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EFFICIENT ALGORITHMS FOR THE p-VERSION OF THE BOUNDARY ELEMENT METHOD

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ABSTRACT. We investigate the p-version of the boundary element Galerkin method for a first kind integral equation. We present a-priori driven algorithms which yield sparse Galerkin matrices and do not destroy the convergence properties of the boundary element method. Further, we show that the additive Schwarz method is nearly an optimal preconditioner for the Galerkin systems. Numerical results confirm the efficiency of our methods.

1. Introduction. The p-version of the boundary element method (BEM) is known to be very efficient in view of its convergence properties. For problems with singularities it converges twice as fast as the usual h-version, see, e.g., [**13**]. To exploit these advantages in practice one has also to take care of an efficient implementation. One aspect is the treatment of generally fully occupied system matrices which are characteristic of the BEM. Another aspect is the fast solution of the linear systems which is of course not peculiar to the BEM. In case of the h-version these aspects have been investigated by several authors, see, e.g., [**1, 8, 7, 20, 18**]. In contrast, the structures of the system matrices of the p-version are not known to be under investigation so far. Also the construction of optimal preconditioners for the p-version has just started to be under investigation, see [**15**].

For simplicity we will concentrate on the weakly singular integral equation

$$V\Phi(x) = g(x), \qquad x \in \Gamma \tag{1}$$

where V is the single layer operator defined as

$$V\Phi(x) := -\frac{1}{\pi} \int_\Gamma \Phi(y) \log |x - y| \, ds_y.$$

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Here Γ is the boundary of a polygonal domain $\Omega \subset \mathbf{R}^2$. To ensure injectivity we assume $\mathrm{cap}(\Gamma) \neq 1$. Equation (1) models an interior Dirichlet problem for the Laplacian in Ω.

Before introducing the boundary element method let us recall the definition of the Sobolev spaces, see, e.g., [**12**]. Let J denote a straight line covered by an edge of the polygon Γ.

$$
\begin{aligned}
H^s(\Omega) &= \{\phi|_\Omega\,;\ \phi \in H^s(\mathbf{R}^2)\}, \quad s \in \mathbf{R},\\
H^s(\Gamma) &= \begin{cases} \{\phi|_\Gamma\,;\ \phi \in H^{s+1/2}(\mathbf{R}^2)\} & s > 0,\\ L^2(\Gamma) & s = 0,\\ (H^{-s}(\Gamma))' \quad \text{(dual space)} & s < 0, \end{cases}\\
H^s(J) &= \{\phi|_J\,;\ \phi \in H^s(\Gamma)\} \quad s \geq 0,\\
\tilde{H}^s(J) &= \{\phi \in H^s(J)\,;\ \tilde{\phi} \in H^s(\Gamma)\} \qquad s \geq 0.
\end{aligned}
$$

Here

$$
\tilde{\phi} = \begin{cases} \phi & \text{on } J\\ 0 & \text{on } \Gamma\backslash J \end{cases}
$$

means the extension of ϕ by 0 outside J. Finally we define the dual spaces on J

$$
\begin{aligned}
H^s(J) &= (\tilde{H}^{-s}(J))' \qquad s < 0,\\
\tilde{H}^s(J) &= (H^{-s}(J))' \qquad s < 0.
\end{aligned}
$$

In order to solve equation (1) for a given $g \in H^{1/2}(\Gamma)$ via the Galerkin method we introduce a family of finite-dimensional subspaces H_N of $\tilde{H}^{-1/2}(\Gamma)$. Then the Galerkin method for (1) reads: *Find* $\Phi_N \in H_N$ *such that for all* $\psi \in H_N$

$$
a(\Phi_N, \psi) := \langle V\Phi_N, \psi\rangle_{L^2(\Gamma)} = \langle g, \psi\rangle_{L^2(\Gamma)}. \tag{2}
$$

The choice of specific subspaces H_N leads to various versions of the Galerkin method. The h-version keeps a low degree p (usually $p = 0$ or $p = 1$) fixed and uses piecewise polynomials of degree p on a sequence of refined meshes to approximate Φ. The p-version keeps the mesh fixed and increases uniformly the polynomial degrees. In the following we will use scaled piecewise Legendre polynomials to construct H_N. As shown in [**5**] V is a strongly elliptic operator mapping

$\tilde{H}^{-1/2}(\Gamma)$ continuously into $H^{1/2}(\Gamma)$. The results in [**17**] guarantee the quasi-optimal convergence of the Galerkin solution in the energy space $\tilde{H}^{-1/2}(\Gamma)$.

The outline of this paper is as follows. In Section 2 we study the local behavior of the single layer operator to justify a modified version of the boundary element method with sparse matrices. Therefore we present an a-priori estimate of the elements of the system matrices (Lemma 1). This estimate can be used to implement the given rule for making the matrices sparse and retaining the convergence rate of the BEM (Theorem 1). In Section 3 we show that the condition number of the additive Schwarz operator is growing only logarithmically in the degree p (Theorem 3). It is therefore a nearly optimal preconditioner for the p-version. Section 4 presents various numerical results regarding the sparsity of the system matrices and the additive Schwarz method as preconditioner for the conjugate gradient method. In this section we also give numerical results for a modified additive Schwarz preconditioner. Both types of the additive Schwarz preconditioner do not need any overlapping of the blocks and can therefore be performed in parallel very easily.

Throughout the paper C denotes a generic constant which is independent of the polynomial degree p.

2. The sparsity of the system matrix. Generally, system matrices arising from the boundary element method are fully occupied. This is disadvantageous for applying fast solvers and for efficient implementations. Here we study local properties of the single layer operator and show, in contrast to the above, that one can deal with sparse matrices instead of using the whole set of entries in the stiffness matrices. The only requirement is the use of piecewise Legendre polynomials as basis functions.

Let $l_{p,I}$ be the Legendre polynomial of degree p linearly transformed onto the open line segment $I \subset \mathbf{R}^2$. By $l^\star_{p,I}$ we denote the scaled function $\sqrt{2p+1}\, l_{p,I}$. $l^\star_{p,I}$ is assumed to be extended by 0 outside I on the entire line containing I where necessary. The usual Legendre polynomials $l_{p,(-1,1)\times\{0\}}$ are denoted by l_p. Let $J \subset \mathbf{R}^2$ be another open line segment with $\bar{I} \cap \bar{J} = \varnothing$. Then the following estimate holds.

Lemma 1. *There exists a constant C depending on Γ such that*

$$\langle V l^\star_{p_j,J}, l^\star_{p_i,I}\rangle_{L^2(I)} \le \frac{C}{\pi}\frac{\max\{|I|,\,|J|\}^{p_i+p_j}}{2^{p_i+p_j}\,\mathrm{dist}\,(I,J)^{p_i+p_j}}. \tag{3}$$

Proof. Using the Taylor expansion of $\log|x-y|$ and the orthogonal properties of the Legendre polynomials we obtain

$$\begin{aligned}\langle V l^\star_{p_j,J}, l^\star_{p_i,I}\rangle_{L^2(I)} &= -\frac{1}{\pi}\int_I\int_J l^\star_{p_i,I}(x)\, l^\star_{p_j,J}(y)\,\log|x-y|\,ds_y\,ds_x \\ &= -\frac{1}{\pi}\int_I\int_J l^\star_{p_i,I}(x)\, l^\star_{p_j,J}(y)\, R_{I,p_i,J,p_j}(x,y)\,ds_y\,ds_x\end{aligned} \tag{4}$$

where R_{I,p_i,J,p_j} is the remainder. The estimate

$$\left|\frac{\partial^{|\alpha|}\partial^{|\beta|}}{\partial x^\alpha \partial y^\beta}\log|x-y|\right| \le C(\Gamma)\,\frac{(|\alpha|+|\beta|-1)!}{|x-y|^{|\alpha|+|\beta|}}$$

for multi-indices α and β yields the inequality

$$|R_{I,p_i,J,p_j}(x,y)| \le C(\Gamma)\,\frac{\max\{|I|,\,|J|\}^{p_i+p_j}}{2^{p_i+p_j}\,\mathrm{dist}\,(I,J)^{p_i+p_j}}.$$

Therefore we obtain from (4) by applying the Cauchy-Schwarz inequality two times

$$\begin{aligned}&|\langle V l^\star_{p_j,J}, l^\star_{p_i,I}\rangle_{L^2(I)}| \\ &\le \frac{1}{\pi}\left(\int_I l^\star_{p_i,I}(x)^2\,ds_x\int_J l^\star_{p_j,J}(y)^2\,ds_y\int_I\int_J R^2_{I,p_i,J,p_j}(x,y)\,ds_y\,ds_x\right)^{1/2} \\ &\le \frac{C}{\pi}|I|^{1/2}|J|^{1/2}\frac{\max\{|I|,\,|J|\}^{p_i+p_j}}{2^{p_i+p_j}\,\mathrm{dist}\,(I,J)^{p_i+p_j}}|I|^{1/2}|J|^{1/2} \\ &\le \frac{C}{\pi}\frac{\max\{|I|,|J|\}^{p_i+p_j}}{2^{p_i+p_j}\,\mathrm{dist}\,(I,J)^{p_i+p_j}},\end{aligned}$$

which completes the proof. □

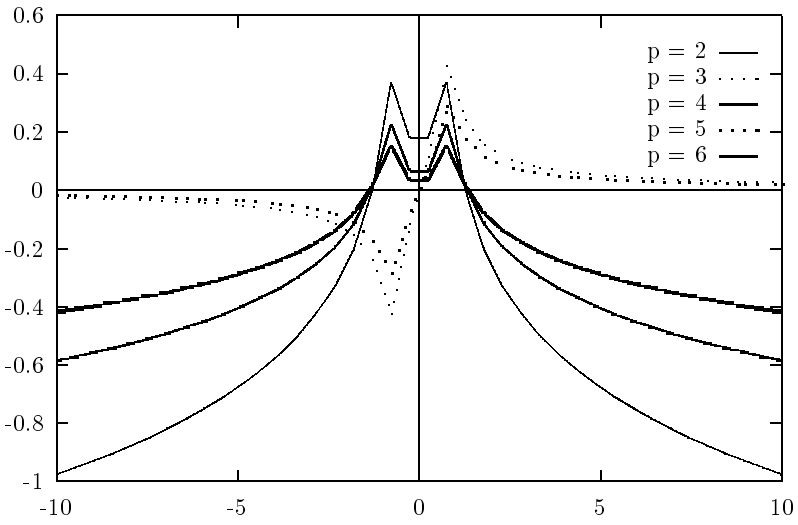

FIGURE 1. The functions $V(x^p|(-1,1))$ for the degrees $p = 2, \ldots, 6$.

Lemma 1 indicates a very local behavior of the single layer operator applied to piecewise Legendre polynomials. Figures 1 and 2 demonstrate this property. Figure 1 shows the values along the real line of the single layer operator acting on monomials restricted to the interval $(-1, 1)$. Figure 2 does the same for the Legendre polynomials. It can be seen that the use of Legendre polynomials improves considerably the locality of this integral operator.

In fact, many of the matrix elements are neglectible if the degree p of the underlying finite-dimensional subspace $H_N \subset \tilde{H}^{-1/2}(\Gamma)$ is high enough. For a certain subspace H_N we obtain an approximation Φ_N to our exact solution Φ by the Galerkin method. Usually this approximation is not exact, and the practical question arises how many of the matrix elements can be neglected without deteriorating the error too much. A natural criterion for the additional error to fulfill is to retain the convergence rate of the performed Galerkin procedure.

That means, if the Galerkin error behaves like

$$\|\Phi - \Phi_N\|_{\tilde{H}^{-1/2}(\Gamma)} \leq CN^{-\alpha}$$

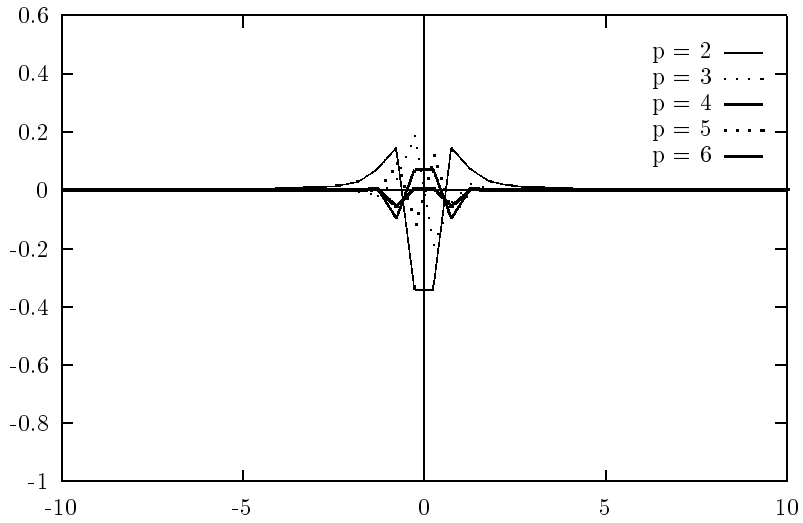

FIGURE 2. The functions $V(l_{p,(-1,1)})$ for the degrees $p = 2, \ldots, 6$.

for $N := \dim(H_N)$ and some $\alpha > 0$ we want to achieve

$$\|\Phi - \tilde{\Phi}_N\|_{\tilde{H}^{-1/2}(\Gamma)} \leq \tilde{C} N^{-\alpha} \tag{5}$$

where $\tilde{\Phi}_N$ is the solution of a sparse linear system which is written as

$$\tilde{a}_N(\tilde{\Phi}_N, \psi) = \langle g, \psi \rangle_{L^2(\Gamma)} \qquad \forall\, \psi \in H_N. \tag{6}$$

In order to derive a sufficient condition to ensure (5) we need to estimate the condition number of the Galerkin matrix A_N, i.e. of the matrix of the linear system (2). For the h-version this is already stated, see, e.g., [**11**]. In the following we assume that the scaled Legendre polynomials $l^{\star}_{p,I}$ are used as basis functions for the p-version. This scaling already improves the condition of A_N. If one uses pure piecewise Legendre polynomials as basis functions the condition number of A_N behaves as p^{α}, $\alpha \approx 2.5$ (c.f. [**9**]). The scaling reduces this behavior at least to p^2 as is shown below. For the finite element method the significance of using special basis functions has been investigated in [**3**].

Lemma 2. *For the smallest and largest eigenvalues of the Galerkin matrix A_N there holds*

$$\lambda_{\min}(A_N) \geq CN^{-2} \quad \textit{and} \quad \lambda_{\max}(A_N) \leq CN^{-1} \tag{7}$$

for the h-version and

$$\lambda_{\min}(A_N) \geq CN^{-2} \quad \textit{and} \quad \lambda_{\max}(A_N) \leq C \tag{8}$$

for the p-version.

Proof. Let $\phi \in H_N$ and $\vec{\phi} \in \mathbf{R}^N$ its N-dimensional representation. Due to the equivalence

$$a(\phi,\phi) = \langle V\phi,\phi\rangle_{L^2(\Gamma)} \simeq \|\phi\|^2_{\tilde{H}^{-1/2}(\Gamma)}$$

it suffices to estimate $\|\phi\|_{\tilde{H}^{-1/2}(\Gamma)}$. As an upper bound we can use

$$\|\phi\|_{\tilde{H}^{-1/2}(\Gamma)} \leq \|\phi\|_{L^2(\Gamma)}.$$

By the inverse assumption (see [**14**]) we obtain (with h being the minimal mesh-size)

$$\|\phi\|_{\tilde{H}^{-1/2}(\Gamma)} \geq Ch^{1/2}\|\phi\|_{L^2(\Gamma)} \sim N^{-1/2}\|\phi\|_{L^2(\Gamma)}$$

for the h-version and

$$\|\phi\|_{\tilde{H}^{-1/2}(\Gamma)} \geq Cp^{-1}\|\phi\|_{L^2(\Gamma)} \sim N^{-1}\|\phi\|_{L^2(\Gamma)}$$

for the p-version as a lower bound. That means there exist constants $c_1, c_2 > 0$ such that

$$c_1 N^{-1} \leq \frac{\langle V\phi,\phi\rangle_{L^2(\Gamma)}}{\langle \phi,\phi\rangle_{L^2(\Gamma)}} \leq c_2,$$

i.e.,

$$\begin{aligned} c_1 N^{-1} \inf_{\phi\in H_N} \frac{\langle \phi,\phi\rangle_{L^2(\Gamma)}}{\vec{\phi}^T\vec{\phi}} &\leq \lambda_{\min}(A_N) \leq \frac{\langle V\phi,\phi\rangle_{L^2(\Gamma)}}{\vec{\phi}^T\vec{\phi}} \\ &\leq \lambda_{\max}(A_N) \leq c_2 \sup_{\phi\in H_N} \frac{\langle \phi,\phi\rangle_{L^2(\Gamma)}}{\vec{\phi}^T\vec{\phi}} \end{aligned} \tag{9}$$

for the h-version and

$$c_1 N^{-2} \le \frac{\langle V\phi,\phi\rangle_{L^2(\Gamma)}}{\langle \phi,\phi\rangle_{L^2(\Gamma)}} \le c_2,$$

i.e.,

$$\begin{aligned} c_1 N^{-2} \inf_{\phi\in H_N} \frac{\langle \phi,\phi\rangle_{L^2(\Gamma)}}{\vec{\phi}^T\vec{\phi}} \le \lambda_{\min}(A_N) \le \frac{\langle V\phi,\phi\rangle_{L^2(\Gamma)}}{\vec{\phi}^T\vec{\phi}} \\ \le \lambda_{\max}(A_N) \le c_2 \sup_{\phi\in H_N} \frac{\langle \phi,\phi\rangle_{L^2(\Gamma)}}{\vec{\phi}^T\vec{\phi}} \end{aligned} \tag{10}$$

for the p-version. Now we have to estimate $\langle \phi,\phi\rangle_{L^2(\Gamma)}/(\vec{\phi}^T\vec{\phi})$. Let $\Gamma = \cup_{j=1}^m \Gamma_j$ be the decomposition of Γ into elements and p_j the respective degrees. Then $\phi \in H_N$ can be written as $\phi = \sum_{j=1}^m \sum_{i=0}^{p_j} c_{ij}\, l^\star_{i,\Gamma_j}$, and there holds

$$\langle \phi,\phi\rangle_{L^2(\Gamma)} = \sum_{j=1}^m \sum_{i=0}^{p_j} c_{ij}^2 |\Gamma_j| \sim h \sum_{j=1}^m \sum_{i=0}^{p_j} c_{ij}^2 = h\,\vec{\phi}^T\vec{\phi} \tag{11}$$

for a quasiuniform mesh where $h = \min\{|\Gamma_j|;\ j = 1,\dots,m\}$. Using (9) this yields for the h-version

$$\begin{aligned} \lambda_{\min}(A_N) &\ge c_1 N^{-1} h \sim N^{-2} \\ \lambda_{\max}(A_N) &\le c_2 h \sim N^{-1} \end{aligned}$$

and using (10) for the p-version

$$\lambda_{\min}(A_N) \ge c_1 N^{-2}, \qquad \lambda_{\max}(A_N) \le c_2. \qquad \square$$

Corollary 1. *The condition number of the Galerkin matrix with respect to the l^2-norm can be estimated as*

$$\kappa(A_N) \le CN$$

for the h-version and as

$$\kappa(A_N) \le CN^2$$

for the p-version.

Proof. This follows by dividing the bounds for the eigenvalues given by Lemma 2. □

Now we investigate the allowed perturbation of the system matrix, i.e., we give a sufficient condition for (5) to hold. Let $\tilde{A}_N$ denote the perturbed matrix of the linear system (2), i.e., A_N where some elements have been neglected, and $\delta A_N = A_N - \tilde{A}_N$. Then there holds

Theorem 1. *Let ω be the maximum of the internal angles at the corners of the polygon Ω and assume $g \in H^1(\Gamma)$ for the righthand side function g of (1). Suppose $\tilde{A}_N$ is constructed such that*

$$\frac{\|\delta A_N\|_2}{\|A_N\|_2^{1/2}} \leq C N^{-\pi/\omega - 3/2 - \varepsilon} \tag{12}$$

in case of the h-version and such that

$$\|\delta A_N\|_2 \leq C N^{-2\pi/\omega - 2 - \varepsilon} \tag{13}$$

in case of the p-version for some $C > 0$ and $\varepsilon > 0$. Then the rate of convergence of $\tilde{\Phi}_N \to \Phi$ is the same as the rate of $\Phi_N \to \Phi$, i.e.,

$$\|\Phi - \tilde{\Phi}_N\|_{\tilde{H}^{-1/2}(\Gamma)} \leq \tilde{C} N^{-\pi/\omega - \varepsilon}$$

in case of the h-version and

$$\|\Phi - \tilde{\Phi}_N\|_{\tilde{H}^{-1/2}(\Gamma)} \leq \tilde{C} N^{-2\pi/\omega - \varepsilon}$$

in case of the p-version.

Proof. Let $\vec{\phi} \in \mathbf{R}^N$ denote the vector of coefficients of a function $\phi \in H_N$ for our basis of scaled piecewise Legendre polynomials. Then there holds for $\delta\Phi_N := \Phi_N - \tilde{\Phi}_N$

$$\|\delta\Phi_N\|_V^2 = \delta\vec{\Phi}_N^T A_N \delta\vec{\Phi}_N \leq \|A_N\|_2 \|\delta\vec{\Phi}_N\|_2^2.$$

Here $\|\cdot\|_V \simeq \|\cdot\|_{\tilde{H}^{-1/2}(\Gamma)}$ denotes the norm induced by the bilinear form $a(\cdot,\cdot)$. Using the condition number $\kappa(A_N)$ of A_N the last term can be estimated with the help of

$$\|\delta\vec{\Phi}_N\|_2 \leq \kappa(A_N)\frac{\|\delta A_N\|_2}{\|A_N\|_2}\|\vec{\Phi}_N\|_2,$$

and we obtain

$$\|\Phi_N - \tilde{\Phi}_N\|_V \leq \kappa(A_N)\frac{\|\delta A_N\|_2}{\|A_N\|_2^{1/2}}\|\vec{\Phi}_N\|_2. \tag{14}$$

Due to (11) we have

$$\|\vec{\Phi}_N\|_2 \leq C\, h^{-1/2}\|\Phi_N\|_{L^2(\Gamma)}.$$

Since $g \in H^1(\Gamma)$ and since Γ is a polygon (with internal angles $\omega_j < 2\pi$) we know that the exact solution Φ of (2) belongs to $L^2(\Gamma)$, and therefore

$$\|\Phi_N\|_{L^2(\Gamma)} \leq \|\Phi - \Phi_N\|_{L^2(\Gamma)} + \|\Phi\|_{L^2(\Gamma)} < C$$

for a constant C independent of N, cf. [**14**, Theorem 3.7]. Using this bound (14) can be estimated with the help of Corollary 1 together with assumption (12) as

$$\|\Phi_N - \tilde{\Phi}_N\|_V \leq C\, N^{-\pi/\omega-\varepsilon} \tag{15}$$

for the h-version and together with assumption (13) as

$$\|\Phi_N - \tilde{\Phi}_N\|_V \leq C\frac{N^{-2\pi/\omega-\varepsilon}}{\|A_N\|_2^{1/2}} \tag{16}$$

for the p-version. Because A_N is a normal matrix we have $\|A_N\|_2 = \lambda_{\max}(A_N)$. Due to the hierarchical construction of the subspaces H_N for the p-version $\lambda_{\max}(A_N)$ and therefore also $\|A_N\|_2$ is a nondecreasing function of N. That means in that case $\|A_N\|_2$ is bounded from below by a positive constant and we obtain from (16)

$$\|\Phi_N - \tilde{\Phi}_N\|_V \leq C\, N^{-2\pi/\omega-\varepsilon} \tag{17}$$

for the p-version.

In fact, (15) and (17) lead to the same convergence rates as those of $\|\Phi - \Phi_N\|_V$, see [**14**]. Therefore, the triangle inequality gives the desired estimate

$$\begin{aligned}\|\Phi - \tilde{\Phi}_N\|_{\tilde{H}^{-1/2}(\Gamma)} &\simeq \|\Phi - \tilde{\Phi}_N\|_V \leq \|\Phi - \Phi_N\|_V + \|\Phi_N - \tilde{\Phi}_N\|_V \\ &\leq C\|\Phi - \Phi_N\|_V \simeq \|\Phi - \Phi_N\|_{\tilde{H}^{-1/2}(\Gamma)}\end{aligned}$$

where C is a constant independent of N. □

Theorem 1, together with Lemma 1, serves as an a-priori criterion to make the system matrices sparse. Numerical results will be presented in Section 4.

3. The additive Schwarz method. Due to the positive definiteness of the Galerkin matrix A_N the conjugate gradient method is the method of choice to solve (2). In order to reduce the number of iterations which are necessary to reach a given accuracy one needs a preconditioner. In view of parallel computer architectures with distributed memory the additive Schwarz method (ASM) received much attention, see e.g. [**4**]. For the h-version several variants of the additive Schwarz preconditioner have been considered, see [**7, 18**] for the single layer operator and [**18**] for a hypersingular operator. Here we investigate the p-version of the BEM and consider two different types of decompositions of the underlying discretization to define the additive Schwarz preconditioner. The first type which is based on a domain decomposition is investigated theoretically (cf. Theorem 3) and for the second type we only present numerical results in Section 4. For further variants which deal with overlapping decompositions and with global terms in the decompositions we refer to [**15**]. In fact, we prove that no overlapping and no global block (which stems from functions with support on the whole boundary Γ) is necessary in our case. However, we note that then the condition numbers may depend on the number of subdomains.

Let us recall the abstract setting of the additive Schwarz method. Let

$$H_N = S_1 \cup S_2 \cup \cdots \cup S_k \tag{18}$$

be a decomposition of H_N into k subspaces and

$$P_j : H_N \longrightarrow S_j, \qquad j = 1, \dots, k,$$

the corresponding projections:

$$a(P_j\phi, \psi) = a(\phi, \psi) \qquad \forall\, \psi \in S_j.$$

The ASM consists in solving the equation

$$P\Phi_N := \sum_{j=1}^{k} P_j \Phi_N = g^\star, \tag{19}$$

where $g^\star = \sum_{j=1}^{k} P_j \Phi_N$ can be computed without knowing the solution Φ_N of (2) by

$$a(P_j\Phi_N, \psi) = a(\Phi_N, \psi) = \langle g, \psi \rangle_{L^2(\Gamma)} \qquad \forall\, \psi \in S_j.$$

The usual way to choose the decomposition (18) is to divide the boundary element space H_N into subspaces where each two of them have different supports (intersecting or nonintersecting), e.g., in case of discontinuous functions

$$S_j := \{\psi|_{\tilde{\Gamma}_j};\ \psi \in H_N\}, \qquad j = 1, \dots, k, \tag{20}$$

where $\Gamma = \cup_{j=1}^{k} \tilde{\Gamma}_j$ is some decomposition of the underlying mesh into sets of elements. This is the so-called domain decomposition (dd) used in most finite element applications and recently also for the h-version of the boundary element method (see [**7**]).

But, especially in view of the p-version of the boundary element method, there is another obvious way to choose a decomposition (18), in the following referred to as degree decomposition (pd):

$$S_j := \operatorname{span}\{\psi_j\ ;\ \psi_j|_{\Gamma_i} = l_{j,\Gamma_i},\ i = 1, \dots, m\}, \qquad j = 0, \dots, p. \tag{21}$$

Here l_{j,Γ_i} is the Legendre polynomial of degree j linearly transformed onto Γ_i, Γ_i is a single element of the m elements defining the mesh, and p stands for the largest degree of the basis functions defining H_N (here k in (18) would equal $p + 1$). Clearly, this is a special choice for

a subspace H_N built of discontinuous piecewise Legendre polynomials and fits well to the case of the single layer potential.

Note that one can obtain also for the degree decomposition a constant number of subspaces even for increasing p by collecting different S_js. This is similar to the classical domain decomposition applied to the h-version where one has a fixed number of domains and gets increasing subspaces by decreasing the element sizes.

For the h-version with piecewise constant ansatz functions the dd-type ASM with overlapping has already been proved to be an optimal preconditioner (compare Corollary 1):

Theorem 2 (M. Hahne, E.P. Stephan [**7**]). *Let H_N be decomposed into subspaces S_j according to* (18) *and* (20). *Let the subdomains $\tilde{\Gamma}_j$, $j = 1, \ldots, k$, fulfill the overlapping condition*

$$\operatorname{dist}\left(\tilde{\Gamma}_i \cap \partial\tilde{\Gamma}_j, \tilde{\Gamma}_j \cap \partial\tilde{\Gamma}_i\right) \geq \gamma k^{-1}$$

for neighboring subdomains $\tilde{\Gamma}_i$, $\tilde{\Gamma}_j$ and a constant γ independent of k. Then the condition number of the additive Schwarz operator P in (19) *is bounded independently of N if the h-version with piecewise constant functions is performed for* (2), *i.e.,*

$$\kappa(P) \leq C.$$

Now we present the main result of this section which proves the efficiency of the additive Schwarz method in the case of the p-version. We note that no overlapping of the subspaces S_j and no block of global functions in the decomposition (18) is used.

Theorem 3. *Let H_N be decomposed into subspaces S_j according to* (18) *and* (20). *The sets $\tilde{\Gamma}_j$ of elements Γ_i are supposed to be distinct. Then the condition number of the additive Schwarz operator P in* (19) *is bounded by*

$$\kappa(P) \leq C\,(1 + \log(1+p))^2$$

if the p-version of the boundary element method is performed. Here C is a positive constant independent of p.

Proof. Let $\phi \in H_N$ and define $\phi_j := \phi|_{\bar{\Gamma}_j} \in S_j$, $j = 1, \ldots, k$. Then we have

$$\phi = \sum_{j=1}^{k} \phi_j$$

and because the $\tilde{\Gamma}_j$s are nonintersecting this representation is unique. Applying

$$\left\| \sum_{j=1}^{k} \phi_j \right\|^2_{\tilde{H}^{-1/2}(\Gamma)} \leq \sum_{j=1}^{k} \|\phi_j\|^2_{\tilde{H}^{-1/2}(\bar{\Gamma}_j)}$$

(cf. [**19**, Lemma 3.2]) we obtain

$$\begin{aligned} a(\phi,\phi) \simeq \|\phi\|^2_{\tilde{H}^{-1/2}(\Gamma)} &= \left\| \sum_{j=1}^{k} \phi_j \right\|^2_{\tilde{H}^{-1/2}(\Gamma)} \\ &\leq \sum_{j=1}^{k} \|\phi_j\|^2_{\tilde{H}^{-1/2}(\bar{\Gamma}_j)} \simeq \sum_{j=1}^{k} a(\phi_j,\phi_j). \end{aligned}$$

This means that the largest eigenvalue of P is bounded,

$$\lambda_{\max}(P) \leq C.$$

Now we look for a lower bound to the smallest eigenvalue $\lambda_{\min}$ of P. Let

$$\phi_j = \sum_{i=0}^{p} \phi_{j,i} =: \phi_{j,0} + \omega_j \quad \text{and} \quad \phi_0 := \sum_{j=1}^{k} \phi_{j,0}$$

where $\phi_{j,i}$ is piecewise exactly of degree i, i.e., $\phi_{j,i} \in S_i$ defined by (21).

First let us consider $\phi_{j,0}$. Let $\vec{c}_{j,0} \in \mathbf{R}^N$ and $\vec{c}_0 \in \mathbf{R}^N$ denote the N-dimensional representation of $\phi_{j,0}$ and ϕ_0, respectively. Then there holds $\|\vec{c}_{j,0}\|_2 \leq \|\vec{c}_0\|_2$ and therefore

$$\begin{aligned} \|\phi_{j,0}\|^2_{\tilde{H}^{-1/2}(\bar{\Gamma}_j)} &\simeq a(\phi_{j,0},\phi_{j,0}) = \vec{c}^{\,T}_{j,0} A_N \vec{c}_{j,0} \simeq \|\vec{c}_{j,0}\|^2_2 \\ &\leq \|\vec{c}_0\|^2_2 \simeq \|\phi_0\|^2_{\tilde{H}^{-1/2}(\Gamma)} \end{aligned}$$

since only a fixed block of the positive definite stiffness matrix A_N is involved. Due to the inequality

$$\|\phi_0\|^2_{\tilde{H}^{-1/2}(\Gamma)} \leq C(1 + \log(1+p))\|\phi\|^2_{\tilde{H}^{-1/2}(\Gamma)}$$

(cf. [**15**, 24]) this yields

$$\|\phi_{j,0}\|^2_{\tilde{H}^{-1/2}(\bar{\Gamma}_j)} \le C(1+\log(1+p))\|\phi\|^2_{\tilde{H}^{-1/2}(\Gamma)}. \tag{22}$$

It remains to estimate the norm of ω_j. Using the antiderivative operator

$$I^{1/2}:\ \tilde{H}^{-1/2}(\tilde{\Gamma}_j) \longrightarrow \tilde{H}^{1/2}(\tilde{\Gamma}_j)$$

of [**7**, Lemma 3 (ii)] and estimating the $\tilde{H}^{1/2}$-norm in terms of the $H^{1/2}$-norm by [**2**, Theorem 6.5] we obtain

$$\begin{aligned} \|\omega_j\|_{\tilde{H}^{-1/2}(\bar{\Gamma}_j)} &\simeq \|I^{1/2}\omega_j\|_{\tilde{H}^{1/2}(\bar{\Gamma}_j)} \\ &\le C(1+\log(1+p))\|I^{1/2}\omega_j\|_{H^{1/2}(\bar{\Gamma}_j)} \\ &\simeq (1+\log(1+p))\|\omega_j\|_{H^{-1/2}(\bar{\Gamma}_j)}. \end{aligned} \tag{23}$$

The norm of $\phi_j = \phi_{j,0} + \omega_j$ can be estimated the following way:

$$\begin{aligned} \|\phi\|_{\tilde{H}^{-1/2}(\Gamma)} = \|\phi\|_{H^{1/2}(\Gamma)'} &= \sup_{\psi\in H^{1/2}(\Gamma)} \frac{\langle\phi,\psi\rangle_{L^2(\Gamma)}}{\|\psi\|_{H^{1/2}(\Gamma)}} \\ &= \sup_{\psi\in H^{1/2}(\Gamma)} \frac{\sum_{j=1}^k \langle\phi_j,\psi\rangle_{L^2(\Gamma)}}{\|\psi\|_{H^{1/2}(\Gamma)}} \\ &\ge \sup_{\psi\in \tilde{H}^{1/2}(\bar{\Gamma}_j)} \frac{\sum_{j=1}^k \langle\phi_j,\psi\rangle_{L^2(\Gamma)}}{\|\psi\|_{H^{1/2}(\Gamma)}} \\ &= \sup_{\psi\in \tilde{H}^{1/2}(\bar{\Gamma}_j)} \frac{\langle\phi_j,\psi\rangle_{L^2(\bar{\Gamma}_j)}}{\|\psi\|_{\tilde{H}^{1/2}(\bar{\Gamma}_j)}} \\ &= \|\phi_j\|_{\tilde{H}^{1/2}(\bar{\Gamma}_j)'} = \|\phi_j\|_{H^{-1/2}(\bar{\Gamma}_j)}, \qquad j=1,\dots,k. \end{aligned}$$

Thus we have together with (22) and (23)

$$\begin{aligned} \|\omega_j\|_{\tilde{H}^{-1/2}(\bar{\Gamma}_j)} &\le C(1+\log(1+p))\|\omega_j\|_{H^{-1/2}(\bar{\Gamma}_j)} \\ &\le C(1+\log(1+p))(\|\omega_j+\phi_{j,0}\|_{H^{-1/2}(\bar{\Gamma}_j)} + \|\phi_{j,0}\|_{H^{-1/2}(\bar{\Gamma}_j)}) \\ &\le C(1+\log(1+p))\|\phi\|_{\tilde{H}^{-1/2}(\Gamma)}. \end{aligned} \tag{24}$$

Combining (22) and (24) we obtain

$$\begin{aligned}\sum_{j=1}^{k} a(\phi_j,\phi_j) &\simeq \sum_{j=1}^{k} \|\phi_j\|^2_{\tilde{H}^{-1/2}(\bar{\Gamma}_j)} \\ &\le C\sum_{j=1}^{k}(\|\phi_{j,0}\|^2_{\tilde{H}^{-1/2}(\bar{\Gamma}_j)} + \|\omega_j\|^2_{\tilde{H}^{-1/2}(\bar{\Gamma}_j)}) \\ &\le C(k)(1+\log(1+p))^2\|\phi\|^2_{\tilde{H}^{-1/2}(\Gamma)} \\ &\simeq (1+\log(1+p))^2 a(\phi,\phi).\end{aligned}$$

Note that the number of domains k is constant for the p-version. Therefore, we have

$$\lambda_{\min}(P) \ge C(1+\log(1+p))^{-2}$$

and $\kappa(P) = \lambda_{\max}(P)/\lambda_{\min}(P) \le C(1+\log(1+p))^2$. □

4. Numerical results. We consider the Dirichlet problem for the L-shaped domain Ω (see Figure 3)

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \Gamma = \partial\Omega \end{cases}$$

where f is chosen such that

$$u(x,y) = \Im(z^{2/3}) \quad \text{for} \quad z = x+iy.$$

This problem is substituted by our integral equation (1). The finite-dimensional subspaces H_N of $\tilde{H}^{-1/2}(\Gamma)$ are constructed by discontinuous piecewise Legendre polynomials on a decomposition of Γ. For more details see [**6**].

First we present numerical results for the boundary element method where we used sparse matrices for our Galerkin equations instead of calculating all the entries, see (6). To this end we neglect as many matrix elements as possible (starting with the smallest ones) until the bounds (12) for the h-version and (13) for the p-version are reached.

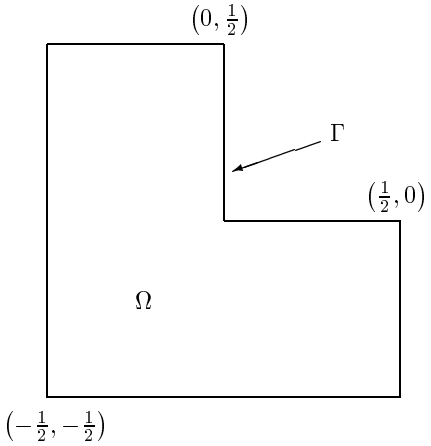

FIGURE 3. The L-shaped domain.

It has to be emphasized that the corresponding matrix elements need not to be calculated. Instead, we use the inequality

$$\|A\|_2^2 \leq \|A\|_F^2 = \sum_{i,j} a_{ij}^2$$

to estimate the spectral norm of A and apply Lemma 1 to get an estimation of the elements of δA_N. The constant in Lemma 1 is simply replaced by 1 and the constants in (12) and (13) are chosen such that the first Galerkin matrix we start with is nearly diagonal. For the h-version we have to take $\|A_N\|_2$ into consideration, cf. (12). Since A_N is a normal matrix there holds $\|A_N\|_2 = \lambda_{\max}(A_N) \leq cN^{-1}$, cf. Lemma 2. Numerically it turns out that this estimate is asymptotically exact (cf. [**16**]) and $\|A_N\|_2$ is implemented this way. The so obtained matrices are in fact rather sparse. Figure 4 shows a typical system matrix for 16 elements and degree 4. The unknowns are ordered with respect to the degrees (as indicated at the margin of the matrix) and for each degree with respect to the boundary elements. Therefore we have 5×5 blocks for the pairs of degrees and in each block 16×16 entries corresponding to testing each boundary element against each other. All the entries which are zero or have been neglected by our procedure are replaced with spaces. The remaining entries are characterized by

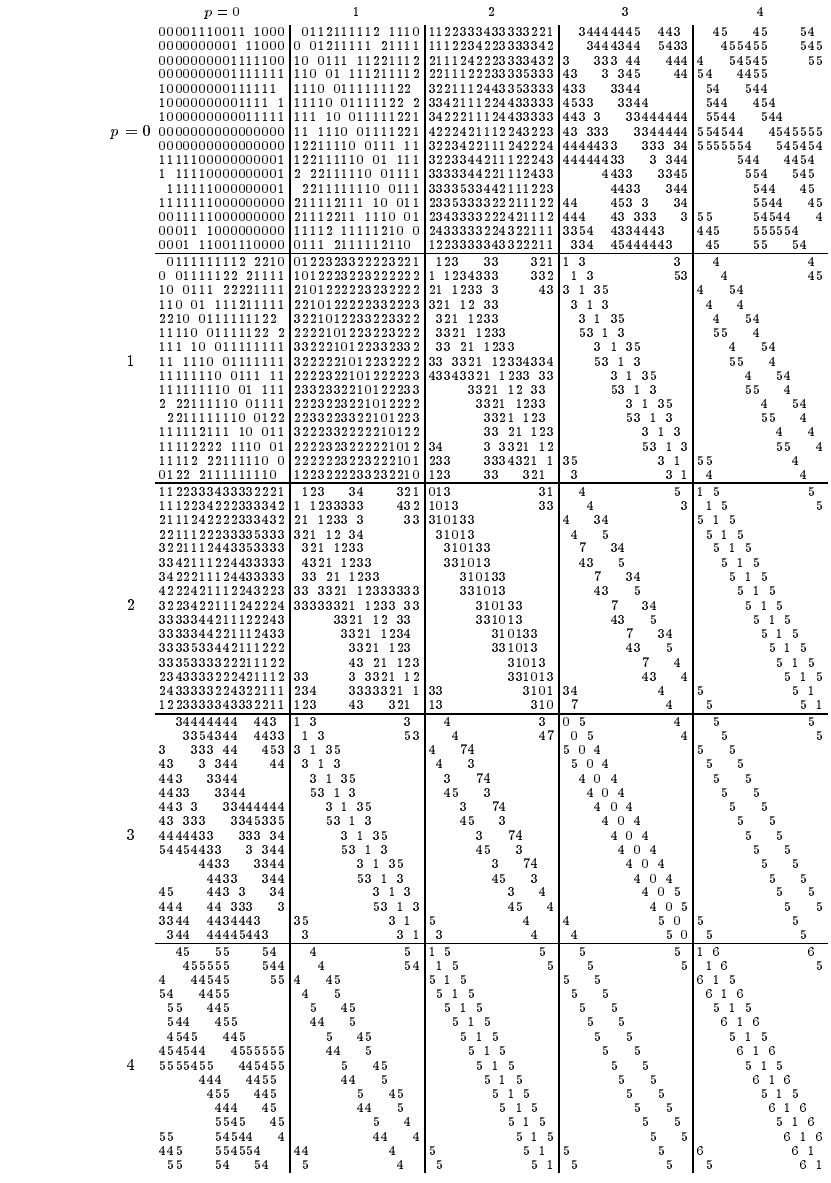

FIGURE 4. The system matrix for 16 elements and degree 4. An entry n means that the corresponding matrix element has an absolute value $\leq 10^{-n}$ times the maximal one.

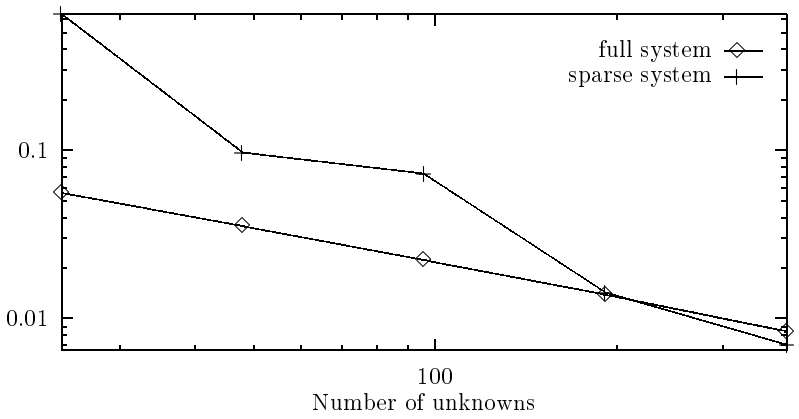

FIGURE 5. The relative error in the energy norm for the h-version.

numbers $0 \leq n \leq 9$ to give an overview of the absolute values. A number n stands for a matrix entry a_{ij} with $|a_{ij}| \leq 10^{-n} a_{\max}$ where $a_{\max} := \max\{|a_{ij}|;\ i,j = 1,\ldots,N\}$.

Figures 5 and 6 show that even for the h-version one can neglect many matrix elements without degrading the convergence properties of the Galerkin method and without using sophisticated basis functions for constructing the subspaces H_N. In this example we used the fixed degree $p = 2$.

Taking a closer look at the matrix in Figure 4 one observes a dominant band structure in the blocks for higher degree elements. This confirms the natural guess that the lower degrees and neighboring elements represent the largest, i.e., most important, matrix elements. To investigate this observation we test another method for the p-version to make the Galerkin matrices sparse. We take a fixed degree $p^\star$ and calculate all the matrix elements $\langle V\, l^\star_{p_i,I},\, l^\star_{p_j,J}\rangle_{L^2(\Gamma)}$ for $\max\{p_i, p_j\} \leq p^\star$. The remaining matrix elements are just calculated if $\operatorname{dist}(I,J) = 0$. That means we have a very simple method which neglects many matrix elements if the chosen degree p is larger than $p^\star$. It turns out that it suffices to choose $p^\star = 2$ to retain the original convergence rate for our example. Figure 7 presents the relative errors in the energy norm for

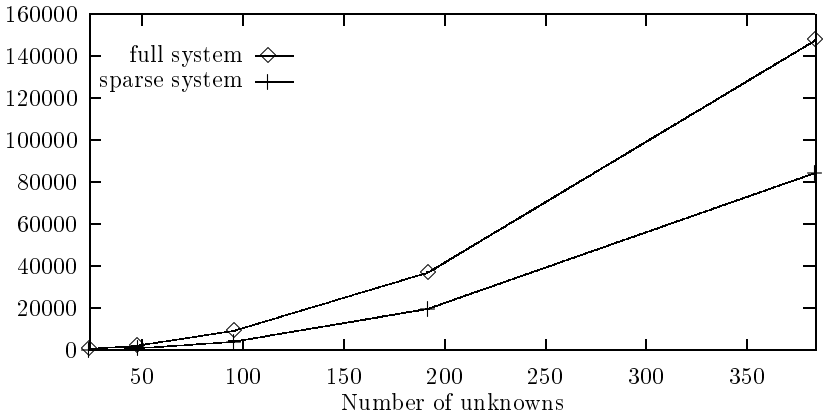

FIGURE 6. The numbers of matrix elements used for the Galerkin matrices of the h-version.

the original p-version and the two sparse variants. The underlying mesh consists of 16 elements. Figure 8 shows the corresponding numbers of matrix elements which were used for the Galerkin systems. As can be seen, both of our sparse methods considerably reduce the density of the Galerkin matrices.

Now we consider the additive Schwarz preconditioner. Recall that we mentioned two different types. The usual dd-type preconditioner implicitly defined by a decomposition of the discretized boundary Γ (cf. (18) and (20)) results in a fixed number k of local linear systems. The pd-type preconditioner consists of an increasing number k of small local linear systems for increasing p. Therefore the pd-type preconditioner can be implemented efficiently in a natural manner in parallel. Table 1 shows the computed smallest and largest eigenvalues of the original system (using scaled piecewise Legendre polynomials) and of the preconditioned systems for the two types. The parameter α describes the behavior $\lambda_{\min} = c\,(1+p)^{\alpha}$. The largest eigenvalues are bounded in either case. The smallest eigenvalue of the original system seems to asymptotically behave as p^{-2}. This rate is covered by the predicted bound given by Lemma 2. The smallest eigenvalues of the ASM-preconditioned systems decrease much slower. In case of the dd-

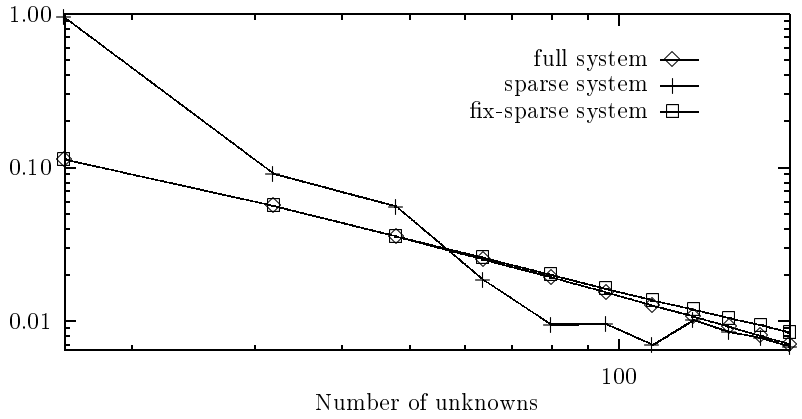

FIGURE 7. The relative error in the energy norm for the p-version.

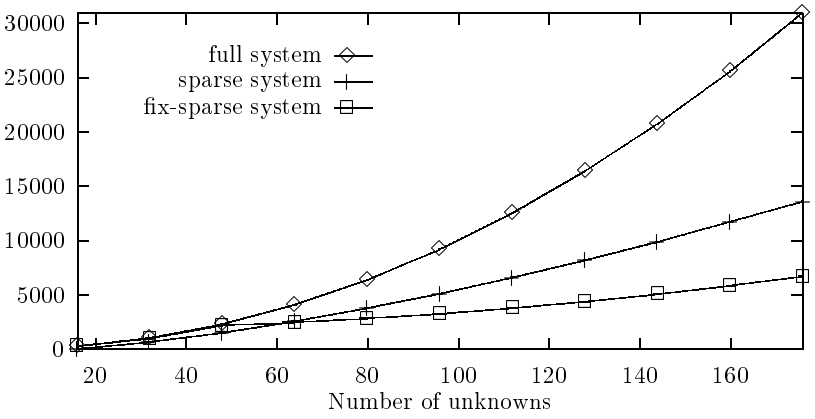

FIGURE 8. The numbers of matrix elements used for the Galerkin matrices of the p-version.

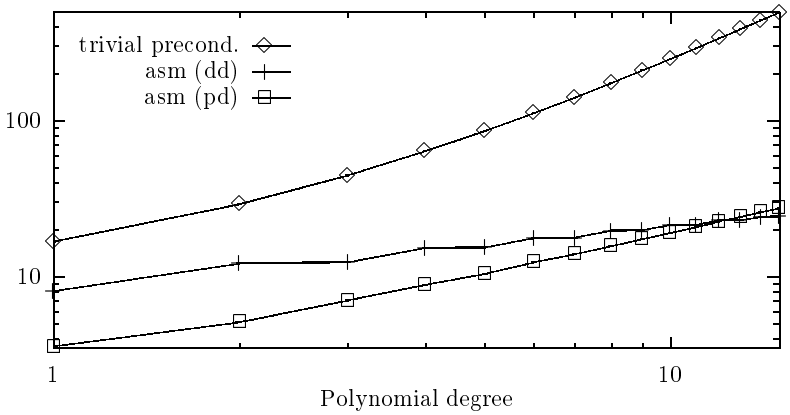

FIGURE 9. The condition numbers of the Galerkin matrices of the p-version for different preconditioners.

type preconditioner they decrease as $p^{-\alpha}$ with $\alpha < 0.5$ and the rate α becomes smaller for higher polynomial degrees. This is not far off the asymptotic prediction by Theorem 3. The smallest eigenvalue obtained by the pd-type preconditioner appears to behave like p^{-1}.

Figure 9 shows the behavior of the improved condition numbers compared to the original ones in a log-log plot. The curve for the trivially preconditioned system tends to a straight line with about two times the slope as that of the line for the pd-type preconditioned system. The curve for the dd-type preconditioned system becomes flatter for higher degrees as predicted by Theorem 3.

The numbers of iterations of the conjugate gradient method which are required to solve the Galerkin system up to the accuracy of the Galerkin error are given also in Table 1. Note that for our example the exact solution is known and therefore the Galerkin error is computable. The numbers for the pd-type preconditioned CG-method are generally smaller than those for the original system using scaled Legendre polynomials. The number of dd-type preconditioned CG-iterations is even constant for the degrees 8 up to 14. This confirms the efficiency particularly of the dd-type preconditioner for the CG-method.

Of course, the application of the ASM-preconditioners is not as simple as the use of scaled Legendre polynomials. But we point the reader to the possibility of combining both types. This would result in the very simple diagonal preconditioner since the decomposition of the ansatz space H_N with respect to all elements and with respect to all degrees yields N subspaces, and each of them is defined by exactly one basis function. Then, one would expect the condition number to behave like $N(\log N)^2$. For theoretical investigations of this preconditioner we refer to a forthcoming paper [**10**].

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