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LOCALIZATION AND POST PROCESSING FOR THE GALERKIN BOUNDARY ELEMENT METHOD APPLIED TO THREE-DIMENSIONAL SCREEN PROBLEMS

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ABSTRACT. We study local error estimates for various Galerkin schemes (Galerkin schemes with quasi-uniform or graded meshes, and the augmented Galerkin method) applied to weakly singular and hypersingular integral equations on open surfaces in $\mathbf{R}^3$. The results are given for a large scale of Sobolev norms, even in some norms that are not defined globally. In the case of the weakly singular integral equation where the highest orders of convergence achieved are in negative Sobolev norms, we establish from the Galerkin solutions new solutions that converge in the L^2-norm to the exact solution in these orders.

1. Introduction. The solutions of elliptic boundary value problems in $\mathbf{R}^3\backslash\Gamma$, where Γ is an open surface in $\mathbf{R}^3$, have special singular forms at the boundary γ of Γ, regardless of whether γ is a smooth or polygonal curve. When those problems are reformulated, via the direct method, into boundary integral equations, the solutions of the latter inherit those singularities. These singularities affect the rate of global convergence of numerical schemes, e.g., the Galerkin boundary element methods. To recover the high order of convergence associated with smooth and closed surfaces, either augmented boundary elements or mesh grading is necessary. However, if the given data are sufficiently smooth, the solutions to the integral equations are smooth locally, i.e., away from the singularities. Then there arises the following question. Is the accuracy of the approximation better in regions of smooth behavior of the exact solutions? Another problem is faced when we want to observe the highest order of global convergence when it is achieved in a negative norm. Is there any effective post-processing method to obtain that highest order for the local convergence in the L^2-norm, which can

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be easily observed? The purpose of this article is to give answers to the above questions.

For strongly elliptic integral equations on a smooth and closed curve in $\mathbf{R}^2$, local error estimates were proved by Saranen [**14**]. The proof was then modified to achieve local estimates for equations on open arcs [**19**]. We will adapt the techniques of [**14**] and [**19**] to prove estimates for the case of open surfaces in $\mathbf{R}^3$.

In the boundary element literature, the K-operator method has been used effectively as a post-processing method to increase the order of local convergence in the L^2-norm of the Galerkin and qualocation methods applied to strongly elliptic equations on smooth curves, closed or open [**18, 20**]. The original idea is due to Bramble and Schatz [**2, 4**] and Thomée [**17**] in the finite element environment. We shall study the effectiveness of that method for the Galerkin approximation to integral equations on an open surface in $\mathbf{R}^3$.

We will particularly be concerned with the weakly-singular and hypersingular equations given by

$$V\psi(x) := -\frac{1}{4\pi}\int_\Gamma \frac{\psi(y)}{|x-y|}\,ds(y) = g(x), \qquad x\in\Gamma, \tag{1}$$

$$D\xi(x) := -\frac{1}{4\pi}\frac{\partial}{\partial n_x}\int_\Gamma \frac{\partial}{\partial n_y}\frac{\xi(y)}{|x-y|}\,ds(y) = f(x), \qquad x\in\Gamma. \tag{2}$$

Here the surface Γ is assumed to be smooth, and its boundary γ is a smooth or polygonal curve, except when otherwise identified. Equation (1), respectively (2), is the integral reformulation (via the direct method) for the Dirichlet, respectively Neumann, boundary value problem in $\mathbf{R}^3\backslash\Gamma$ with vanishing condition at infinity (see, e.g., [**5, 15**]).

It was proved in [**15**] that, for $0<\sigma<\sigma'<1/2$, if $g\in H^{3/2+\sigma}(\Gamma)$ and $f\in H^{1/2+\sigma}(\Gamma)$, then ψ and ξ have the form

$$\psi = \beta(s)\rho^{-1/2}\chi(\rho)+\psi_r \quad \text{on } \Gamma,$$
$$\xi = \alpha(s)\rho^{1/2}\chi(\rho)+\xi_r \quad \text{on } \Gamma,$$

where

$$\begin{aligned}\beta &\in H^{1/2+\sigma}(\gamma),\\ \psi_r &\in \tilde{H}^{1/2+\sigma'}(\Gamma),\\ \alpha &\in H^{1/2+\sigma}(\gamma),\\ \xi_r &\in L^2(I;H^{1/2+\sigma}(\gamma))\cap \tilde{H}^{3/2+\sigma'}(I;L^2(\gamma)),\\ I &= (0,1),\end{aligned}$$

and where χ is a cutoff function with $\chi \equiv 1$ for $|\rho| < 1/2$ and $\chi \equiv 0$ for $|\rho| > 1$. Here s denotes the parameter of arclength of γ and ρ corresponds to the Euclidean distance to γ. (For the definition of the Sobolev spaces, see Section 2.)

The plan of the paper is as follows: in Section 2 we give some known results on the Galerkin approximation methods to (1) and (2). These include the standard Galerkin method with regular and graded meshes and the augmented-Galerkin method. The results on local error estimates for these Galerkin approximation schemes are proved in Section 3. The K-operator method for equation (1) is studied in Section 4. However, the same method can also be used for any Galerkin approximation to equation (2), provided that an error estimate in a negative Sobolev norm exists. Section 5 is devoted to some numerical experiments.

2. The Galerkin approximations. Let $\tilde{\Gamma}$ be a smooth and closed surface containing Γ. We recall from [**11, 12**] the function spaces to be used in this paper:

$$\begin{aligned}H^s(\tilde{\Gamma}) &= \begin{cases}\{u|_{\tilde{\Gamma}} : u \in H^{s+1/2}_{\mathrm{loc}}(\mathbf{R}^3)\} & \text{for } s>0,\\ L^2(\Gamma) & \text{for } s=0,\\ (H^{-s}(\tilde{\Gamma}))',\ \text{(dual space)} & \text{for } s<0,\end{cases}\\ H^s(\Gamma) &= \{u|_\Gamma : u\in H^s(\tilde{\Gamma})\} \quad \text{for } s>0,\\ \tilde{H}^s(\Gamma) &= \{u\in H^s(\Gamma) : u^* \in H^s(\tilde{\Gamma})\} \quad \text{for } s>0,\\ H^s(\Gamma) &= (\tilde{H}^{-s}(\Gamma))' \quad \text{for } s<0,\\ \tilde{H}^s(\Gamma) &= (H^{-s}(\Gamma))' \quad \text{for } s<0,\end{aligned}$$

where

$$u^* = \begin{cases} u & \text{on } \Gamma, \\ 0 & \text{on } \tilde{\Gamma}\backslash\Gamma. \end{cases}$$

We assume that the surface Γ is given by a regular parametric representation $x = X(v)$ with $v \in U$, where U is a compact region in $\mathbf{R}^2$ whose boundary is mapped onto γ and which also has a regular parametric representation. By choosing a sequence of regular triangulations of U with maximal meshsize h, we can define a regular system $S^{r,k}$, $0 \le k < r$, of finite elements in the sense of Babuška and Aziz [**1**]. These elements can then be transplanted onto Γ by the above parametrization to form the system $S_h^{r,k}(\Gamma)$. The parameters in $S_h^{r,k}(\Gamma)$ have the following meanings: $h \in (0, h_0]$ is the mesh size of the partition of Γ, e.g., h is the longest side of a triangle in a uniform triangulation; $r-1$ is the degree of piecewise polynomials constituting the corresponding finite elements; and k describes the conformity $S_h^{r,k}(\Gamma) \subset H^k(\Gamma)$. Analogously, we can define $S_h^{r,k}(\gamma)$. Moreover, we define

$$\overset{\circ}{S}{}_h^{r,k}(\Gamma) = \{\phi \in S_h^{r,k}(\Gamma) \mid \phi = 0 \text{ on } \gamma\}.$$

In particular, $\overset{\circ}{S}{}_h^{2,1}(\Gamma)$ is the space of piecewise-linear, continuous functions on Γ vanishing on γ and $S_h^{1,0}(\Gamma)$ is the space of piecewise-constant functions on Γ. Note that $S_h^{1,0}(\Gamma) \subset \tilde{H}^{-1/2}(\Gamma)$ and $\overset{\circ}{S}{}_h^{2,1}(\Gamma) \subset \tilde{H}^{1/2}(\Gamma)$.

2.1. *Standard Galerkin schemes with regular mesh.* Here we assume that Γ is a smooth open surface with a smooth or polygonal boundary γ. The standard Galerkin schemes for the integral equations (1) and (2) read as

Find $\psi_h \in S_h^{1,0}(\Gamma)$ such that for all $\varphi \in S_h^{1,0}(\Gamma)$

$$\langle V\psi_h, \varphi\rangle = \langle g, \varphi\rangle, \tag{3}$$

and as

Find $\xi_h \in \overset{\circ}{S}{}_h^{2,1}(\Gamma)$ such that for all $\phi \in \overset{\circ}{S}{}_h^{2,1}(\Gamma)$

$$\langle D\xi_h, \phi\rangle = \langle f, \phi\rangle. \tag{4}$$

The following results were proved in [**6**, Theorem 4.1].

Theorem A. *There is a meshwidth* $h_0 > 0$ *such that, for* $0 < h \leq h_0$, *the Galerkin equations* (3) *and* (4) *are uniquely solvable in* $S_h^{1,0}(\Gamma)$ *and* $\mathring{S}_h^{2,1}(\Gamma)$, *respectively. Moreover, there hold*

$$\|\psi_h - \psi\|_{\tilde{H}^{-1/2}(\Gamma)} \leq c \inf\{\|\varphi - \psi\|_{\tilde{H}^{-1/2}(\Gamma)} : \varphi \in S_h^{1,0}(\Gamma)\},$$
$$\|\xi_h - \xi\|_{\tilde{H}^{1/2}(\Gamma)} \leq c \inf\{\|\phi - \xi\|_{\tilde{H}^{1/2}(\Gamma)} : \phi \in \mathring{S}_h^{2,1}(\Gamma)\}.$$

Furthermore, for sufficiently smooth g *and* f, *there hold, with* $\varepsilon > 0$,

$$\|\psi_h - \psi\|_{\tilde{H}^{t}(\Gamma)} \leq ch^{-\varepsilon - t}\|\psi\|_{\tilde{H}^{-\varepsilon}(\Gamma)} \quad \text{for } -1 + \varepsilon \leq t \leq -\varepsilon,$$
$$\|\xi_h - \xi\|_{\tilde{H}^{s}(\Gamma)} \leq ch^{1-s-\varepsilon}\|\xi\|_{\tilde{H}^{1-\varepsilon}(\Gamma)} \quad \text{for } \varepsilon \leq s \leq 1 - \varepsilon.$$

2.2. *Standard Galerkin schemes with graded mesh.* For simplicity, we assume now that Γ is the square $[-1,1]\times[-1,1]$. We introduce a mesh, that is uniform on $\Gamma^* = [-3/4, 3/4] \times [-3/4, 3/4]$ and graded on the rest, by the lines x_1 and x_2 defined as follows:

$$x_1 = \begin{cases} -1 + 4^{\varrho-1}(ih)^{\varrho} & \text{if } 0 \leq i \leq N/8 - 1, \\ -1 + ih & \text{if } N/8 \leq i \leq 7N/8 - 1, \\ 1 - 4^{\varrho-1}(2 - ih)^{\varrho} & \text{if } 7N/8 \leq i \leq N, \end{cases}$$
$$x_2 = \begin{cases} -1 + 4^{\varrho-1}(jh)^{\varrho} & \text{if } 0 \leq j \leq N/8 - 1, \\ -1 + jh & \text{if } N/8 \leq j \leq 7N/8 - 1, \\ 1 - 4^{\varrho-1}(2 - jh)^{\varrho} & \text{if } 7N/8 \leq j \leq N, \end{cases}$$

where $N = 8n$, $n = 1, 2, \dots, h = 2/N$ and $\varrho \geq 1$. Note that when $\varrho = 1$ the mesh is uniform. Local uniformity of the mesh is essential for the application of the K-operator since it is required that a spline still be a spline locally after being translated by a mesh step (see the proof of Theorem 4.1). Besides, this requirement allows us to use, in the proof of local error estimates in Section 3, the inverse property on Γ^*.

On this mesh, we define $H_h^{0,\varrho}$ to be the space of piecewise-constant functions and $H_h^{1,\varrho}$ to be the space of piecewise-linear continuous functions vanishing on γ. The Galerkin schemes now read as:

Find $\psi_h \in H_h^{0,\varrho}$ such that for all $\varphi \in H_h^{0,\varrho}$

$$\langle V\psi_h, \varphi\rangle = \langle g, \varphi\rangle, \tag{5}$$

and as

Find $\xi_h \in H_h^{1,\varrho}$ such that for all $\phi \in H_h^{1,\varrho}$

$$\langle D\xi_h, \phi\rangle = \langle f, \phi\rangle. \tag{6}$$

The following results hold (see [**10**, Theorem 1.5] and [**9**, Theorem 2.2]):

Theorem B. *There exists a meshwidth $h_0 > 0$ such that, for $0 < h \le h_0$, the Galerkin equations* (5) *and* (6) *are uniquely solvable in $H_h^{0,\varrho}$ and $H_h^{1,\varrho}$, respectively. Moreover, for any $\varepsilon > 0$ there exists a constant $c = c(\varrho)$ independent of h such that*

$$\|\psi_h - \psi\|_{\tilde{H}^{-1/2}(\Gamma)} \le \begin{cases} ch^{\varrho/2-\varepsilon} & \text{if } 1 \le \varrho \le 3, \\ ch^{3/2} & \text{if } \varrho > 3, \end{cases}$$

$$\|\xi_h - \xi\|_{\tilde{H}^{1/2}(\Gamma)} \le \begin{cases} ch^{\varrho/2-\varepsilon} & \text{if } 1 \le \varrho < 3, \\ ch^{3/2-\varepsilon} & \text{if } \varrho \ge 3. \end{cases}$$

Moreover, in the deepest negative norms, we have

$$\|\psi_h - \psi\|_{\tilde{H}^{-1/2-\varrho/2}(\Gamma)} \le ch^{\varrho-\varepsilon} \quad \text{if } 1 \le \varrho \le 3,$$
$$\|\psi_h - \psi\|_{\tilde{H}^{-2}(\Gamma)} \le ch^{3} \quad \text{if } \varrho > 3,$$
$$\|\xi_h - \xi\|_{\tilde{H}^{1/2-\varrho/2}(\Gamma)} \le ch^{\varrho-\varepsilon} \quad \text{if } 1 \le \varrho < 3.$$

2.3. *The augmented-Galerkin method.* For this method, Γ is assumed to be a smooth open surface with a smooth boundary γ. We define, for $s < 0$,

$$Z^s(\Gamma) = \tilde{H}^s(\Gamma).$$

For $0 \le s < 1$, we define

$$Z^s(\Gamma) = \{\psi = \beta\rho^{-1/2}\chi(\rho) + \psi_r : \beta \in H^s(\gamma) \text{ and } \psi_r \in \tilde{H}^s(\Gamma)\}, \tag{7}$$

equipped with

$$\|\psi\|_{Z^s(\Gamma)} = \|\beta\|_{H^s(\gamma)} + \|\psi_r\|_{\tilde{H}^s(\Gamma)},$$

where ρ corresponds to the Euclidean distance to γ, and χ is a C^∞ cut-off function with $\chi \equiv 1$ for $|\rho| < 1/2$ and $\chi \equiv 0$ for $|\rho| > 1$.

Similarly, we define, for $t < 1$,

$$Y^t(\Gamma) = \tilde{H}^t(\Gamma).$$

For $1 \leq t < 2$, we define

$$Y^t(\Gamma) = \{\xi = \alpha\rho^{1/2}\chi(\rho) + \xi_r : \alpha \in H^t(\gamma) \text{ and } \xi_r \in \tilde{H}^t(\Gamma)\}, \tag{8}$$

equipped with

$$\|\xi\|_{Y^t(\Gamma)} = \|\alpha\|_{H^t(\gamma)} + \|\xi_r\|_{\tilde{H}^t(\Gamma)}.$$

The augmented finite element spaces are defined as

$$\begin{aligned} Z_h(\Gamma) &= \{\varphi = \beta\rho^{-1/2}\chi(\rho) + \varphi_r : \beta \in S_h^{2,1}(\gamma) \text{ and } \varphi_r \in \mathring{S}_h^{2,1}(\Gamma)\}, \\ Y_h(\Gamma) &= \{\phi = \alpha\rho^{1/2}\chi(\rho) + \phi_r : \alpha \in S_h^{3,2}(\gamma) \text{ and } \phi_r \in \mathring{S}_h^{3,2}(\Gamma)\}. \end{aligned} \tag{9}$$

We note that $Z_h(\Gamma) \subset Z^{1-\varepsilon}(\Gamma) \subset \tilde{H}^{-1/2}(\Gamma)$ and $Y_h(\Gamma) \subset Y^{2-\varepsilon}(\Gamma) \subset \tilde{H}^{1/2}(\Gamma)$ for any $\varepsilon > 0$. The augmented-Galerkin schemes for (1) and (2) now read as

Find $\psi_h \in Z_h(\Gamma)$ such that for all $\varphi \in Z_h(\Gamma)$

$$\langle V\psi_h, \varphi\rangle = \langle g, \varphi\rangle, \tag{10}$$

and as

Find $\xi_h \in Y_h(\Gamma)$ such that for all $\phi \in Y_h(\Gamma)$

$$\langle D\xi_h, \phi\rangle = \langle f, \phi\rangle. \tag{11}$$

The following theorem is a consequence of [**15**, Theorem 3.2] and [**7**, Theorem 3.1] (see also [**6**, Theorem 3.4]):

Theorem C. *There exists a mesh width $h_0 > 0$ such that, for $0 < h \le h_0$, the Galerkin equations* (10) *and* (11) *are uniquely solvable in $Z_h(\Gamma)$ and $Y_h(\Gamma)$, respectively. Moreover, there hold*

$$\|\psi_h - \psi\|_{\tilde{H}^{-1/2}(\Gamma)} \le c \inf\{\|\varphi - \psi\|_{\tilde{H}^{-1/2}(\Gamma)} : \varphi \in Z_h(\Gamma)\},$$
$$\|\xi_h - \xi\|_{\tilde{H}^{1/2}(\Gamma)} \le c \inf\{\|\phi - \xi\|_{\tilde{H}^{1/2}(\Gamma)} : \phi \in Y_h(\Gamma)\}.$$

Furthermore, for sufficiently smooth g and f, there hold, with $\varepsilon > 0$,

$$\|\psi_h - \psi\|_{\tilde{H}^t(\Gamma)} \le ch^{1-t-\varepsilon}\|\psi\|_{Z^{1-\varepsilon}(\Gamma)} \quad \text{for } -2+\varepsilon \le t \le 1-\varepsilon,$$
$$\|\xi_h - \xi\|_{\tilde{H}^s(\Gamma)} \le ch^{2-s-\varepsilon}\|\xi\|_{Y^{2-\varepsilon}(\Gamma)} \quad \text{for } -1+\varepsilon \le s \le 2-\varepsilon.$$

3. Local error estimates. In the analysis of local error estimates, we shall use the following nested sub-pieces of the surface Γ

$$\Gamma_0 \subset\subset \Gamma_1 \subset\subset \cdots \subset\subset \Gamma_J \subset\subset \Gamma_* \subset\subset \Gamma^* \subset \Gamma, \tag{12}$$

and the following cut-off functions:

$$\omega_j \in C_0^\infty(\Gamma_{j+1}) \quad \text{and} \quad \omega_j \equiv 1 \quad \text{on } \Gamma_j \quad \text{for } j = 0, \dots, J-1. \tag{13}$$

Here $X \subset\subset Y$ means that the closure of X is contained in the interior of Y.

We will frequently use the following properties of the splines, the proofs of which can be found in, e.g., [**1, 6, 13, 14, 21**].

Lemma 3.1 (Approximation property). *Let $t \le s \le r$, $t \le k$ and $j = 0, \dots, J-2$. There exists a constant c such that, for any $v \in \tilde{H}^s(\Gamma)$, there exists $\zeta \in S_h^{r,k}(\Gamma)$ such that* $\operatorname{supp}\zeta \subset \Gamma_{j+2}$ *and*

$$\|\omega_j v - \zeta\|_{\tilde{H}^t(\Gamma)} \le ch^{s-t}\|v\|_{H^s(\Gamma_{j+1})}.$$

Lemma 3.2 (Inverse property). *Let $t \le s \le k$ and $j = 0, \dots, J-2$. There exists a constant c such that, for any $\phi \in S_h^{r,k}(\Gamma)$,*

$$\|\omega_j \phi\|_{\tilde{H}^s(\Gamma)} \le ch^{t-s}\|\phi\|_{H^t(\Gamma_{j+1})}.$$

Lemma 3.3 (Super-approximation property). *Let $t \le s \le k$, and let $j = 0, \dots, J-2$. There exists a constant c such that, for any $\phi \in S_h^{r,k}(\Gamma)$, there exists $\zeta \in S_h^{r,k}(\Gamma)$ such that* $\operatorname{supp}\zeta \subset \Gamma_{j+2}$ *and*

$$\|\omega_j\phi - \zeta\|_{\tilde{H}^t(\Gamma)} \le ch^{s-t+1}\|\phi\|_{H^s(\Gamma_{j+1})}.$$

In order that the singular terms in the definitions (7) and (8) do not affect our local analysis, we shall assume in the sequel that the cut-off function χ defined in (7) vanishes on Γ^*. The main results in this section concerning equation (1) are given in the following theorems:

Theorem 3.4. *Assume that the solution of equation* (1) *satisfies $\psi \in H^s(\Gamma^*) \cap \tilde{H}^{-1/2}(\Gamma)$ for some $s \in (-1/2, 1]$. For any $t \in \mathbf{R}$, let*

$$\sigma = \sigma(t) = \begin{cases} 0 & \text{if } t \le -1/2, \\ -t - 1/2 & \text{if } t > -1/2. \end{cases}$$

Let $\varepsilon > 0$ be given, sufficiently small.

(i) *If $\psi_h \in S_h^{1,0}(\Gamma)$ is the solution of* (3)*, then for any t with $-1 < t \le \min(s, 0)$*

$$\|\psi_h - \psi\|_{H^t(\Gamma_0)} \le c\{h^{s-t}\|\psi\|_{H^s(\Gamma^*)} + h^\sigma\|\psi_h - \psi\|_{\tilde{H}^{-1+\varepsilon}(\Gamma)}\}.$$

(ii) *If $\psi_h \in H_h^{0,\varrho}$ is the solution of* (5)*, then for any t with $-2 < t \le \min(s, 0)$,*

$$\|\psi_h - \psi\|_{H^t(\Gamma_0)} \le c\{h^{s-t}\|\psi\|_{H^s(\Gamma^*)} + h^\sigma\|\psi_h - \psi\|_{\tilde{H}^{-2+\varepsilon}(\Gamma)}\}.$$

In particular, for equation (3) with regular mesh, if g, the righthand side of the equation, is in $H^{3/2}(\Gamma)$, then $\psi \in H^{1/2}(\Gamma^*) \cap \tilde{H}^{-\varepsilon}(\Gamma)$ for some $\varepsilon > 0$ (see, e.g., [**8**, Lemma 4.1, **19**, Lemma 4.1]); therefore, we have, in the $H^{-1/2}$-norm, convergence of order $O(h^{1-2\varepsilon})$ locally, compared to order $O(h^{1/2-\varepsilon})$ globally, and in the L^2-norm, convergence of order $O(h^{1/2-\varepsilon})$ locally, whereas the global L^2-norm of the error is undefined. Meanwhile, for equation (5) using graded mesh with $\varrho = 3$, if $g \in H^2(\Gamma)$, then $\psi \in H^1(\Gamma^*) \cap \tilde{H}^{-\varepsilon}(\Gamma)$ (see, e.g., [**8**, Lemma 4.1, **19**, Lemma 4.1]). Hence, there are local convergence in orders $O(h^{3/2})$ and $O(h)$ in the $H^{-1/2}$-norm and L^2-norm, respectively. The global convergence in the $H^{-1/2}$-norm is of order $O(h^{3/2-\varepsilon})$.

Theorem 3.5. *Let $\varepsilon > 0$ be given. Assume that the solution of equation* (1) *satisfies $\psi \in H^2(\Gamma^*) \cap Z^{1-\varepsilon}(\Gamma)$. Let $\psi_h \in Z_h(\Gamma)$ be the solution of* (10). *If $-2+\varepsilon \le t \le 1$, then*

$$\|\psi_h - \psi\|_{H^t(\Gamma_0)} \le c\{h^{2-t}\|\psi\|_{H^2(\Gamma^*)} + h^{3-2\varepsilon+\sigma}\|\psi\|_{Z^{1-\varepsilon}(\Gamma)}\},$$

where

$$\sigma = \begin{cases} 0 & \text{if } -2+\varepsilon \le t \le -1/2, \\ -t-1/2 & \text{if } -1/2 < t \le 1. \end{cases}$$

In particular, in the $H^{-1/2}$-norm and L^2-norm, we have local convergence of order $O(h^{5/2})$ and $O(h^2)$, respectively, compared to the global order $O(h^{3/2-\varepsilon})$ and $O(h^{1-\varepsilon})$, respectively. Moreover, we have local convergence of order $O(h)$ in the H^1-norm, which is not achieved in the global sense.

We follow the same approach as in [**14, 19**]. The proofs for Theorems 3.4 and 3.5 are similar, with that for Theorem 3.5 slightly more complicated since for the latter we will use the augmented finite element space instead of the standard one. We thus prove only Theorem 3.5.

Proof. Introducing the notation $\tilde{v} = \omega_0 v$ for any function v, we decompose the error $\tilde{e} = \tilde{\psi} - \tilde{\psi}_h$ as

$$\tilde{e} = (\tilde{\psi} - G\tilde{\psi}) + (G\tilde{\psi} - G\tilde{\psi}_h) + (G\tilde{\psi}_h - \tilde{\psi}_h), \tag{14}$$

and estimate each of the terms in the parentheses separately. Here, for any v, Gv is the Galerkin approximation to v, i.e., $Gv \in S_h^{1,0}(\Gamma)$ satisfies

$$\langle V(Gv - v), \varphi\rangle = 0 \quad \text{for any } \varphi \in S_h^{1,0}(\Gamma). \tag{15}$$

Step 1. Estimate $\|\tilde{\psi} - G\tilde{\psi}\|_{\tilde{H}^{-1/2}(\Gamma)}$. Note that the cutoff function χ in the expansion (7) is chosen such that it vanishes on Γ^*. Also note that $\tilde\psi \in \tilde H^2(\Gamma)$. Then, from Theorem C, the fact that $\overset{\circ}{S}{}_h^{2,1}(\Gamma) \subset Z_h(\Gamma)$ and Lemma 3.1, we obtain

$$\begin{aligned}\|\tilde\psi - G\tilde\psi\|_{\tilde H^{-1/2}(\Gamma)} &\le c\inf\{\|\tilde\psi-\eta\|_{\tilde H^{-1/2}(\Gamma)} : \eta \in Z_h(\Gamma)\}\\ &\le c\inf\{\|\tilde\psi-\eta\|_{\tilde H^{-1/2}(\Gamma)} : \eta \in \overset{\circ}{S}{}_h^{2,1}(\Gamma)\}\\ &\le ch^{5/2}\|\tilde\psi\|_{\tilde H^{2}(\Gamma)} \le ch^{5/2}\|\psi\|_{H^{2}(\Gamma^*)}.\end{aligned} \tag{16}$$

Step 2. Estimate $\|G(\tilde\psi-\tilde\psi_h)\|_{\tilde H^{-1/2}(\Gamma)}$. It is known from the stability condition that

$$\|G\tilde e\|_{\tilde H^{-1/2}(\Gamma)} \le c \sup_{\varphi\in Z_h(\Gamma)} \frac{|\langle VG\tilde e,\varphi\rangle|}{\|\varphi\|_{\tilde H^{-1/2}(\Gamma)}}. \tag{17}$$

From (15), we deduce, for any $\varphi \in S_h^{1,0}(\Gamma)$,

$$\langle VG\tilde e,\varphi\rangle = \langle V\tilde e,\varphi\rangle = \langle Ve,\tilde\varphi\rangle + \langle [V,\omega_0]e,\varphi\rangle, \tag{18}$$

where $[V,\omega_0] = V\omega_0 - \omega_0 V$. From the definition (9) we note that for any $\varphi \in Z_h(\Gamma)$, there exists $\varphi_r \in S_h^{2,1}(\Gamma)$ such that $\varphi_r = \varphi$ on Γ^*. Hence $\tilde\varphi = \tilde\varphi_r$. By using Lemma 3.3 for φ_r we can choose $\zeta \in \overset{\circ}{S}{}_h^{2,1}(\Gamma)$ with $\operatorname{supp}\zeta \subset \Gamma_2$ such that

$$\begin{aligned}\|\tilde\varphi-\zeta\|_{\tilde H^{-1/2}(\Gamma)} &= \|\tilde\varphi_r-\zeta\|_{\tilde H^{-1/2}(\Gamma)}\\ &\le ch\|\varphi_r\|_{H^{-1/2}(\Gamma_1)}\\ &= ch\|\varphi\|_{H^{-1/2}(\Gamma_1)}\\ &\le ch\|\varphi\|_{\tilde H^{-1/2}(\Gamma)}.\end{aligned} \tag{19}$$

Since $\overset{\circ}{S}{}_h^{2,1}(\Gamma) \subset Z_h(\Gamma)$ and since $\omega_2 \equiv 1$ on $\operatorname{supp}(\tilde\varphi-\zeta)$, equation (10) implies

$$\begin{aligned}\langle Ve,\tilde\varphi\rangle &= \langle Ve,\tilde\varphi-\zeta\rangle = \langle \omega_2 Ve,\tilde\varphi-\zeta\rangle\\ &= \langle \omega_2 V\omega_3 e,\tilde\varphi-\zeta\rangle + \langle \omega_2 V(1-\omega_3)e,\tilde\varphi-\zeta\rangle.\end{aligned}$$

Since $\omega_2(1-\omega_3)\equiv 0$, the kernel of $\omega_2 V(1-\omega_3)$ is a C^∞ function, and hence from the Cauchy-Schwarz inequality and (19), we infer

$$\begin{aligned}|\langle Ve,\tilde\varphi\rangle| \le c\{&\|\omega_2 V\omega_3 e\|_{H^{1/2}(\Gamma)}\\ &+ \|\omega_2 V(1-\omega_3)e\|_{H^{1/2}(\Gamma)}\}\|\tilde\varphi-\zeta\|_{\tilde H^{-1/2}(\Gamma)}\\ \le ch\{&\|e\|_{H^{-1/2}(\Gamma^*)} + \|e\|_{\tilde H^{-2+\varepsilon}(\Gamma)}\}\|\varphi\|_{\tilde H^{-1/2}(\Gamma)}.\end{aligned} \tag{20}$$

The last term of (18) can be rewritten as

$$\begin{aligned}\langle [V,\omega_0]e,\varphi\rangle &= \langle [V,\omega_0]\omega_2 e,\varphi\rangle + \langle [V,\omega_0](1-\omega_2)e,\varphi\rangle\\ &= \langle [V,\omega_0]\omega_2 e,\varphi\rangle - \langle \omega_0 V(1-\omega_2)e,\varphi\rangle.\end{aligned}$$

Since $[V,\omega_0]$ is a pseudo-differential operator of order -2 (see [**16**]) and the kernel of $\omega_0 V(1-\omega_2)$ is a C^∞-function, we obtain, by using the Cauchy-Schwarz inequality,

$$\begin{aligned}|\langle [V,\omega_0]e,\varphi\rangle| &\le \{\|[V,\omega_0]\omega_2 e\|_{H^{1/2}(\Gamma)}\\ &\qquad + \|\omega_0 V(1-\omega_2)e\|_{H^{1/2}(\Gamma)}\}\|\varphi\|_{\tilde H^{-1/2}(\Gamma)}\\ &\le c\{\|e\|_{H^{-3/2}(\Gamma^*)} + \|e\|_{\tilde H^{-2+\varepsilon}(\Gamma)}\}\|\varphi\|_{\tilde H^{-1/2}(\Gamma)}.\end{aligned}\tag{21}$$

Inequalities (17), (18), (20) and (21) now give

$$\begin{aligned}\|G(\tilde\psi - \tilde\psi_h)\|_{\tilde H^{-1/2}(\Gamma)} &\le c\{h\|e\|_{H^{-1/2}(\Gamma^*)}\\ &\qquad + \|e\|_{H^{-3/2}(\Gamma^*)} + \|e\|_{\tilde H^{-2+\varepsilon}(\Gamma)}\}.\end{aligned}\tag{22}$$

Step 3. Estimate $\|G\tilde\psi_h - \tilde\psi_h\|_{\tilde H^{-1/2}(\Gamma)}$. From the definition (9), there exists $\psi_{h,r} \in \mathring{S}_h^{2,1}(\Gamma)$ such that $\psi_{h,r} = \psi_h$ on Γ^*. We then deduce from Theorem C, the fact that $\mathring{S}_h^{2,1}(\Gamma) \subset Z_h(\Gamma)$, and Lemma 3.3 that

$$\begin{aligned}\|G\tilde\psi_h - \tilde\psi_h\|_{\tilde H^{-1/2}(\Gamma)} &\le c\inf\{\|\tilde\psi_h - \eta\|_{\tilde H^{-1/2}(\Gamma)} : \eta \in Z_h(\Gamma)\}\\ &\le c\inf\{\|\tilde\psi_h - \eta\|_{\tilde H^{-1/2}(\Gamma)} : \eta \in \mathring{S}_h^{2,1}(\Gamma)\}\\ &= c\inf\{\|\tilde\psi_{h,r} - \eta\|_{\tilde H^{-1/2}(\Gamma)} : \eta \in \mathring{S}_h^{2,1}(\Gamma)\}\\ &\le ch^{5/2}\|\psi_{h,r}\|_{H^1(\Gamma_1)} = ch^{5/2}\|\psi_h\|_{H^1(\Gamma_1)}.\end{aligned}\tag{23}$$

Note that for any $\phi \in S_h^{2,1}(\Gamma)$, we have

$$\begin{aligned}\|\psi_h\|_{H^1(\Gamma_1)} &\le \|\omega_1\psi_h\|_{\tilde H^1(\Gamma)}\\ &\le \|\omega_1(\psi_h - \phi)\|_{\tilde H^1(\Gamma)} + \|\omega_1\phi\|_{\tilde H^1(\Gamma)}\\ &\le \|\omega_1(\psi_h - \phi)\|_{\tilde H^1(\Gamma)} + \|\phi\|_{\tilde H^1(\Gamma)}\\ &\le \|\omega_1(\psi_h - \phi)\|_{\tilde H^1(\Gamma)} + \|\omega_2\psi - \phi\|_{\tilde H^1(\Gamma)} + \|\omega_2\psi\|_{\tilde H^1(\Gamma)}.\end{aligned}$$

We can choose $\phi \in S_h^{2,1}(\Gamma)$ such that

$$
\begin{aligned}
&\|\omega_2\psi - \phi\|_{\tilde{H}^t(\Gamma)} \le ch^{s-t}\|\psi\|_{H^s(\Gamma_3)} \\
&\quad \text{for } -1/2 \le t \le s \le 2, \quad t \le 1.
\end{aligned} \tag{24}
$$

Moreover, since $\psi_h = \psi_{h,r}$ on Γ^*, by using the inverse property (Lemma 3.2) we have

$$
\begin{aligned}
\|\omega_1(\psi_h - \phi)\|_{\tilde{H}^1(\Gamma)} &= \|\omega_1(\psi_{h,r} - \phi)\|_{\tilde{H}^1(\Gamma)} \\
&\le ch^{-3/2}\|\psi_{h,r} - \phi\|_{H^{-1/2}(\Gamma_2)} \\
&= ch^{-3/2}\|\psi_h - \phi\|_{H^{-1/2}(\Gamma_2)}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|\psi_h\|_{H^1(\Gamma_1)} &\le c\{h^{-3/2}\|\psi_h - \phi\|_{H^{-1/2}(\Gamma_2)} + \|\psi\|_{H^1(\Gamma_3)}\} \\
&\le c\{h^{-3/2}\|e\|_{H^{-1/2}(\Gamma_2)} + h^{-3/2}\|\psi - \phi\|_{H^{-1/2}(\Gamma_2)} \\
&\qquad + \|\psi\|_{H^1(\Gamma_3)}\} \\
&\le c\{h^{-3/2}\|e\|_{H^{-1/2}(\Gamma_2)} + h^{-3/2}\|\omega_2\psi - \phi\|_{\tilde{H}^{-1/2}(\Gamma)} \\
&\qquad + \|\psi\|_{H^1(\Gamma_3)}\} \\
&\le c\{h^{-3/2}\|e\|_{H^{-1/2}(\Gamma_2)} + \|\psi\|_{H^1(\Gamma_3)}\}.
\end{aligned} \tag{25}
$$

Inequalities (23) and (25) give

$$
\|G\tilde{\psi}_h - \tilde{\psi}_h\|_{\tilde{H}^{-1/2}(\Gamma)} \le c\{h^{5/2}\|\psi\|_{H^1(\Gamma^*)} + h\|e\|_{H^{-1/2}(\Gamma^*)}\}. \tag{26}
$$

Combining inequalities (16), (22) and (26) given in Steps 1, 2 and 3, we obtain

$$
\begin{aligned}
\|e\|_{H^{-1/2}(\Gamma_0)} \le c\{&h^{5/2}\|\psi\|_{H^2(\Gamma^*)} + h\|e\|_{H^{-1/2}(\Gamma^*)} \\
&+ \|e\|_{H^{-3/2}(\Gamma^*)} + \|e\|_{\tilde{H}^{-2+\varepsilon}(\Gamma)}\}.
\end{aligned} \tag{27}
$$

Step 4. Estimate $\|e\|_{H^t(\Gamma_0)}$ for $-2+\varepsilon \le t \le -1/2$. We have by the definition of the Sobolev norms

$$
\|\tilde{e}\|_{\tilde{H}^t(\Gamma)} = \sup_{w \in H^{-t}(\Gamma)} \frac{|\langle \tilde{e}, w\rangle|}{\|w\|_{H^{-t}(\Gamma)}}. \tag{28}
$$

For any $w \in H^{-t}(\Gamma)$, $1/2 \le -t \le 2-\varepsilon$, there exists $y \in Z^{-t-1}(\Gamma)$ such that $Vy = w$ and that

$$\|y\|_{Z^{-t-1}(\Gamma)} \le c\|w\|_{H^{-t}(\Gamma)},$$

(see [**15**]). Moreover, since V is symmetric we can write

$$\langle \tilde{e}, w\rangle = \langle \tilde{e}, Vy\rangle = \langle V\tilde{e}, y\rangle = \langle Ve, \tilde{y}\rangle + \langle [V,\omega_0]e, y\rangle. \tag{29}$$

Since $\tilde{y} \in \tilde{H}^{-t-1}(\Gamma)$, see definition (7), there exists $\zeta \in \overset{\circ}{S}{}_h^{2,1}(\Gamma)$ with $\operatorname{supp}\zeta \subset \Gamma_2$ such that

$$\begin{aligned} \|\tilde{y}-\zeta\|_{\tilde{H}^{-1/2}(\Gamma)} &\le ch^{-t-1/2}\|\tilde{y}\|_{\tilde{H}^{-t-1}(\Gamma)} \\ &\le ch^{-t-1/2}\|\tilde{y}_r\|_{\tilde{H}^{-t-1}(\Gamma)} \\ &\le ch^{-t-1/2}\|y\|_{Z^{-t-1}(\Gamma)}, \end{aligned} \tag{30}$$

where the last two inequalities are obtained from the definition (7) (when $-t-1<0$ we will take $y_r = y$ on Γ). Hence, the first term on the righthand side of (29) can be estimated as

$$\begin{aligned} |\langle Ve, \tilde{y}\rangle| &= |\langle Ve, \tilde{y}-\zeta\rangle| = |\langle \omega_2 Ve, \tilde{y}-\zeta\rangle| \\ &\le |\langle \omega_2 V\omega_3 e, \tilde{y}-\zeta\rangle| + \langle \omega_2 V(1-\omega_3)e, \tilde{y}-\zeta\rangle| \\ &\le (\|\omega_2 V\omega_3 e\|_{H^{1/2}(\Gamma)} \\ &\qquad + \|\omega_2 V(1-\omega_3)e\|_{H^{1/2}(\Gamma)})\|\tilde{y}-\zeta\|_{\tilde{H}^{-1/2}(\Gamma)} \\ &\le ch^{-t-1/2}(\|e\|_{H^{-1/2}(\Gamma^*)} + \|e\|_{\tilde{H}^{-2+\varepsilon}(\Gamma)})\|y\|_{Z^{-t-1}(\Gamma)} \\ &\le ch^{-t-1/2}(\|e\|_{H^{-1/2}(\Gamma^*)} + \|e\|_{\tilde{H}^{-2+\varepsilon}(\Gamma)})\|w\|_{H^{-t}(\Gamma)}. \end{aligned} \tag{31}$$

The last term of (29) can be estimated as:

$$\begin{aligned} |\langle [V,\omega_0]e, y\rangle| &= |\langle [V,\omega_0]\omega_2 e, y\langle + \langle [V,\omega_0](1-\omega_2)e, y\rangle| \\ &= \langle [V,\omega_0]\omega_2 e, y\langle - \langle \omega_0 V(1-\omega_2)e, y\rangle| \\ &\le (\|[V,\omega_0]\omega_2 e\|_{H^s(\Gamma)} + \|\omega_0 V(1-\omega_2)e\|_{H^2(\Gamma)})\|y\|_{\tilde{H}^{-s}(\Gamma)} \\ &\le c(\|e\|_{H^{s-2}(\Gamma^*)} + \|e\|_{\tilde{H}^{-2+\varepsilon}(\Gamma)})\|w\|_{H^{-s+1}(\Gamma)} \\ &\le c(\|e\|_{H^{s-2}(\Gamma^*)} + \|e\|_{\tilde{H}^{-2+\varepsilon}(\Gamma)})\|w\|_{H^{-t}(\Gamma)}, \end{aligned} \tag{32}$$

where $s = \max(t+1, \varepsilon)$ and therefore $-1/2 \le -s \le -\varepsilon$. Inequalities (28), (29), (31) and (32) now give

$$\|e\|_{H^t(\Gamma_0)} \le c\{h^{-t-1/2}\|e\|_{H^{-1/2}(\Gamma^*)} + \|e\|_{H^\tau(\Gamma^*)} + \|e\|_{\tilde{H}^{-2+\varepsilon}(\Gamma)}\}, \tag{33}$$

where $\tau = \max(t-1, -2+\varepsilon)$.

Step 5. An explicit estimate in the $H^{-1/2}$ norm. From (27), we have

$$\begin{aligned}\|e\|_{H^{-1/2}(\Gamma_0)} \le c\{&h^{5/2}\|\psi\|_{H^2(\Gamma^*)} + h\|e\|_{H^{-1/2}(\Gamma_1)} \\ &+ \|e\|_{H^{-3/2}(\Gamma_1)} + \|e\|_{\tilde{H}^{-2+\varepsilon}(\Gamma)}\}.\end{aligned}$$

Using (33) with $t = -3/2$, we then deduce

$$\|e\|_{H^{-3/2}(\Gamma_1)} \le c\{h\|e\|_{H^{-1/2}(\Gamma_2)} + \|e\|_{\tilde{H}^{-2+\varepsilon}(\Gamma)}\}.$$

Hence,

$$\|e\|_{H^{-1/2}(\Gamma_0)} \le c\{h^{5/2}\|\psi\|_{H^2(\Gamma^*)} + h\|e\|_{H^{-1/2}(\Gamma_1)} + \|e\|_{\tilde{H}^{-2+\varepsilon}(\Gamma)}\}. \tag{34}$$

Repeated use of (34) for $\|e\|_{H^{-1/2}(\Gamma_1)}$ then gives

$$\|e\|_{H^{-1/2}(\Gamma_0)} \le c\{h^{5/2}\|\psi\|_{H^2(\Gamma^*)} + h^J\|e\|_{H^{-1/2}(\Gamma_J)} + \|e\|_{\tilde{H}^{-2+\varepsilon}(\Gamma)}\}.$$

With J chosen sufficiently large so that

$$h^J\|e\|_{H^{-1/2}(\Gamma_J)} \le c\{h^{5/2}\|\psi\|_{H^2(\Gamma^*)} + \|e\|_{\tilde{H}^{-2+\varepsilon}(\Gamma)}\}, \tag{35}$$

we will have

$$\|e\|_{H^{-1/2}(\Gamma_0)} \le c\{h^{5/2}\|\psi\|_{H^2(\Gamma^*)} + \|e\|_{\tilde{H}^{-2+\varepsilon}(\Gamma)}\}. \tag{36}$$

This can be done by using Lemma 3.2:

$$\begin{aligned}h^J\|e\|_{H^{-1/2}(\Gamma_J)} &\le h^J\{\|\psi\|_{H^{-1/2}(\Gamma_J)} + \|\psi_h\|_{H^{-1/2}(\Gamma_J)}\} \\ &\le h^J\{\|\psi\|_{H^2(\Gamma^*)} + \|\omega_J\psi_h\|_{\tilde{H}^{-1/2}(\Gamma)}\} \\ &\le h^J\{\|\psi\|_{H^2(\Gamma^*)} + h^{-3/2+\varepsilon}\|\psi_h\|_{H^{-2+\varepsilon}(\Gamma_{J+1})}\} \\ &\le h^{J-3/2+\varepsilon}\{\|\psi\|_{H^2(\Gamma^*)} + \|e\|_{\tilde{H}^{-2+\varepsilon}(\Gamma)}\}.\end{aligned}$$

By choosing J so that $J-3/2+\varepsilon \geq 5/2$, we obtain the required estimate (35) and hence (36).

Step 6. Estimates in other norms. First, consider the case $-2+\varepsilon \leq t \leq -1/2$. Inequalities (33) and (36) give

$$\|e\|_{H^t(\Gamma_0)} \leq c\{h^{2-t}\|\psi\|_{H^2(\Gamma^*)} + \|e\|_{H^\tau(\Gamma_1)} + \|e\|_{\tilde{H}^{-2+\varepsilon}(\Gamma)}\},$$

where $\tau = \max(t-1, -2+\varepsilon)$. Repeating the same argument for the middle term on the righthand side, we can eliminate that term and then use Theorem C to obtain

$$\|e\|_{H^t(\Gamma_0)} \leq c\{h^{2-t}\|\psi\|_{H^2(\Gamma^*)} + h^{3-2\varepsilon}\|\psi\|_{Z^{1-\varepsilon}(\Gamma)}\}.$$

Finally, consider the case $-1/2 < t \leq 1$. Let $\phi \in S_h^{2,1}(\Gamma)$ be defined as in (24). By using Lemma 3.2, noting that $\omega_i \equiv 1$ on Γ_i, we obtain

$$\begin{aligned}
\|e\|_{H^t(\Gamma_0)} &\leq \|\omega_2\psi - \phi\|_{H^t(\Gamma_0)} + \|\psi_h - \phi\|_{H^t(\Gamma_0)} \\
&\leq \|\omega_2\psi - \phi\|_{\tilde{H}^t(\Gamma)} + \|\omega_0(\psi_h - \phi)\|_{\tilde{H}^t(\Gamma)} \\
&\leq c\{h^{2-t}\|\psi\|_{H^2(\Gamma^*)} + h^{-t-1/2}\|\psi_h - \phi\|_{H^{-1/2}(\Gamma_1)}\} \\
&\leq c\{h^{2-t}\|\psi\|_{H^2(\Gamma^*)} + h^{-t-1/2}(\|\psi - \phi\|_{H^{-1/2}(\Gamma_1)} \\
&\qquad + \|e\|_{H^{-1/2}(\Gamma_1)})\} \\
&\leq c\{h^{2-t}\|\psi\|_{H^2(\Gamma^*)} + h^{-t-1/2}(\|\omega_2\psi - \phi\|_{\tilde{H}^{-1/2}(\Gamma)} \\
&\qquad + \|e\|_{H^{-1/2}(\Gamma_1)})\} \\
&\leq c\{h^{2-t}\|\psi\|_{H^2(\Gamma^*)} + h^{-t-1/2}\|e\|_{H^{-1/2}(\Gamma_1)}\}.
\end{aligned} \tag{37}$$

Using (36) and Theorem C for the last term of (37), we achieve

$$\begin{aligned}
\|e\|_{H^t(\Gamma_0)} &\leq c\{h^{2-t}\|\psi\|_{H^2(\Gamma^*)} + h^{-t-1/2}\|e\|_{\tilde{H}^{-2+\varepsilon}(\Gamma)}\} \\
&\leq c\{h^{2-t}\|\psi\|_{H^2(\Gamma^*)} + h^{3-2\varepsilon-t-1/2}\|\psi\|_{Z^{1-\varepsilon}(\Gamma)}\}.
\end{aligned}$$

The theorem is proved. □

The following local estimates for the Galerkin schemes to approximate the solution of (2) can be proved in the same manner if we note that

D and $[D, \omega_0] = D\omega_0 - \omega_0 D$ are pseudo-differential operators of orders 1 and 0, respectively.

Theorem 3.6. *Assume that the solution of* (2) *satisfies* $\xi \in H^s(\Gamma^*) \cap \tilde{H}^{1/2}(\Gamma)$ *for some* $s \in (1/2, 2]$. *For any* $t \in \mathbf{R}$, *let*

$$\sigma = \sigma(t) = \begin{cases} 0 & \text{if } t \leq 1/2, \\ -t + 1/2 & \text{if } t > 1/2. \end{cases}$$

Let $\varepsilon > 0$ *be given, sufficiently small.*

(i) *If* ξ_h *is the solution of* (4), *then for any* t *with* $0 < t \leq \min(s, 1)$,

$$\|\xi_h - \xi\|_{H^t(\Gamma_0)} \leq c\{h^{s-t}\|\xi\|_{H^s(\Gamma^*)} + h^\sigma \|\xi_h - \xi\|_{\tilde{H}^\varepsilon(\Gamma)}\}.$$

(ii) *If* ξ_h *is the solution of* (6), *then for any* t *with* $-1 < t \leq \min(s, 1)$,

$$\|\xi_h - \xi\|_{H^t(\Gamma_0)} \leq c\{h^{s-t}\|\xi\|_{H^s(\Gamma^*)} + h^\sigma \|\xi_h - \xi\|_{\tilde{H}^{-2+\varepsilon}(\Gamma)}\}.$$

Theorem 3.7. *Let* $\varepsilon > 0$ *be given. Assume that the solution of equation* (2) *satisfies* $\xi \in H^3(\Gamma^*) \cap Y^{2-\varepsilon}(\Gamma)$. *Let* $\xi_h \in Y_h(\Gamma)$ *be the solution of* (11). *If* $-1 + \varepsilon \leq t \leq 2$, *then*

$$\|\xi_h - \xi\|_{H^t(\Gamma_0)} \leq c\{h^{3-t}\|\xi\|_{H^3(\Gamma^*)} + h^{3-2\varepsilon+\sigma}\|\xi\|_{Y^{2-\varepsilon}(\Gamma)}\},$$

where

$$\sigma = \begin{cases} 0 & \text{if } -1 + \varepsilon \leq t \leq 1/2, \\ -t + 1/2 & \text{if } 1/2 < t \leq 2. \end{cases}$$

4. A post-processing method. In the above section we see that in the case of quasi-uniform meshes the highest order of convergence ($O(h^{1-2\varepsilon})$) achieved for the approximation of equation (1) is in a negative norm ($H^{-1+\varepsilon}$ norm), which is not easily observed. In this section we shall use the K-operator (see [**2, 4, 17, 18, 20, 21**]) as a post-processing for the Galerkin solution ψ_h of (1) so as to achieve that order locally in the L^2-norm. For simplicity, we only consider the case where Γ is a domain in $\mathbf{R}^2$.

As in [**2, 4**], we define the spline $K_h = K_{h,l}^{2q}$ with integers l, q as

$$K_h(x) = \prod_{i=1}^{2} \sum_{j=-(q-1)}^{q-1} h^{-1} k_j \psi^{(l)}(x_i/h - j), \qquad x = (x_1, x_2), \tag{38}$$

where $\psi^{(l)}$ is a B-spline of order l defined by

$$\psi^{(l)} = \chi * \chi * \cdots * \chi, \quad \text{convolution } l-1 \text{ times},$$

with

$$\chi(t) = \begin{cases} 1 & \text{for } |t| \le 1/2, \\ 0 & \text{for } |t| > 1/2, \end{cases}$$

for any real value of t. The coefficients k_j, $j = -(q-1), \dots, q-1$ are chosen such that, for any v,

$$\begin{gathered} \|v - K_h * v\|_{L^2(\Gamma_i)} \le c h^s \|v\|_{H^s(\Gamma_{i+1})}, \\ 0 \le s \le 2q, \quad i = 0, \dots, J-1. \end{gathered} \tag{39}$$

The existence and uniqueness of k_j, $j = -(q-1), \dots, q-1$ is proved in [**3**].

Now for any v defined on Γ, we extend v by 0 onto $\mathbf{R}^2 \backslash \Gamma$, denote the extension by $\tilde{v}$, and then define K-operator as

$$\begin{aligned} [K_h(v)](x) &:= (K_h * \tilde{v})(x) \\ &= h^{-2} \int_{\mathbf{R}^2} \left\{ \prod_{i=1}^{2} \sum_{j=-(q-1)}^{q-1} k_j \psi^{(l)} \left(\frac{x_i - y_i}{h} - j \right) \right\} \tilde{v}(y)\, dy. \end{aligned} \tag{40}$$

We have the following theorem:

Theorem 4.1. (i) *Assume that the mesh is uniform on* Γ^* *and quasi-uniform on* $\Gamma \backslash \Gamma^*$. *Let* $\psi \in \tilde{H}^{-\varepsilon}(\Gamma) \cap H^{1-\varepsilon}(\Gamma^*)$, ψ_h *be the solution of*

(3), *and* K_h *be defined by* (38) *with* $l = q = 1$. *Then there exists an* $h_0 > 0$ *such that, for* $0 < h \le h_0$,

$$\|K_h(\psi_h) - \psi\|_{L^2(\Gamma_0)} \le ch^{1-2\varepsilon}\{\|\psi\|_{H^{1-\varepsilon}(\Gamma^*)} + \|\psi\|_{\tilde{H}^{-\varepsilon}(\Gamma)}\}. \tag{41}$$

(ii) *Assume that the mesh is uniform on* Γ^* *and graded on* $\Gamma\backslash\Gamma^*$ *with* $\varrho \ge 3$. *Let* $\psi \in \tilde{H}^{-\varepsilon}(\Gamma) \cap H^{3-\varepsilon}(\Gamma_*)$, ψ_h *be the solution of* (5), *and* K_h *be defined by* (38) *with* $l = q = 2$. *Then there exists an* $h_0 > 0$ *such that, for* $0 < h \le h_0$,

$$\|K_h(\psi_h) - \psi\|_{L^2(\Gamma_0)} \le ch^{3-\varepsilon}. \tag{42}$$

Proof. (i) Let ω_* be a cut-off function satisfying

$$\omega_* \equiv 1 \quad \text{on } \Gamma_* \quad \text{and} \quad \omega_* \in C_0^\infty(\Gamma^*).$$

Let $\psi_* = \omega_*\tilde{\psi}$. Then

$$\begin{aligned}\|K_h(\psi_h) - \psi\|_{L^2(\Gamma_0)} &= \|K_h * \tilde{\psi}_h - \psi_*\|_{L^2(\Gamma_0)}\\ &\le \|K_h * (\tilde{\psi}_h - \psi_*)\|_{L^2(\Gamma_0)}\\ &\quad + \|K_h * \psi_* - \psi_*\|_{L^2(\Gamma_0)} = I + II.\end{aligned}$$

That II is bounded by the righthand side of (41) is from (39) and the local regularity of ψ. For the term I, we note that from [**4**, Lemmas 2.2 and 5.3] we have

$$\|K_h * (\tilde{\psi}_h - \psi_*)\|_{L^2(\Gamma_0)} \le c\{\|\tilde{\psi}_h - \psi_*\|_{H^{-1}(\Gamma_1)} + \|\partial_h(\tilde{\psi}_h - \psi_*)\|_{H^{-1}(\Gamma_1)}\},$$

where ∂_h is the forward difference operator defined by $\partial_h = (T_h - I)/h$ with $T_h v(x) = v(x+h)$ and $Iv(x) = v(x)$. Note that $\tilde{\psi}_h = \psi_h$ and $\psi_* = \tilde{\psi} = \psi$ on Γ_*. Moreover, since the mesh is uniform on $\Gamma^* \supset \Gamma_1$, there exists an $h_0 > 0$ such that $\partial_h\psi_* = \partial_h\tilde{\psi}$ on Γ_1 for $0 < h \le h_0$. Hence, using Theorem A we infer

$$\|K_h * (\tilde{\psi}_h - \psi_*)\|_{L^2(\Gamma_0)} \le c\{h^{1-2\varepsilon}\|\psi\|_{\tilde{H}^{-\varepsilon}(\Gamma)} + \|\partial_h(\tilde{\psi}_h - \tilde{\psi})\|_{H^{-1}(\Gamma_1)}\}. \tag{43}$$

We now estimate the last term on the righthand side of (43). Let Γ' be a plane domain containing Γ, and let $\delta = \operatorname{dist}(\Gamma', \Gamma)$. Let $V_{\Gamma'}$ be defined as V in (1) with Γ replaced by Γ'. Then for $0 < h < \delta$ and $\phi \in \overset{\circ}{S}{}_h^{1,0}(\Gamma)$ with support in Γ_1, we have

$$\begin{aligned}
&\langle V_{\Gamma'} \partial_h(\tilde{\psi}_h - \tilde{\psi}), \phi \rangle_{L^2(\Gamma')} \\
&\qquad = \frac{1}{h} \langle V_{\Gamma'} T_h(\tilde{\psi}_h - \tilde{\psi}), \phi \rangle_{L^2(\Gamma')} \\
&\qquad = -\frac{1}{4\pi h} \int_{\Gamma'} \int_{\Gamma'} \frac{1}{|x-y|} (\tilde{\psi}_h - \tilde{\psi})(y+h) \phi(x) \, dy \, dx \\
&\qquad = -\frac{1}{4\pi h} \int_{\Gamma'+h} \int_{\Gamma'+h} \frac{1}{|x-y|} (\tilde{\psi}_h - \tilde{\psi})(y) \phi(x-h) \, dy \, dx,
\end{aligned}$$

where $\Gamma' + h = \{x + h : x \in \Gamma'\}$. Since $0 < h < \delta$, and since $\tilde{\psi}_h, \tilde{\psi}$ and ϕ vanish outside Γ, we deduce

$$\begin{aligned}
&\langle V_{\Gamma'} \partial_h(\tilde{\psi}_h - \tilde{\psi}), \phi \rangle_{L^2(\Gamma')} \\
&\qquad\qquad = -\frac{1}{4\pi h} \int_{\Gamma} \int_{\Gamma} \frac{1}{|x-y|} (\tilde{\psi}_h - \tilde{\psi})(y) \phi(x-h) \, dy \, dx \\
&\qquad\qquad = \langle V(\psi_h - \psi), \phi(\cdot - h) \rangle.
\end{aligned}$$

Moreover, since the mesh is uniform on Γ^*, we can choose $h_0 < \delta$ such that if $\phi \in \overset{\circ}{S}{}_h^{1,0}(\Gamma)$ with support in Γ_1, then $\phi(\cdot - h) \in \overset{\circ}{S}{}_h^{1,0}(\Gamma)$ with support in Γ_2. Hence, by (3),

$$\langle V_{\Gamma'} \partial_h(\tilde{\psi}_h - \tilde{\psi}), \phi \rangle_{L^2(\Gamma')} = 0. \tag{44}$$

From (44) we conclude that $\partial_h \tilde{\psi}_h$ is the *interior* Galerkin approximation to $\partial_h \tilde{\psi}$ in the sense that:

1. $\partial_h \tilde{\psi}_h$ is a spline only on Γ^* (since the mesh is only uniform on Γ^*);

2. Equation (44) is satisfied only with $\phi \in \overset{\circ}{S}{}_h^{1,0}(\Gamma)$, $\operatorname{supp} \phi \subset \Gamma_1$.

However, as in the 1-dimensional case (see [**19**]), we can slightly modify the proof of Theorem 3.1 to obtain the estimate:

(45)

$$\begin{aligned}
\|\partial_h \tilde{\psi}_h - \partial_h \tilde{\psi}\|_{H^{-1}(\Gamma_1)} &\le c\{h^{1-2\varepsilon} \|\partial_h \tilde{\psi}\|_{H^{-\varepsilon}(\Gamma^*)} + \|\partial_h \tilde{\psi}_h - \partial_h \tilde{\psi}\|_{\tilde{H}^{\beta}(\Gamma')}\} \\
&\le c\{h^{1-2\varepsilon} \|\tilde{\psi}\|_{H^{1-\varepsilon}(\Gamma^*)} + \|\partial_h \tilde{\psi}_h - \partial_h \tilde{\psi}\|_{\tilde{H}^{\beta}(\Gamma')}\} \\
&= c\{h^{1-2\varepsilon} \|\psi\|_{H^{1-\varepsilon}(\Gamma^*)} + \|\partial_h \tilde{\psi}_h - \partial_h \tilde{\psi}\|_{\tilde{H}^{\beta}(\Gamma')}\},
\end{aligned}$$

where $\beta \leq -1/2$ is arbitrary but fixed. Note that we do not have a direct estimate for $\|\partial_h \tilde{\psi}_h - \partial_h \tilde{\psi}\|_{\tilde{H}^\beta(\Gamma')}$. However,

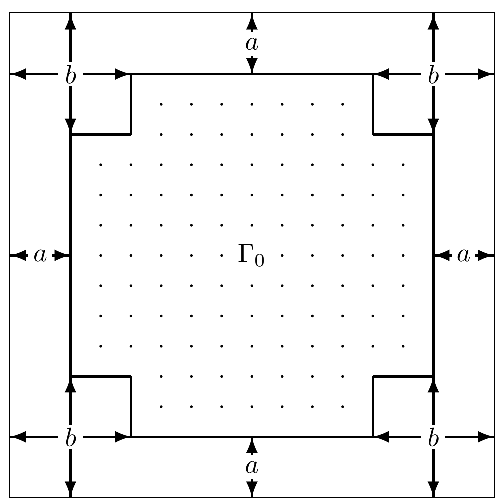
$$\|\partial_h \tilde{\psi}_h - \partial_h \tilde{\psi}\|_{\tilde{H}^\beta(\Gamma')} \leq \|\tilde{\psi}_h - \tilde{\psi}\|_{\tilde{H}^{\beta+1}(\Gamma')} \leq \|\psi_h - \psi\|_{\tilde{H}^{\beta+1}(\Gamma)}.$$

Choosing $\beta = -2$ and using Theorem A, we obtain

$$\|\partial_h \tilde{\psi}_h - \partial_h \tilde{\psi}\|_{\tilde{H}^\beta(\Gamma')} \leq ch^{1-2\varepsilon} \|\psi\|_{\tilde{H}^{-\varepsilon}(\Gamma)}. \tag{46}$$

From (43), (45) and (46) we see that I is bounded by the righthand side of (41) and thus complete the proof of (i). The proof of (ii) can proceed similarly by making use of Theorem B instead of Theorem A. $\square$

(-1,1) (1,1)



(-1,-1) (1,-1)

FIGURE 1. Tested domain for the L^2-errors.

TABLE 1. Empirical orders of convergence for the Galerkin solution.

	$a=0.10$ $b=0.10$		$a=0.10$ $b=0.20$		$a=0.20$ $b=0.20$		$a=0.20$ $b=0.40$	
	$N'=20$	$N'=25$	$N'=20$	$N'=25$	$N'=20$	$N'=25$	$N'=20$	$N'=25$
N	α_N	α_N	α_N	α_N	α_N	α_N	α_N	α_N
45	0.73	0.62	0.69	0.58	0.48	0.67	0.48	0.65
55	0.62	0.50	0.58	0.47	0.49	0.63	0.49	0.62
60	0.75	0.68	0.71	0.65	0.54	0.67	0.54	0.66
65	0.73	0.66	0.68	0.62	0.51	0.63	0.49	0.59

5. Numerical results. We carried out the numerical experiment for the weakly singular integral equation (1) for the Dirichlet screen problem with $\Gamma = [-1,1]^2$ and $g \equiv 1$. The discrete problem (3) is obtained by using piecewise-constant test and trial functions with respect to a uniform mesh of square elements with side length h. The approximate solution ψ_h can be expressed as

$$\psi_h(x) = \sum_{i,j=1}^{N} c_{i,j}\psi^{(1)}\left(\frac{x_1+1}{h} - i + 1/2\right)\psi^{(1)}\left(\frac{x_2+1}{h} - j + 1/2\right), \tag{47}$$

where $x = (x_1, x_2)$, $h = 2/N$ and where $\psi^{(l)}$ are B-splines defined in Section 4.

To test the results on interior estimates, we observed the errors on Γ_0 where Γ_0 is the shaded domain given in Figure 1. Since the exact solution of the equation (1) is not known for the given data $g \equiv 1$, we compare ψ_h with the Galerkin solution ψ_{h_0}, where $h_0 = 2/N$ with $N = 240$, assuming that

$$\|\psi_h - \psi_{h_0}\|_{L^2(\Gamma_0)} \sim \|\psi_h - \psi\|_{L^2(\Gamma_0)}.$$

This can be done since the convergence $\psi_h \to \psi$ is guaranteed by Theorem A. In Table 1, we list the experimental convergence rate α_N computed as

$$\alpha_N = \frac{\log(e_{N'}/e_N)}{\log(N/N')},$$

where N' is a fixed integer and $e_N = \|\psi_h - \psi_{h_0}\|_{L^2(\Gamma_0)}$ for any integer N. It can be seen from this table that local convergence of order $O(h^{1/2})$ in the L^2-norm is achieved as we predicted from our analysis.

To test the effectiveness of the K-operator, we observed the error $\|K_h * \tilde{\psi}_h - \psi\|_{L^2(\Gamma_d)}$ where $\Gamma_d = [-1+d, 1-d]^2$ for $0 < d < 1$ and where $K_h = K^2_{h,1} = (1/h)\psi^{(1)}(x/h)$. Recall that $\tilde{\psi}_h$ is the extension of ψ_h by 0 outside Γ. Since K_h and $\tilde{\psi}_h$ (which is also expressible by (47)) are piecewise-constant functions, the convolution $K_h * \tilde{\psi}_h$ is piecewise bilinear and can be computed as

$$K_h * \tilde{\psi}_h(x) = \sum_{i,j=1}^{N} c_{i,j}\psi^{(2)}\left(\frac{x_1+1}{h} - i + 1/2\right)\psi^{(2)}\left(\frac{x_2+1}{h} - j + 1/2\right),$$

where again $x = (x_1, x_2)$ and $h = 2/N$. We assume here again that

$$e^*_{N_i} := \|K_{h_i} * \tilde{\psi}_{h_i} - \psi_{h_0}\|_{L^2(\Gamma_d)} \sim \|K_{h_i} * \tilde{\psi}_{h_i} - \psi\|_{L^2(\Gamma_d)},$$

where $h_0 = 2/N$ with $N = 240$ and $h_i = 2/N_i$, $i = 1, 2, \ldots$. The empirical order of convergence is now given by

$$\alpha^*_{N_i} = \frac{\log(e_{N_i}/e_{N_{i-1}})}{\log(N_{i-1}/N_i)}, \qquad i = 2, 3, \ldots .$$

The results given in Table 2 show us that the post-processing Galerkin solution gives local convergence of order almost $O(h)$ in the L^2-norm, which matches our analysis.

TABLE 2. Errors and empirical orders of convergence for the post-processing Galerkin solution.

	$d = 0.05$		$d = 0.10$		$d = 0.20$		$d = 0.40$		$d = 0.70$	
N	e^*_N	α^*_N	e^*_N	α^*_N	e^*_N	α^*_N	e^*_N	α^*_N	e^*_N	α^*_N
20	4.21		1.68		2.65e-1		7.13e-2		2.81e-2	
30	3.25	0.64	5.61e-1	2.70	1.21e-1	1.93	4.99e-2	0.88	1.87e-2	1.00
40	1.79	2.08	2.90e-1	2.29	8.04e-2	1.42	3.74e-2	1.00	1.37e-2	1.08
50	8.36e-1	3.41	1.74e-1	2.29	6.76e-2	0.78	2.96e-2	1.05	1.06e-2	1.15
60	6.21e-1	1.63	1.39e-1	1.25	5.57e-2	0.85	2.44e-2	1.04	8.53e-3	1.20

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