\rho_{tt}(t,x) = \phi(u^\rho_x(t,x))_x & \\ + \int_{-\infty}^{t} F(t-s)\phi(u^\rho_x(s,x))_x \, ds & \\ + \rho g(t,x) + f(x), & \qquad t \geq 0, \quad x \in [0,1], \end{aligned} \tag{1.5}$$

$$\begin{aligned} u^\rho(t,0) = u^\rho(t,1) = 0, & \qquad t \geq 0 \\ u^\rho(s,x) = v^\rho(s,x), & \qquad s \leq 0, \quad x \in [0,1], \end{aligned} \tag{1.6}$$

which can be found in, e.g., Dafermos and Nohel [**2**] and MacCamy [**13**]. Here u is the displacement, ρg is the body force, f is the external force, and ρ is the density of the material. Same as in MacCamy [**13**], we assume that ϕ on $\Re$ is nonlinear, $\phi(0) = 0$, and there is a constant $c_0 > 0$ such that $\phi' \geq c_0$ on $\Re$.

For Equations (1.5)–(1.6), we propose the singular perturbation problem in the following sense: show that when $\rho \downarrow 0$, the solutions of (1.5)–(1.6) approach the solutions of the equation obtained from (1.5)–(1.6) with $\rho = 0$. It will be shown that the solution of (1.5)–(1.6) with $\rho = 0$ exists uniquely and is independent of t, i.e., in static-state. Thus, this singular perturbation can also be regarded as a quasi-static approximation.

When ϕ is linear, (1.5)–(1.6) is studied in Grimmer and Liu [**6**], where linearity is used to subtract the solution w of (1.5)–(1.6) with $\rho = 0$

from the solutions u^ρ of (1.5)–(1.6). Then an equation for $Q^\rho \equiv u^\rho - w$ is formulated and the method of energy estimate is employed to show that $(u^\rho - w =)\, Q^\rho \to 0$ as $\rho \to 0$.

When ϕ is nonlinear but $f = 0$, it is shown in [**6**] that the solution w of (1.5)–(1.6) with $\rho = 0$ is $w = 0$. Thus the equation for $Q^\rho \equiv u^\rho - w = u^\rho$ is the same as Equations (1.5)–(1.6) (with $f = 0$). Therefore, it is indicated in [**6**] that the energy estimate method can be modified to show that $(u^\rho - w = u^\rho =)\, Q^\rho \to 0$ as $\rho \to 0$.

Now, in this paper, we look at the case where ϕ is nonlinear and $f \neq 0$. It will be seen that this case is more complicated than the previous cases. For example, the equation for $Q^\rho \equiv u^\rho - w$ also involves w. However, after some trials and errors, we found an appropriate energy function for Q^ρ so that the method of the energy estimate used in [**6**] can also be extended here to show that $(u^\rho - w =)\ Q^\rho \to 0$ as $\rho \to 0$. Moreover, the rate of convergence in ρ is obtained to be $\|u^\rho - w\|_{L^2} \le K\sqrt{\rho}$ and $\|u_x^\rho - w_x\|_{L^2} \le K\sqrt{\rho}$ for some constant K independent of ρ, as a by-product of our energy estimate in this paper. (The rate of convergence was not discovered in [**6**].)

Related studies of singular perturbations can be found in, for example, Chow and Lu [**1**], Fattorini [**5**], Hale and Raugel [**8**], Grimmer and Liu [**6**], and Liu [**9, 10**].

2. Singular perturbations. Note that the existence and uniqueness of solutions of Equations (1.5)–(1.6) (with $\rho > 0$) were obtained in [**2, 7, 12, 13**], and we are only interested in singular perturbations in this paper, so we will assume that Equations (1.5)–(1.6) (with $\rho > 0$) has a unique solution u^ρ for every $\rho > 0$. Also note that we first assume that the "history" v^ρ satisfies Equation (1.5) on $\Re^-$. Then we will see that if v^ρ is only specified on $\Re^-$ (may not satisfy (1.5)), then with essentially the same proof, we can obtain the similar results.

Now we can state and prove our main results with the following hypothesis:

(H) $1 + \hat{F}(\lambda) \neq 0$ for $\operatorname{Re}\lambda \ge 0$. F and $F' \in L^1(\Re^+)$. $F = 0$ on $\Re^-$. $f \in C[0,1]$. $\|v_t^\rho(s,\cdot)\|_{L^2}$ and $\|g(-s)\|_{L^2}$ are bounded for $s \le 0$.

Here $\hat{F}$ is the Laplace transform of F, and $L^2 = L^2[0,T]$.

Theorem 2.1. *Assume that the hypothesis* (H) *is satisfied. Then there is a unique* w*, which is independent of* t*, such that*

$$0 = \phi(w_x(x))_x + \int_{-\infty}^{t} F(t-s)\phi(w_x(x))_x ds + f(x), \quad t \in \Re, \quad x \in [0,1], \tag{2.1}$$

$$w(0) = w(1) = 0. \tag{2.2}$$

(*This equation is obtained from* (1.5)–(1.6) *with* $\rho = 0$.)

Proof. Similar to [**6**], we let R be the function such that $R(s) = 0$, $s \leq 0$ and

$$R(t) = -F(t) - \int_0^t R(t-s)F(s)ds, \qquad t \geq 0, \tag{2.3}$$

whose existence is studied in, e.g., [**2, 3, 7**]. Note that (2.3) can be written as

$$(\delta + R) * (\delta + F) = \delta, \tag{2.4}$$

where

$$R * F(t) = \int_{-\infty}^{t} R(t-s)F(s)\, ds \quad \text{and} \quad \delta * H = H. \tag{2.5}$$

Now, write (1.5) with $\rho = 0$ as

$$-f(x) = (\delta + F) * \phi(u_x(t,x))_x. \tag{2.6}$$

This implies

$$\begin{aligned} \phi(u_x(t,x))_x &= -(\delta + R) * f(x) \\ &= -\left[1 + \int_0^\infty R(s)\, ds\right] f(x) \\ &= -\left[1 + \int_0^\infty F(s)\, ds\right]^{-1} f(x) \stackrel{\text{def}}{=} f_0(x). \end{aligned} \tag{2.7}$$

Thus we have

$$\phi(u_x(t,x)) = \int_0^x f_0(r)\,dr + C, \tag{2.8}$$

$$u_x(t,x) = \phi^{-1}\bigg(\int_0^x f_0(r)\,dr + C\bigg). \tag{2.9}$$

Therefore, the solution takes the following form

$$w(x) \stackrel{\text{def}}{=} u(t,x) = \int_0^x \phi^{-1}\bigg(\int_0^s f_0(r)\,dr + C\bigg)\,ds + C_1. \tag{2.10}$$

Taking into account of the boundary condition (1.6), we see that $C_1 = 0$ and that we only need to verify that there is a unique constant C such that

$$\int_0^1 \phi^{-1}\bigg(\int_0^s f_0(r)\,dr + C\bigg)\,ds = 0. \tag{2.11}$$

For this purpose, we first note that since $\phi' \geq c_0 > 0$ on $\Re$, one has $\phi^{-1}(-\infty) = -\infty$ and $\phi^{-1}(\infty) = \infty$. Thus there exists at least one C such that (2.11) is true.

Next, taking a derivative in C of the function

$$G(C) \equiv \int_0^1 \phi^{-1}\bigg(\int_0^s f_0(r)\,dr + C\bigg)\,ds, \tag{2.12}$$

one gets

$$\frac{1}{c_0} \geq G'(C) = \int_0^1 \frac{1}{\phi'(\phi^{-1}(\int_0^s f_0(r)dr + C))}\,ds > 0. \tag{2.13}$$

Therefore $G(C)$ is strictly increasing in C. Hence, there exists a unique C such that (2.11) is true. □

Theorem 2.2. *Assume that the hypothesis* (H) *is satisfied and that Equations* (1.5)–(1.6) *have a unique solution* u^ρ *(on* $\Re$*) for* $\rho > 0$*, i.e.,* v^ρ *satisfies Equations* (1.5)–(1.6) *on* $\Re^-$*. Let* w *be the unique solution* *of* (1.5)–(1.6) *with* $\rho = 0$ (*from Theorem* 2.1). *For* $T > 0$ *fixed and* $t \in [0,T]$, $x \in [0,1]$, *define* $Q^\rho(t,x) \equiv u^\rho(t,x) - w(x)$ *and*

$$E(t;\rho) \equiv \int_0^1 [Q_t^\rho(t,x)]^2\,dx + \frac{2}{\rho}\int_0^1\int_0^{Q_x^\rho(t,x)} [\phi(r + w_x(x)) - \phi(w_x(x))]\,dr\,dx. \tag{2.14}$$

If there exists a constant K_0 *independent of* ρ *such that* $E(0,\rho) \le K_0$, $\rho > 0$, *then as* $\rho \to 0$, *we have* $u^\rho(t,\cdot) \to w(\cdot)$ *and* $u_x^\rho(t,\cdot) \to w_x(\cdot)$ *in* $C([0,T], L^2[0,T])$. *Moreover, there exists a constant* K *independent of* ρ *such that*

$$\begin{gathered} \|u^\rho(t,\cdot) - w(\cdot)\|_{L^2} \le K\sqrt{\rho}, \\ \|u_x^\rho(t,\cdot) - w_x(\cdot)\|_{L^2} \le K\sqrt{\rho}, \\ t \in [0,T], \qquad \rho > 0. \end{gathered} \tag{2.15}$$

Remark 2.1. $E(0,\rho)$ is bounded when, for example, $v_t^\rho(0,x)$ is bounded and $Q_x^\rho(0,x) = 0$, i.e., $v_x^\rho(0,x) = w_x(x)$, independently of ρ.

Proof of Theorem 2.2. We first verify that

$$\int_0^t [\phi(r+s) - \phi(s)]\,dr \ge \frac{c_0}{2}t^2, \qquad t, s \in \Re. \tag{2.16}$$

For this purpose let us use the mean value theorem and get

$$\int_0^t [\phi(r+s) - \phi(s)]\,dr = \int_0^t \phi'(\xi) r\,dr. \tag{2.17}$$

If $t > 0$, then $r \ge 0$ and

$$\int_0^t \phi'(\xi) r\,dr \ge c_0 \int_0^t r\,dr = \frac{c_0}{2}t^2. \tag{2.18}$$

If $t < 0$, then $r \leq 0$ and

$$\begin{aligned}\int_0^t \phi'(\xi) r\,dr &= \int_t^0 \phi'(\xi)(-r)\,dr \\ &\geq c_0 \int_t^0 (-r)\,dr = \frac{c_0}{2} t^2.\end{aligned} \tag{2.19}$$

Next, we show that for the $E(t;\rho)$ defined by (2.14) with $E(0;\rho) \leq K_0$, there exists a constant K_1 independent of ρ such that $E(t;\rho) \leq K_1$, $\rho > 0$, $t \in [0,T]$.

For this end we first note that, from (2.16), one has

$$\begin{aligned}\int_0^1 \int_0^{Q_x^\rho(t,x)} &[\phi(r + w_x(x)) - \phi(w_x(x))]\,dr\,dx \\ &\geq \frac{c_0}{2} \int_0^1 [Q_x^\rho(t,x)]^2\,dx \geq 0.\end{aligned} \tag{2.20}$$

Then observe that, since we assumed that u^ρ satisfies Equation (1.5) on $\Re$, the equation for $Q^\rho(t,x) \equiv u^\rho(t,x) - w(x)$ is

$$\begin{aligned}\rho Q_{tt}^\rho(t,x) &= [\phi(Q_x^\rho(t,x) + w_x(x)) - \phi(w_x(x))]_x \\ &\quad + \int_{-\infty}^t F(t-s)[\phi(Q_x^\rho(s,x) + w_x(x)) \\ &\qquad - \phi(w_x(x))]_x\,ds + \rho g(t,x)\end{aligned} \tag{2.21}$$

for $t \in \Re$. Using (2.5), this can be written as

(2.22)

$$\rho(Q_{tt}^\rho(t,x) - g(t,x)) = (\delta + F) * [\phi(Q_x^\rho(t,x) + w_x(x)) - \phi(w_x(x))]_x, \quad t \in \Re.$$

Now, note that from [**6, 14**] one has $R(\infty) = 0$. Hence,

$$
\begin{aligned}
[\phi(Q_x^\rho(t,x)+w_x(x))-\phi(w_x(x))]_x &= \rho(\delta+R)*(Q_{tt}^\rho(t,x)-g(t,x)) \\
&= \rho\bigg(Q_{tt}^\rho(t,x)-g(t,x) \\
&\quad +\int_{-\infty}^t R(t-s)[Q_{tt}^\rho(s,x)-g(s,x)]\,ds\bigg) \\
&= \rho\bigg(Q_{tt}^\rho(t,x)-g(t,x)+R(0)Q_t^\rho(t,x) \\
&\quad +\int_{-\infty}^t R'(t-s)Q_t^\rho(s,x)\,ds \\
&\quad -\int_{-\infty}^t R(t-s)g(s,x)\,ds\bigg).
\end{aligned} \tag{2.23}
$$

Next, take a derivative of $E(t;\rho)$ in t and use the boundary condition (1.6) to get

$$
\begin{aligned}
\frac{d}{dt}E(t;\rho) &= 2\int_0^1 Q_t^\rho(t,x)Q_{tt}^\rho(t,x)\,dx \\
&\quad +\frac{2}{\rho}\int_0^1 [\phi(Q_x^\rho(t,x)+w_x(x))-\phi(w_x(x))]Q_{xt}^\rho(t,x)\,dx \\
&= 2\int_0^1 Q_t^\rho(t,x)Q_{tt}^\rho(t,x)\,dx \\
&\quad -\frac{2}{\rho}\int_0^1 [\phi(Q_x^\rho(t,x)+w_x(x))-\phi(w_x(x))]_x Q_t^\rho(t,x)\,dx.
\end{aligned}
$$

Then, replace (2.23) into it to obtain

$$
\begin{aligned}
\frac{d}{dt}E(t;\rho) &= 2\int_0^1 Q_t^\rho(t,x)Q_{tt}^\rho(t,x)\,dx \\
&\quad -2\int_0^1 \bigg(Q_{tt}^\rho(t,x)-g(t,x)+R(0)Q_t^\rho(t,x) \\
&\qquad +\int_{-\infty}^t R'(t-s)Q_t^\rho(s,x)\,ds \\
&\qquad -\int_{-\infty}^t R(t-s)g(s,x)\,ds\bigg)Q_t^\rho(t,x)\,dx
\end{aligned}
$$

$$
\begin{aligned}
&= 2\int_0^1 \biggl(g(t,x) - R(0)Q_t^\rho(t,x) - \int_{-\infty}^t R'(t-s)Q_t^\rho(s,x)\,ds \\
&\qquad\qquad + \int_{-\infty}^t R(t-s)g(s,x)\,ds\biggr)Q_t^\rho(t,x)\,dx \\
&\le \|g(t,\cdot)\|_{L^2}^2 + (2+2|R(0)|)\|Q_t^\rho(t,\cdot)\|_{L^2}^2 \\
&\quad + \int_{-\infty}^t |R'(t-s)|[\|Q_t^\rho(s,\cdot)\|_{L^2}^2 + \|Q_t^\rho(t,\cdot)\|_{L^2}^2]\,ds \\
&\quad + \int_0^1 \left[\int_{-\infty}^t |R(t-s)g(s,x)|\,ds\right]^2 dx \\
&\le \left(2+2|R(0)| + \int_0^\infty |R'(s)|\,ds\right)\|Q_t^\rho(t,\cdot)\|_{L^2}^2 \\
&\quad + \int_0^t |R'(t-s)|\|Q_t^\rho(s,\cdot)\|_{L^2}^2\,ds \\
&\quad + \|g(t,\cdot)\|_{L^2}^2 + \int_{-\infty}^0 |R'(t-s)|\|Q_t^\rho(s,\cdot)\|_{L^2}^2\,ds \\
&\quad + \int_0^1 \left[\int_{-\infty}^t |R(t-s)g(s,x)|\,ds\right]^2 dx.
\end{aligned}
$$

Now, note that $\|Q_t^\rho(t,\cdot)\|_{L^2}^2 \le E(t;\rho)$ by (2.20). Then from above one gets

$$
\begin{aligned}
\frac{d}{dt}E(t;\rho) &\le \left(2+2|R(0)| + \int_0^\infty |R'(s)|\,ds\right)E(t;\rho) \\
&\quad + \int_0^t |R'(t-s)|E(s;\rho)\,ds \\
&\quad + \|g(t,\cdot)\|_{L^2}^2 + \int_{-\infty}^0 |R'(t-s)|\,\|Q_t^\rho(s,\cdot)\|_{L^2}^2\,ds \\
&\quad + \int_0^1 \left[\int_{-\infty}^t |R(t-s)g(s,x)|ds\right]^2 dx \\
&\le HE(t;\rho) + \int_0^t |R'(t-s)|E(s;\rho)\,ds + P,
\end{aligned}
\tag{2.24}
$$

where H and P are constants defined in a obvious way.

Similar to [**6**], we can use the standard arguments in differential inequality to obtain a constant K_1 independent of ρ such that $E(t;\rho) \le$

K_1, $t \in [0,T]$, $\rho > 0$. Therefore, (2.20) implies

$$\frac{c_0}{\rho}\int_0^1 [Q_x^\rho(t,x)]^2\,dx \le E(t;\rho) \le K_1, \qquad t \in [0,T], \quad \rho > 0. \tag{2.25}$$

Now, note that the boundary condition in (1.6) implies

$$\|Q^\rho(t,\cdot)\|_{L^2} \le \|Q_x^\rho(t,\cdot)\|_{L^2}. \tag{2.26}$$

Thus we can let $K \equiv \sqrt{K_1/c_0}$ and obtain

$$\|Q^\rho(t,\cdot)\|_{L^2} \le \|Q_x^\rho(t,\cdot)\|_{L^2} \le K\sqrt{\rho}, \qquad t \in [0,T], \quad \rho > 0. \tag{2.27}$$

This proves the Theorem. □

Remark 2.2. Here, the proof of $Q^\rho(t,x) \to 0$ as $\rho \to 0$ is different from [**6**], and is short and direct, and can also provide the rate of convergence in ρ.

In the following, we will verify that if v^ρ is only specified on $\Re^-$ and may not satisfy Equation (1.5), then we can still get the similar results. Because now, (2.21) becomes

$$\begin{aligned}\rho Q_{tt}^\rho(t,x) &= [\phi(Q_x^\rho(t,x) + w_x(x)) - \phi(w_x(x))]_x \\ &\quad + \int_0^t F(t-s)[\phi(Q_x^\rho(s,x) + w_x(x)) - \phi(w_x(x))]_x\,ds \\ &\quad + \int_{-\infty}^0 F(t-s)[\phi(Q_x^\rho(s,x) + w_x(x)) - \phi(w_x(x))]_x\,ds \\ &\quad + \rho g(t,x), \qquad t \ge 0.\end{aligned} \tag{2.28}$$

And hence, (2.22) becomes

$$\begin{aligned}&\rho(Q_{tt}^\rho(t,x) - g(t,x)) \\ &\qquad = (\delta + F)\hat{*}[\phi(Q_x^\rho(t,x) + w_x(x)) - \phi(w_x(x))]_x \\ &\qquad\quad + \int_{-\infty}^0 F(t-s)[\phi(Q_x^\rho(s,x) + w_x(x)) - \phi(w_x(x))]_x\,ds\end{aligned} \tag{2.29}$$

where the integration in $\hat{*}$ is from 0 to t. Therefore (2.23) becomes

$$
\begin{aligned}
(2.30)\quad & [\phi(Q_x^\rho(t,x)+w_x(x))-\phi(w_x(x))]_x \\
&= (\delta+R)\hat{*}\Big\{\rho(Q_{tt}^\rho(t,x)-g(t,x)) \\
&\qquad - \int_{-\infty}^0 F(t-s)[\phi(Q_x^\rho(s,x)+w_x(x))-\phi(w_x(x))]_x\,ds\Big\} \\
&= \rho\Big(Q_{tt}^\rho(t,x)-g(t,x)+\int_0^t R(t-s)[Q_{tt}^\rho(s,x)-g(s,x)]\,ds\Big) \\
&\quad - (\delta+R)\hat{*}\int_{-\infty}^0 F(t-s)[\phi(Q_x^\rho(s,x)+w_x(x))-\phi(w_x(x))]_x\,ds \\
&= \rho\Big(Q_{tt}^\rho(t,x)-g(t,x)+R(0)Q_t^\rho(t,x)-R(t)Q_t^\rho(0,x) \\
&\qquad + \int_0^t R'(t-s)Q_t^\rho(s,x)\,ds-\int_0^t R(t-s)g(s,x)\,ds\Big) \\
&\quad - (\delta+R)\hat{*}\int_{-\infty}^0 F(t-s)[\phi(Q_x^\rho(s,x)+w_x(x))-\phi(w_x(x))]_x\,ds.
\end{aligned}
$$

Thus, (2.24) will be changed to

$$
\begin{aligned}
\frac{d}{dt}E(t;\rho) &= 2\int_0^1 Q_t^\rho(t,x)Q_{tt}^\rho(t,x)\,dx \\
&\quad - \frac{2}{\rho}\int_0^1 [\phi(Q_x^\rho(t,x)+w_x(x))-\phi(w_x(x))]_x Q_t^\rho(t,x)\,dx \\
&= 2\int_0^1 Q_t^\rho(t,x)Q_{tt}^\rho(t,x)\,dx \\
&\quad - 2\int_0^1 \Big(Q_{tt}^\rho(t,x)-g(t,x)+R(0)Q_t^\rho(t,x) \\
&\qquad - R(t)Q_t^\rho(0,x)+\int_0^t R'(t-s)Q_t^\rho(s,x)\,ds \\
&\qquad - \int_0^t R(t-s)g(s,x)\,ds\Big)Q_t^\rho(t,x)\,dx \\
&\quad + \frac{2}{\rho}\int_0^1 \Big\{(\delta+R)\hat{*}\int_{-\infty}^0 F(t-s)[\phi(Q_x^\rho(s,x)+w_x(x))
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad - \phi(w_x(x))]_x \, ds\Big\} Q_t^\rho(t,x)\,dx \\
&= 2\int_0^1 \Big(g(t,x) + R(t)Q_t^\rho(0,x) - R(0)Q_t^\rho(t,x) \\
&\qquad - \int_0^t R'(t-s)Q_t^\rho(s,x)\,ds \\
&\qquad + \int_0^t R(t-s)g(s,x)\,ds\Big) Q_t^\rho(t,x)\,dx \\
&\quad + 2\int_0^1 \Big\{\frac{1}{\rho}(\delta+R)\hat{*}\int_{-\infty}^0 F(t-s)[\phi(Q_x^\rho(s,x)+w_x(x)) \\
&\qquad\qquad\qquad\qquad - \phi(w_x(x))]_x \, ds\Big\} Q_t^\rho(t,x)\,dx
\end{aligned}
$$

(2.31)

$$
\begin{aligned}
&\le \|g(t,\cdot) + R(t)Q_t^\rho(0,\cdot)\|_{L^2}^2 + (3 + 2|R(0)|)\|Q_t^\rho(t,\cdot)\|_{L^2}^2 \\
&\quad + \int_0^t |R'(t-s)|[\|Q_t^\rho(s,\cdot)\|_{L^2}^2 + \|Q_t^\rho(t,\cdot)\|_{L^2}^2]\,ds \\
&\quad + \int_0^1 \Big[\int_0^t |R(t-s)g(s,x)|\,ds\Big]^2 dx \\
&\quad + \int_0^1 \Big\{\frac{1}{\rho}(\delta+R)\hat{*}\int_{-\infty}^0 F(t-s)[\phi(Q_x^\rho(s,x) \\
&\qquad\qquad\qquad + w_x(x)) - \phi(w_x(x))]_x \, ds\Big\}^2 dx \\
&\le \Big(3 + 2|R(0)| + \int_0^\infty |R'(s)|\,ds\Big)\|Q_t^\rho(t,\cdot)\|_{L^2}^2 \\
&\quad + \int_0^t |R'(t-s)|\|Q_t^\rho(s,\cdot)\|_{L^2}^2\,ds \\
&\quad + \|g(t,\cdot) + R(t)Q_t^\rho(0,\cdot)\|_{L^2}^2 \\
&\quad + \int_0^1 \Big[\int_0^t |R(t-s)g(s,x)|\,ds\Big]^2 dx \\
&\quad + \int_0^1 \Big\{(\delta+R)\hat{*}\int_{-\infty}^0 F(t-s)\frac{1}{\rho}[\phi(v_x^\rho(s,x))
\end{aligned}
$$

$$- \phi(w_x(x))]_x \, ds \Big\}^2 \, dx$$
$$\leq \hat{H}E(t;\rho) + \int_0^t |R'(t-s)| E(s;\rho) \, ds + \hat{P}.$$

Now, it is clear that we have the following result, which is similar to Theorem 2.2:

Theorem 2.3. *Assume that the hypothesis* (H) *is satisfied and that Equations* (1.5)–(1.6) *have a unique solution* u^ρ (*on* $\Re^+$) *for* $\rho > 0$, *i.e.,* v^ρ *is only specified on* $\Re^-$ *and may not satisfy Equations* (1.5)–(1.6) *on* $\Re^-$. *Let* w *be the unique solution of* (1.5)–(1.6) *with* $\rho = 0$ (*from Theorem* 2.1). *Assume further that, for some constant* C *independent of* ρ,

$$\frac{1}{\rho} |[\phi(v_x^\rho(s,x)) - \phi(w_x(x))]_x| \leq C, \quad s \leq 0, \quad x \in [0,1], \qquad \rho > 0. \tag{2.32}$$

If there exists a constant K_0 *independent of* ρ *such that* $E(0,\rho) \leq K_0$, $\rho > 0$, *then as* $\rho \to 0$, *we have* $u^\rho(t,\cdot) \to w(\cdot)$ *and* $u_x^\rho(t,\cdot) \to w_x(\cdot)$ *in* $C([0,T], L^2[0,T])$. *Moreover, there exists a constant* K *independent of* ρ *such that*

$$\begin{gathered} \|u^\rho(t,\cdot) - w(\cdot)\|_{L^2} \leq K\sqrt{\rho}, \\ \|u_x^\rho(t,\cdot) - w_x(\cdot)\|_{L^2} \leq K\sqrt{\rho}, \\ t \in [0,T], \qquad \rho > 0. \end{gathered} \tag{2.33}$$

Remark 2.3. Equation (2.32) is satisfied if, for example, $v_x^\rho(s,x) = w_x(x)$, $s \leq 0$, $x \in [0,1]$, $\rho > 0$.

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