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IMPROVED CONVERGENCE RATES FOR SOME DISCRETE GALERKIN METHODS

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ABSTRACT. We show how to improve the estimate of the convergence rate of a number of discrete polynomially-based Galerkin methods for Fredholm and Cauchy singular integral equations. This has been accomplished by sharpening the bounds on the quadrature errors in a manner analogous to that of Joe [**14**] for spline-based methods. These results are then extended to establish the convergence of some discrete Galerkin methods for one-dimensional hypersingular equations and some boundary integral equations on the sphere in $\mathbf{R}^3$.

Introduction. In a number of recent papers we have examined the convergence rate of various polynomially-based Galerkin methods for Fredholm and singular integral equations [**8**–**12**]. The convergence analysis took into account the effects of quadrature errors and for Fredholm equations may be seen as complementary to similar results of Atkinson and Bogomolny [**4**], Joe [**14**] and Spence and Thomas [**20**] using spline bases. In the case of splines, the above authors were able to obtain optimal convergence rates, i.e., convergence rates equal to that of the best approximation to the solution by splines of a given order. For polynomial approximations we were unable to do this, in part because of over estimation of various quadrature errors. In this paper, making use of an argument analogous to that of Joe [**14**] for spline approximations, we are able to improve our estimate of the convergence rate from $0(n^{-r+1})$ to $0(n^{-r+\frac{1}{2}})$ where n is the degree of the polynomial approximation. This seems the best that can be done by perturbation techniques.

The paper is divided into five sections. In Section 2 we review our previous results for Fredholm and Cauchy singular equations and indicate where improvements to our prior analysis can be made. In Section 3 we provide new estimates of quadrature errors generalizing those in [**10**–**12**]. These are then applied to improve the convergence rates

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given in [**10**–**12**]. In Section 4 we extend these results to establish the convergence of a discrete Galerkin method for hypersingular integral equations discussed by Frenkel [**6**] and us in [**10, 11**]. In Section 5 we analyze the convergence of some discrete Galerkin methods based on spherical harmonics for integral equations on the sphere.

2. Discrete Galerkin methods.

2.1 *Fredholm equations.* We consider the numerical solution of the integral equation

$$u(x) = \int_{-1}^{1} \kappa(x,t)u(t)\,dt + f(x) \tag{2.1}$$

where $\kappa(x,t)$ is continuous on $[-1,1]\times[-1,1]$ and $f(x)$ is continuous on $[-1,1]$. We assume that (2.1) has a unique solution. Let L_2 be the space of real square-integrable functions on $[-1,1]$ with inner product

$$\langle f,g\rangle = \int_{-1}^{1} f(t)g(t)\,dt \tag{2.2}$$

and norm

$$\|f\| = (\langle f,f\rangle)^{1/2}. \tag{2.3}$$

To solve (2.1) by Galerkin's method let $\{\varphi_k\}$ be the normalized Legendre polynomials and approximate u by

$$u_n = \sum_{k=0}^{n} a_k\varphi_k \tag{2.4}$$

where the coefficients $\{a_k\}_{k=0}^{n}$ are determined by solving

$$a_j - \sum_{k=0}^{n}\langle K\varphi_k,\varphi_j\rangle a_k = \langle f,\varphi_j\rangle, \qquad j=0,1,\dots,n, \tag{2.5}$$

and

$$Ku(x) = \int_{-1}^{1}\kappa(x,t)u(t)\,dt. \tag{2.6}$$

Under the stated conditions on κ and f it is well known that for all n sufficiently large that u_n exists, is unique and

$$\|u - u_n\| \leq cn^{-r} \tag{2.7}$$

if $\kappa \in C^r$ and $f \in C^r$, thus establishing the convergence of u_n to u.

One can also show that u_n converges uniformly to u if $r \geq 2$ and [**12**]

$$\|u - u_n\|_\infty \leq cn^{-r+3/2} \tag{2.8}$$

where $\| \cdot \|_\infty$ is the uniform norm.

In general the integrals in (2.5) cannot be evaluated analytically, so some type of numerical integration method needs to be used to obtain approximations to $\langle f, \varphi_j \rangle$ and $\langle K\varphi_k, \varphi_j \rangle$. For this we define integration rules $\{Q_{M(n)}\}$

$$\int_{-1}^{1} g(t)\, dt \simeq Q_{M(n)}(g) = \sum_{l=1}^{M} w_l g(t_l) \tag{2.9}$$

where

(1) $w_l > 0$, $l = 1, 2, \ldots, M(n)$;

(2) $Q_{M(n)}$ has precision at least $2n$; that is,

$$Q_{M(n)}(g) = \int_{-1}^{1} g(t)\, dt \tag{2.10}$$

for all polynomials of degree $\leq 2n$. (For simplicity we will suppress the dependence of $M(n)$ on n for the remainder of the paper.)

Using Q_M we define the following approximations:

$$\langle f, \varphi_k \rangle \simeq Q_M(f\varphi_k), \qquad 0 \leq k \leq n, \tag{2.11}$$

and

$$\begin{aligned} \langle K\varphi_k, \varphi_j \rangle &= \int_{-1}^{1} \int_{-1}^{1} \kappa(x,t)\varphi_k(t)\varphi_j(x)\, dx\, dt \\ &\simeq \sum_{m=1}^{M} \sum_{l=1}^{M} w_l w_m \kappa(x_m, t_l)\varphi_k(t_l)\varphi_j(x_m) \\ &\equiv Q_M \times Q_M(\kappa\varphi_k\varphi_j) \end{aligned} \tag{2.12}$$

where $\{x_m\}$ and $\{t_l\}$ represent the same sets of points.

Substituting (2.11)–(2.12) into (2.5) and letting v_n be the resulting approximation to u_n

$$v_n = \sum_{k=0}^{n} b_k \varphi_k \tag{2.13}$$

where $\{b_k\}_{k=0}^n$ satisfy

$$b_j - \sum_{k=0}^{n} b_k \bigg[\sum_{m=1}^{M} \sum_{l=1}^{M} w_m w_l \kappa(x_m, t_l) \varphi_k(t_l) \varphi_j(x_m) \bigg] = \sum_{m=1}^{M} w_m f(x_m) \varphi_j(x_m), \qquad 0 \le j \le n. \tag{2.14}$$

The method defined by (2.13)–(2.14) is called a *discrete Galerkin method.*

To prove the convergence of v_n and to obtain rates of convergence we use the theory of perturbed projection methods as in [**12**]. Then some tedious algebra shows that v_n defined by (2.13)–(2.14) satisfies the operator equation

$$v_n = \pi_n K_n v_n + \pi_n f, \tag{2.15}$$

where

$$K_n u(x) = \sum_{l=1}^{M} w_l \kappa(x, t_l) u(t_l) \tag{2.16}$$

and π_n is a 'discrete projection'

$$\pi_n u(x) = \sum_{k=0}^{n} Q_M(u\varphi_k) \varphi_k(x). \tag{2.17}$$

Letting P_n be the operator of orthogonal projection onto $X_n =$ span $(\{\varphi_k\}_{k=0}^n)$, some rearrangement of (2.15) gives

$$v_n = P_n K v_n + R_n v_n + P_n f + r_n \tag{2.18}$$

where

$$R_n v_n = -(P_n K v_n - \pi_n K_n v_n) \tag{2.19}$$

and

$$r_n = -(P_n f - \pi_n f). \tag{2.20}$$

Note that R_n may be viewed as an operator from $X_n \to X_n$ and $v_n \in X_n$.

Using (2.17), (2.20) and the fact that

$$P_n f = \sum_{k=0}^{n} \langle f, \varphi_k \rangle \varphi_k \tag{2.21}$$

$$\begin{aligned} r_n &= -\sum_{k=0}^{n} \langle f, \varphi_k \rangle \varphi_k + \sum_{k=0}^{n} Q_M(f\varphi_k)\varphi_k \\ &= -\left[\sum_{k=0}^{n} (\langle f, \varphi_k \rangle - Q_M(f\varphi_k))\varphi_k \right] \\ &= -\sum_{k=0}^{n} e_k(f\varphi_k)\varphi_k \end{aligned} \tag{2.22}$$

where $e_k(f\varphi_k) = \langle f, \varphi_k \rangle - Q_M(f\varphi_k)$ is the integration error in approximating $\langle f, \varphi_k \rangle$ by $Q_M(f\varphi_k)$.

Similarly

$$R_n v_n = \sum_{k=0}^{n} E_k(\kappa v_n \varphi_k)\varphi_k \tag{2.23}$$

where $E_k(\kappa v_n \varphi_k)$ is the integration error in approximating $\langle K v_n, \varphi_k \rangle$ by $Q_M \times Q_M(\kappa v_n \varphi_k)$.

Letting

$$\|R_n\|_n = lub\{\|R_n w_n\|, w_n \in X_n,\ \|w_n\| = 1\} \tag{2.24}$$

it follows from Theorem 1 of [**18**] that if $\|R_n\|_n \to 0$ and $\|r_n\| \to 0$ that, for all n sufficiently large, $v_n \to u$ and

$$\|u - v_n\| \le c[\|u - u_n\| + \|R_n\|_n + \|r_n\|]. \tag{2.25}$$

Since $\|u - u_n\|$ is bounded by (2.7), it suffices to bound $\|R_n\|_n$ and $\|r_n\|$ in order to bound $\|u - v_n\|$. In [**12**] we used the bounds

$$\|r_n\| = \left[\sum_{k=0}^{n} e_k^2(f\varphi_k)\right]^{1/2} \tag{2.26}$$

and

$$\|R_n\|_n \le \left[\sum_{k=0}^{n}\sum_{j=0}^{n} E_{jk}^2\right]^{1/2}, \tag{2.27}$$

where

$$E_{jk} = \langle K\varphi_k, \varphi_j\rangle - Q_M \times Q_M(\kappa\varphi_k\varphi_j) \tag{2.28}$$

is the integration error in approximating $\langle K\varphi_k, \varphi_j\rangle$ by $Q_M \times Q_M(\kappa\varphi_k\varphi_j)$.

In [**12**] (2.27) was obtained by expanding $w_n = \sum_{k=0}^{n} c_k\varphi_k$, $\sum_{k=0}^{n} c_k^2 = 1$ so that

$$R_n w_n = \sum_{k=0}^{n}\sum_{j=0}^{n} E_{jk}(\kappa\varphi_k\varphi_j)\varphi_k$$

and then using the Cauchy-Schwarz inequality. From this and the results for the integration errors given in [**12**] we found that $\|r_n\| \le cn^{-r+1/2}$ if $f \in C^r$, and $\|R_n\|_n \le cn^{-r+1}$ if $\kappa \in C^r, r > 1$. This gave

$$\|u - v_n\| \le cn^{-r+1}, \qquad r > 1. \tag{2.29}$$

However, if we do not expand w_n as above, then

$$\|R_n\|_n = lub_{\{\|w_n\|=1\}}\left[\sum_{k=0}^{n} E_k^2(\kappa w_n\varphi_k)\right]^{1/2}. \tag{2.30}$$

By modifying the proof in [**12**] it will be shown in Section 3 that

$$|E_k| \le n^{-r} \tag{2.31}$$

for all $w_n \in X_n$ such that $\|w_n\| = 1$. It then follows from (2.30) that $\|R_n\|_n \leq cn^{-r+1/2}$, the same bound as for $\|r_n\|$. Thus (2.25) gives

$$\|u - v_n\| \leq cn^{-r+1/2}, \tag{2.32}$$

an improvement over the bound $\|u - v_n\| \leq cn^{-r+1}$ given in (2.29).

Unfortunately the bound in (2.32) still appears not to be optimal since it was shown in [**12**] that $\|u - v_n\| \leq cn^{-r}$ in the special case that Q_M is $n+1$ point Gaussian quadrature. On the other hand, it seems that (2.32) is the best that can be obtained by perturbation methods.

2.2 *Cauchy singular equations.* Here we consider solving the Cauchy singular integral equation (CSIE)

$$av(x) + \frac{b}{\pi} \int_{-1}^{1} \frac{v(t)}{t-x}\, dt + \int_{-1}^{1} \kappa(x,t) v(t)\, dt = f(x), \tag{2.33}$$

where the first integral in (2.33) is a Cauchy principal value, and for convenience we assume $a^2 + b^2 = 1$. When $\kappa(x,t) = 0$, it is well known that the solution of (2.33) is given by

$$v(x) = w(x) u(x), \tag{2.34}$$

where

$$w(x) = (1-x)^{\alpha}(1+x)^{\beta}, \tag{2.35}$$

$$\alpha = \frac{1}{2(\pi i)} \log\left[\frac{a - ib}{a + ib}\right] + N, \tag{2.36}$$

$$\beta = -\frac{1}{2(\pi i)} \log\left[\frac{a - ib}{a + ib}\right] + M, \tag{2.37}$$

and M and N are integers determined so that the index $\nu = -(\alpha+\beta) = -(N+M)$ is restricted to the values $-1, 0, 1$. This guarantees that the solution to (2.33) is integrable if $f(x)$ is Hölder continuous. For simplicity we only consider the case $\nu = 0$. (For the case $\nu = 1$ we refer the reader to [**10, 11**]. The analysis for $\nu = -1$ seems not to be in the literature.)

The representation $v(x) = w(x)u(x)$ when $\kappa = 0$ suggests the substitution $v(x) = w(x)u(x)$ in (2.33) with $u(x)$ satisfying

$$aw(x)u(x) + \frac{b}{\pi} \int_{-1}^{1} \frac{w(t)u(t)}{t-x}\,dt + \int_{-1}^{1} w(t)\kappa(x,t)u(t)\,dt = f(x). \tag{2.38}$$

For convenience we write (2.38) in operator form

$$Hu + Ku = f, \tag{2.39}$$

where

$$Hu(x) = aw(x)u(x) + \frac{b}{\pi} \int_{-1}^{1} \frac{w(t)u(t)\,dt}{t-x} \tag{2.40}$$

and

$$Ku(x) = \int_{-1}^{1} w(t)\kappa(x,t)u(t)\,dt. \tag{2.41}$$

We assume that (2.39) has a unique solution.

To solve (2.39) by Galerkin's method, let $\{\psi_k\}_{k=0}^{\infty}$ be a set of Jacobi polynomials for $w(x)$ normalized so that

$$\int_{-1}^{1} w(x)\psi_k^2(x)\,dx = 1. \tag{2.42}$$

Similarly let $\{\chi_k\}_{k=0}^{\infty}$ be a set of Jacobi polynomials for $1/w(x)$ normalized so that

$$\int_{-1}^{1} [\chi_k^2(x)/w(x)]\,dx = 1. \tag{2.43}$$

Then it is known that $\{\chi_k\}$ can be chosen so that [**11**]

$$H\psi_k = \chi_k, \qquad k \geq 0. \tag{2.44}$$

Now let L_w denote the space of real functions square-integrable with respect to w, and similarly $L_{1/w}$ is the space of functions square integrable with respect to $1/w$. The inner product on L_ρ, $\rho = w$, or $1/w$ is given by

$$\langle f, g\rangle_\rho = \int_{-1}^{1} \rho(t) f(t) g(t)\, dt \tag{2.45}$$

and the corresponding norm by $\|\cdot\|_\rho$.

From (2.44) it follows that H can be extended as a bounded invertible operator from $L_w \to L_{1/w}$. If $\kappa(x,t)$ is continuous, then k defines a compact operator $K : L_w \to L_{1/w}$. We assume this to be the case for the remainder of the paper. Thus $H+K$ may be considered as a bounded linear operator from $L_w \to L_{1/w}$. Hence, we consider (2.39) as the problem of finding a unique $u \in L_w$ for each $f \in L_{1/w}$.

Using these preliminaries we look for approximations to u given by

$$u_n = \sum_{k=0}^{n} a_k \psi_k \tag{2.46}$$

with $\{a_k\}_{k=0}^n$ determined by setting the residual $\delta_n = (H+K)u_n - f$ orthogonal to $\{\chi_j\}_{j=0}^n$ in the $L_{1/w}$ inner product, i.e.,

$$\langle Hu_n + Ku_n - f, \chi_j\rangle_{1/w} = 0, \qquad 0 \le j \le n. \tag{2.47}$$

Using (2.44) and the fact that $\langle \chi_k, \chi_j\rangle_{1/w} = \delta_{kj}$, (2.47) becomes

$$a_j + \sum_{k=0}^{n} \langle K\psi_k, \chi_j\rangle_{1/w} a_k = \langle f, \chi_j\rangle_{1/w}, \qquad 0 \le j \le n. \tag{2.48}$$

In [**11**] it was shown that if κ is continuous, then for all n sufficiently large u_n exists, is unique and

$$\|u - u_n\|_w \le c\|Hu - \mathcal{Q}_n Hu\|_{1/w} \tag{2.49}$$

where $\mathcal{Q}_n$ is the orthogonal projection of $L_{1/w}$ onto Y_n = span $(\{\chi_j\}_{j=0}^n)$. Since $\mathcal{Q}_n g \to g$ for all $g \in L_{1/w}$, it follows from (2.49) with $g = Hu$ that $u_n \to u$ in L_w.

To get convergence rates, assume that $f \in C^r$ and $k \in C^r$, then $Hu = f - Ku \in C^r$ and it follows from (2.49) and Jackson's theorem [**5**] that

$$\|u - u_n\|_w \leq cn^{-r}, \tag{2.50}$$

which is analogous to (2.7) for Fredholm equations.

However, as for Fredholm equations, the practical implementation of (2.46) and (2.48) requires that the inner products in (2.48) be evaluated numerically. For this we introduce quadrature rules Q_N and Q_M where

$$\int_{-1}^{1} [g(x)/w(x)]\, dx \cong Q_N(g) = \sum_{m=1}^{N} \sigma_m g(x_m), \tag{2.51}$$

and

$$\int_{-1}^{1} w(t)g(t)\, dt \cong Q_M(g) = \sum_{l=1}^{M} w_l g(t_l). \tag{2.52}$$

In addition, we require that

(2.53)

$$(1) \quad \sigma_m > 0, \quad 1 \leq m \leq N, \qquad w_l > 0, \quad 1 \leq l \leq M;$$

(2.54a)

$$(2) \quad Q_N(g) = \int_{-1}^{1} [g(x)/w(x)]\, dx, \quad \forall\, g \in Y_{2n},$$

(2.54b)

$$(3) \quad Q_M(g) = \int_{-1}^{1} w(x)g(x)\, dt, \quad \forall\, g \in X_{2n}.$$

Then we approximate

$$\langle f, \chi_j \rangle_{1/w} \simeq Q_N(f\chi_j), \tag{2.55}$$

and

$$\langle K\psi_k, \chi_j \rangle_{1/w} \simeq Q_M \times Q_N(\kappa\psi_k\chi_j). \tag{2.56}$$

In [**10**] we analyzed the convergence of the approximation v_n to u_n given by replacing the integrals in (2.48) by (2.55)–(2.56). For this we observe that

$$v_n = \sum_{k=0}^{n} b_k \psi_k \tag{2.57}$$

where $\{b_k\}_{k=0}^{n}$ satisfy

$$b_j + \sum_{k=0}^{n} \left[\sum_{m=1}^{N} \sum_{l=1}^{M} \sigma_m w_l \kappa(x_m, t_l) \psi_k(t_l) \chi_j(x_m) \right] b_k = \sum_{m=1}^{N} \sigma_m f(x_m) \chi_j(x_m). \tag{2.58}$$

Letting

$$\pi_n u = \sum_{k=0}^{n} Q_N(u\chi_k)\chi_k \tag{2.59}$$

and

$$K_n u = \sum_{l=1}^{M} w_l \kappa(x, t_l) u(t_l), \tag{2.60}$$

it was shown in [**10**] that (2.57)–(2.58) are equivalent to the operator equation

$$H v_n + \mathcal{Q}_n K v_n + R_n v_n = \mathcal{Q}_n f + r_n, \tag{2.61}$$

where

$$R_n v_n = -[\mathcal{Q}_n K v_n - \pi_n K_n v_n], \tag{2.62}$$

and

$$r_n = -[\mathcal{Q}_n f - \pi_n f], \tag{2.63}$$

where $R_n : X_n \to Y_n$ and $r_n \in Y_n$.

As for the Fredholm case, it can be shown that

$$R_n v_n = -\sum_{k=0}^{n} E_k(\kappa \chi_k v_n)\chi_k \tag{2.64}$$

and

$$r_n = -\sum_{k=0}^{n} e_k(f\chi_k)\chi_k \tag{2.65}$$

where $E_k(\kappa\chi_k v_n)$ is the integration error in approximating $\langle Kv_n, \chi_k\rangle_{1/w}$ by $Q_M \times Q_N(\kappa\chi_k v_n)$ and $e_k(f\chi_k)$ is the error in approximating $\langle f, \chi_k\rangle_{1/w}$ by $Q_N(f\chi_k)$.

Using the result in [**18**] convergence of $v_n \to u$ is guaranteed if $\|R_n\|_n \to 0$ and $\|r_n\|_{1/w} \to 0$ where

$$\|R_n\|_n = lub_{\{\|w_n\|_w=1\}}\{\|R_n w_n\|_{1/w}\}. \tag{2.66}$$

In [**10**] we used the bound

$$\|R_n\|_n \le \left[\sum_{k=0}^{n}\sum_{j=0}^{n} E_{jk}^2\right]^{1/2} \tag{2.67}$$

where $E_{jk} = \langle K\psi_j, \chi_k\rangle_{1/w} - Q_M \times Q_N(\kappa\psi_j\chi_k)$. For $k(x,t) \in C^r$ this gave the error bound $\|R_n\|_n \le cn^{-r+1}$ [**10**]. Similarly, $\|r_n\|_{1/w} \le cn^{-r+1/2}$ and using this in the estimate

$$\|u - v_n\|_w \le c[\|u - u_n\|_w + \|R_n\|_n + \|r_n\|_{1/w}] \tag{2.68}$$

gives

$$\|u - v_n\|_w \le cn^{-r+1}. \tag{2.69}$$

To improve on this here we use

$$\|R_n\|_n = lub_{\{\|w_n\|_w=1\}}\left[\sum_{k=0}^{n} E_k^2\right]^{1/2}. \tag{2.70}$$

From Theorem 3.2 and (2.70) it will be shown that

$$\|R_n\|_n \leq cn^{-r+1/2}, \tag{2.71}$$

so that

$$\|u - v_n\|_w \leq cn^{-r+1/2}. \tag{2.72}$$

Again, this appears to be the best that can be obtained using perturbation theory.

We now proceed to obtain the bounds on the integration errors needed to establish (2.32) and (2.72).

3. Bounds on integration errors.

Theorem 3.1. *Let $w(x) \geq 0$ be a nonnegative integrable weight function, and let $\{\varphi_k\}$ be the orthogonal polynomials associated with w. Let $X_n = \operatorname{span}(\{\varphi_k\}_{k=0}^n)$ and $v_n \in X_n$ with $\|v_n\| = 1$. Consider the integral*

$$I_n = \int_{-1}^{1} w(x) f(x) v_n(x)\, dx \tag{3.1}$$

where $f \in C^r$. Suppose I_n is approximated by $Q_M(fv_n)$ where Q_M is an integration rule satisfying (2.53)–(2.54)*. Then the error*

$$e = I_n - Q_M(fv_n) \tag{3.2}$$

satisfies

$$|e| \leq cn^{-r}, \tag{3.3}$$

where c depends only on f.

Proof. Since $\varphi_k v_n \in X_{2n}$, $0 \leq k \leq n$, and $Q_M(g) = I_n$ for all $g \in X_{2n}$, $e(fv_n) = 0$ for all $f \in X_n$. By Lemma 4.1 of [**12**]

$$|e| \leq \inf_{y \in X_n} \|f - y\|_\infty \|e\| \tag{3.4}$$

where $\|e\|$ is the norm of e considered as a linear functional $C[-1,1]$.

By Jackson's theorem [**2**]

$$\inf_{y\in X_n} \|f-y\|_\infty \le cn^{-r}, \qquad n>r.$$

Thus we need to bound $\|e\|$. Hence

$$|e| \le \int_{-1}^{1} w(x)\,|f(x)|\,|v_n(x)|\,dx + \sum_{m=1}^{M} w_m |f(t_m)|\,|v_n(t_m)|. \tag{3.5}$$

By the Cauchy-Schwarz inequality for integrals and sums

$$\begin{aligned} &\int_{-1}^{1} w(x)|f(x)|\,|v_n(x)|\,dx \\ &\qquad \le \left[\int_{-1}^{1} w(x)f^2(x)\,dx\right]^{1/2}\left[\int_{-1}^{1} w(x)v_n^2(x)\,dx\right]^{1/2} &(3.6)\\ &\qquad = \left[\int_{-1}^{1} w(x)f^2(x)\,dx\right]^{1/2} \\ &\qquad \le \|f\|_\infty\left[\int_{-1}^{1} w(x)\,dx\right]^{1/2} &(3.7)\end{aligned}$$

since $\int_{-1}^{1} w(x)v_n^2(x)\,dx = 1$. Also

$$\begin{aligned} &\sum_{m=1}^{M} w_m|f(x_m)|\,|v_n(x_m)| \\ &\qquad \le \left[\sum_{m=1}^{M} w_m|f(x_m)|^2\right]^{1/2}\left[\sum_{m=1}^{M} w_m v_n^2(x_m)\right]^{1/2} &(3.8)\\ &\qquad \le \|f\|_\infty\left[\sum_{m=1}^{M} w_m\right]^{1/2} \\ &\qquad = \|f\|_\infty\left[\int_{-1}^{1} w(x)\,dx\right]^{1/2} &(3.9)\end{aligned}$$

since $v_n^2 \in X_{2n}$ and Q_M is exact for all $g \in X_{2n}$. Thus

$$|e| \le 2\left[\int_{-1}^{1} w(x)\,dx\right]^{1/2}\|f\|_\infty \tag{3.10}$$

giving

$$\|e\| \le 2\left[\int_{-1}^{1} w(x)\,dx\right]^{1/2}. \tag{3.11}$$

Hence $|e| \le 2c[\int_{-1}^{1} w(x)\,dx]^{1/2} n^{-r}$, $n > r$. □

Theorem 3.2. *Let $\rho \ge 0$, $\gamma \ge 0$ be nonnegative weight functions on $[-1,1]$, and let $\{\varphi_k\}$ and $\{\chi_k\}$ be the orthogonal polynomials associated with ρ and γ, respectively. Let $\kappa(x,t) \in C^r([-1,1]\times[-1,1])$ and consider the integrals*

$$I_n = \int_{-1}^{1}\int_{-1}^{1} \rho(x)\gamma(t)\kappa(x,t)z_n(x)v_n(t)\,dx\,dt \tag{3.12}$$

where $v_n \in \operatorname{span}(\{\chi_k\}_{k=0}^n)$ and $z_n \in \operatorname{span}(\{\varphi_k\}_{k=0}^n)$ with $\|v_n\|_\gamma = \|z_n\| - \rho = 1$. Then the error

$$E = I_n - Q_M \times Q_N(\kappa v_n z_n) \tag{3.13}$$

satisfies

$$|E| \le cn^{-r}, \qquad n > r \tag{3.14}$$

where c depends only on $k(x,t)$ and not on n.

Proof. By definition

$$\begin{aligned} E = &\int_{-1}^{1}\int_{-1}^{1} \rho(x)\gamma(t)\kappa(x,t)v_n(t)z_n(x)\,dx\,dt \\ &- \sum_{l=1}^{N}\sum_{m=1}^{M} \sigma_l w_m \kappa(x_l,t_m)v_n(t_m)z_n(x_l). \end{aligned} \tag{3.15}$$

Letting

$$h_m(x) = \kappa(x,t_m), \qquad 1 \le m \le M, \tag{3.16}$$

and

$$h(t) = \int_{-1}^{1} \rho(x)\kappa(x,t)z_n(x)\,dx, \tag{3.17}$$

$$E = \int_{-1}^{1} \gamma(t)h(t)v_n(t)\,dt - \sum_{m=1}^{M} w_m v_n(t_m)Q_N(h_m z_n). \tag{3.18}$$

But

$$\begin{aligned} Q_N(h_m z_n) &= \int_{-1}^{1} \rho(x)\kappa(x,t_m)z_n(x)\,dx - e(h_m z_n) \\ &= h(t_m) - e(h_m z_n). \end{aligned} \tag{3.19}$$

Using (3.19) in (3.18) gives

$$\begin{aligned} E &= \int_{-1}^{1} \gamma(t)h(t)v_n(t)\,dt \\ &\quad - \sum_{m=1}^{M} w_m v_n(t_m)[h(t_m) - e(h_m z_n)] \\ &= \int_{-1}^{1} \gamma(t)h(t)v_n(t) + \sum_{m=1}^{M} w_m v_n(t_m)e(h_m z_n) \\ &\quad - \sum_{m=1}^{M} w_m v_n(t_m)h(t_m) \end{aligned} \tag{3.20}$$

$$= e(hv_n) + \sum_{m=1}^{M} w_m v_n(t_m)e(h_m z_n). \tag{3.21}$$

By the proof of Theorem 3.1 $|e(hv_n)| \le c_1 n^{-r}$ and $|e(h_m z_n)| \le c_2 n^{-r}$ uniformly in m. (The uniformity in m follows from the error formula in Jackson's theorem [**3**].) Thus

$$|E| \le c_1 n^{-r} + c_2 n^{-r} \sum_{m=1}^{M} w_m |v_n(t_m)|.$$

By the argument in Theorem 3.1

$$\sum_{m=1}^{M} w_m |v_n(t_m)| \le \left[\int_{-1}^{1} \gamma(t)\, dt \right]^{1/2}, \tag{3.22}$$

so that

$$|E| \le c_1 n^{-r} + c_3 n^{-r} \le c_4 n^{-r}. \qquad \square \tag{3.23}$$

Summarizing our arguments in Section 2 and using Theorem 3.2 we get the following theorems.

Theorem 3.3. *Let v_n be the discrete Galerkin approximation to the solution of the Fredholm equation* (2.1). *If $f \in C^r$ and $\kappa \in C^r$, $r \ge 1$, then v_n converges to u in L_2 and $\|u - v_n\| \le cn^{-r+1/2}$.*

Proof. This follows from the error bound (2.25) and Theorem 3.2 with $\rho = \gamma = 1$, $v_n \in X_n$ and $z_n = \varphi_k$, $0 \le k \le n$. $\square$

Theorem 3.4. *Let v_n be the discrete Galerkin approximation to the* CSIE (2.33). *If $f \in C^r$, $\kappa \in C^r$, $r \ge 1$, then v_n converges to u in L_w and $\|u - v_n\|_w \le n^{-r+1/2}$.*

Proof. This follows from the error bound (2.68) and Theorem 3.2 with $\gamma = w$, $\rho = 1/w$, $v_n \in X_n$ and $z_n = \chi_k$, $0 \le k \le n$. $\square$

Remark. In [**10**] we analyzed a collocation method for the CSIE (2.33) with $u \cong u_n = \sum_{k=0}^{n} a_k \psi_k$ with $\{a_k\}$ obtained by setting $(H+K)u_n(x_j) - f(x_j) = 0$, $0 \le j \le n$ where $\{x_j\}$ are the zeros of χ_{n+1}. When the integrals $\{Ku_n(x_j)\}$ were approximated by using Q_M in (2.52), it was shown that the corresponding discrete collocation method v_n converges to u in L_w and $\|u - v_n\|_w \le cn^{-r+1/2}$. Theorem 3.4 shows that the discrete Galerkin and collocation methods appear to converge at the same rate. In [**13**] we showed that the discrete Galerkin method can be implemented using an 'almost sparse' matrix. This suggests that the discrete Galerkin method may be more efficient than the corresponding discrete collocation method. This is in contrast to the conventional wisdom on this subject [**11**].

One can define a similar discrete collocation method for (2.1) with the same $0(n^{-r+1/2})$ convergence rate as for v_n.

5. A hypersingular equation. In recent years there has been considerable interest in hypersingular equations as a means for solving a variety of boundary value problems [**16–18**].

For the one-dimensional equation

$$\frac{1}{\pi}\frac{d}{dx}\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}u(t)\,dt+\int_{-1}^{1}\sqrt{1-t^2}\kappa(x,t)u(t)\,dt=f(x) \tag{4.1}$$

a Galerkin method was developed by Frenkel in [**6**] and analyzed by Golberg in [**8, 9**]. In those papers it was assumed that all the integrals were evaluated analytically. Here, using the results in Theorems 3.1 and 3.2, we analyze the convergence of a discrete version of this algorithm where quadrature errors are taken into account. We begin by reviewing the analysis given in [**8**].

For this we write (4.1) in operator form as

$$Hu+Ku=f \tag{4.2}$$

where

$$Hu=\frac{1}{\pi}\frac{d}{dx}\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}u(t)\,dt \tag{4.3}$$

and

$$Ku=\int_{-1}^{1}\sqrt{1-t^2}\kappa(x,t)u(t)\,dt. \tag{4.4}$$

If

$$U_n=\sin(n+1)\theta/\sin\theta,\qquad \theta=\cos^{-1}x, \tag{4.5}$$

are the Chebyshev polynomials of the second kind, then [**8**]

$$HU_n=-(n+1)U_n,\qquad n\geq 0. \tag{4.6}$$

It then follows from (4.6) that H can be extended as a bounded linear operator from $L_1(\rho) \to L(\rho)$ where $L(\rho)$ is the space of functions square integrable with respect to $\rho = (1-x^2)^{1/2}$ and $L_1(\rho)$ is the subspace of functions $u \in L(\rho)$ satisfying

$$\|u\|_1^2 = \sum_{k=0}^{\infty}(k+1)^2\langle u, \varphi_k\rangle_\rho^2 < \infty \tag{4.7}$$

where

$$\varphi_k = \sqrt{\frac{2}{\pi}}U_k \tag{4.8}$$

and

$$\langle f, g\rangle_\rho = \int_{-1}^{1} \rho(t)f(t)g(t)\,dt. \tag{4.9}$$

If $k(x,t)$ is continuous, then K defines a compact operator from $L_1(\rho) \to L(\rho)$ [**8**]. Thus, we consider solving (4.1) for $u \in L_1(\rho)$ for $f \in L(\rho)$. We assume that (4.1) has a unique solution for each $f \in L(\rho)$.

To solve (4.1) by Galerkin's method we approximate u by

$$u_n = \sum_{k=0}^{n} a_k\varphi_k$$

where $\{a_k\}_{n=0}^{n}$ are obtained by solving

$$\langle (H+K)u_n - f, \varphi_j\rangle_\rho = 0, \qquad 0 \le j \le n. \tag{4.10}$$

Using (4.6) and (4.9) gives

$$-(j+1)a_j + \sum_{k=0}^{n} a_k\langle K\varphi_k, \varphi_j\rangle_\rho = \langle f, \varphi_j\rangle_\rho, \qquad 0 \le j \le n. \tag{4.11}$$

In [**9**] it was shown that $u_n \to u$ in $L(\rho)$ and

$$\|u - u_n\|_\rho \le cn^{-r} \tag{4.12}$$

if $f \in C^r$ and $k \in C^r$. It was also shown that if $r > 3$ that $\|u - u_n\|_\infty \leq cn^{-r+2}$. These results were obtained under the assumption that $\{\langle K\varphi_k, \varphi_j\rangle_\rho\}$ and $\{\langle f, \varphi_j\rangle_\rho\}$ were evaluated analytically. For practical implementation we consider quadrature rules $\{Q_M\}$ where

$$\int_{-1}^{1} \rho(t)g(t)\,dt \simeq Q_M(g) = \sum_{m=1}^{M} w_m g(t_m) \tag{4.13}$$

satisfying (2.53) and (2.54) and approximate

$$\langle K\varphi_k, \varphi_j\rangle_\rho \simeq Q_M \times Q_M(\kappa\varphi_k\varphi_j) \tag{4.14}$$

and

$$\langle f, \varphi_j\rangle_\rho \simeq Q_M(f\varphi_j). \tag{4.15}$$

Then using (4.14) and (4.15) in (4.11) u_n is approximated by

$$v_n = \sum_{k=0}^{n} b_k\varphi_k \tag{4.16}$$

where $\{b_k\}_{k=0}^n$ satisfy

$$\begin{aligned} -(j+1)b_j + \sum_{k=0}^{n} b_k\Big[\sum_{m=1}^{M}\sum_{l=1}^{M} w_l w_m \kappa(x_l, t_m)\varphi_k(t_m)\varphi_j(x_l)\Big] \\ = \sum_{l=1}^{M} w_l f(x_l)\varphi_j(x_l), \end{aligned} \tag{4.17}$$

and $\{t_m\}$ and $\{t_l\}$ represent the same set of points.

Proceeding as in the Fredholm and singular cases, one can show that v_n satisfies the operator equation

$$Hv_n + P_nKv_n + R_nv_n = P_nf + r_n \tag{4.18}$$

where P_n is the operator of orthogonal projection onto span $(\{\varphi_k\}_{k=0}^n)$,

(4.19)
$$R_n v_n = [\pi_n K_n v_n - P_n K v_n] = -\sum_{k=0}^{n} E_k(\kappa \varphi_k v_n)\varphi_k,$$

(4.20)
$$K_n v_n = \sum_{m=1}^{M} w_m \kappa(x, t_m) v_n(t_m),$$

(4.21)
$$\pi_n u = \sum_{k=0}^{n} Q_M(u\varphi_k)\varphi_k,$$

and

$$r_n = \pi_n f - P_n f = -\sum_{k=0}^{n} e_k(f\varphi_k)\varphi_k \tag{4.22}$$

where E_k and e_k are the quadrature errors in approximating $\langle K v_n, \varphi_k\rangle_\rho$ and $\langle f, \varphi_k\rangle_\rho$ by $Q_M \times Q_M(\kappa\varphi_k v_n)$ and $Q_M(f\varphi_k)$, respectively.

Using Theorem 1 of [**18**] to establish convergence of v_n, it suffices to show

$$\|r_n\|_\rho \to 0 \tag{4.23}$$

and

$$\|R_n\|_n \to 0 \tag{4.24}$$

where

$$\|R_n\|_n = lub_{\{\|w_n\|_1 = 1\}}\{\|R_n w_n\|_\rho\}. \tag{4.25}$$

Since $\|w_n\|_\rho \le \|w_n\|_1$

$$\begin{aligned} \|R_n\|_n &\le lub\{\|R_n w_n\|_\rho, w_n \in X_n,\ \|w_n\|_\rho \le 1\} \\ &= lub\{\|R_n w_n\|_\rho, w_n \in X_n,\ \|w_n\|_\rho = 1\} \\ &= lub_{\{\|w_n\|_\rho = 1\}} \sum_{k=0}^{n} |E_k(\kappa\varphi_k w_n)|^2. \end{aligned} \tag{4.26}$$

Using Theorems 3.1 and 3.2, we find that

$$\|R_n\|_n \le c_1 n^{-r+1/2} \tag{4.27}$$

and

$$\|r_n\|_\rho \le c_2 n^{-r+1/2} \tag{4.28}$$

if $\kappa \in C^r$ and $f \in C^r$. Using (4.27), (4.28) and (4.12) and the error estimate [**18**]

$$\begin{aligned}\|u - v_n\|_\rho &\le \|u - v_n\|_1 \\ &\le c[\|u - u_n\|_1 + \|R_n\|_n + \|r_n\|_\rho]\end{aligned} \tag{4.29}$$

we find that

$$\|u - v_n\|_\rho \le c n^{-r+1/2}. \tag{4.30}$$

Thus, the discrete Galerkin method for (4.1) has a convergence rate analogous to that for Fredholm equations and CSIEs.

5. Fredholm equations on the sphere. The results given in Sections 2–4 are for one-dimensional equations. In this section we show how our previous analysis can be extended to analyze an algorithm developed by Atkinson for solving Fredholm equations on the sphere when the data are smooth.

For this, let

$$D = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

be the unit sphere in $\mathbf{R}^3$ and consider the integral equation

$$u(\mathbf{P}) = \int_D \kappa(\mathbf{P}, \mathbf{Q}) u(\mathbf{Q})\, ds + f(\mathbf{P}), \qquad \mathbf{P} \in D, \tag{5.1}$$

where $f \in C^r(D)$ and $\kappa(\mathbf{P}, \mathbf{Q}) \in C^r(D \times D)$, $r \ge 1$. We assume that (5.1) has a unique solution in C^r for each $f \in C^r$.

In a series of papers Atkinson considered solving (5.1) using Galerkin's method with spherical harmonics as a basis [**1**–**3**]. Here we analyze the convergence of the discrete version of this algorithm using integration rules analogous to (2.9) for the one-dimensional case.

Now let $X_n = \operatorname{span}(\{Y_{lm}, 0 \le l \le n, 1 \le m \le 2l+1\})$ be the set of spherical polynomials of degree $\le n$ where $\{Y_{lm}\}$ are the normalized spherical harmonics on D [**2**]. The dimension of X_n is $N(n) = (n+1)^2$. As is well known, $\{Y_{lm}\}$ is an orthonormal basis for X_n.

Let $\left\{\hat{Y}_1, \cdots, \hat{Y}_N\right\}$ be an ordering of $\{Y_{lm}\}$ and approximate u by

$$u_N = \sum_{k=1}^{N} a_k \hat{Y}_k \tag{5.2}$$

where $\{a_k\}_{k=1}^N$ are determined by solving

$$a_j = \sum_{k=1}^{N} \langle K\hat{Y}_k, \hat{Y}_j\rangle a_k + \langle f, \hat{Y}_j\rangle, \qquad 1 \le j \le N, \tag{5.3}$$

and the inner product $\langle\ ,\rangle$ is given by

$$\langle f, g\rangle = \int_D f(\mathbf{P}) g(\mathbf{P})\, ds. \tag{5.4}$$

In [**2**] it was shown that $u_N \to u$ in $L_2(D)$ and

$$\|u - u_N\| \le c n^{-r}. \tag{5.5}$$

(In fact in [**2**] only $K : L_2 \to C^r$ was needed, so that singular potential kernels satisfy the theory.)

As for the one-dimensional cases practical implementation of (5.2)–(5.3) requires the numerical evaluation of the integrals in (5.3). To do this, suppose that Q_M is a quadrature rule

$$\int_D g(\mathbf{P})\, ds \simeq Q_M(g) = \sum_{m=1}^{M} w_m g(\mathbf{P}_m), \qquad \mathbf{P}_m \in D, \tag{5.6}$$

satisfying

$$(1) \quad w_m > 0, \quad 1 \le m \le M \tag{5.7}$$

$$(2) \quad Q_M(g) = \int_D g(\mathbf{P})\, ds, \quad \forall\, g \in X_{2n}. \tag{5.8}$$

One example of such a rule is

$$Q_M(g) = \delta \sum_{i=1}^{2(n+1)} \sum_{j=1}^{n+1} w_j g(\theta_j, \phi_j),$$

where $\delta = \pi/(n+1)$, $\phi_i = i\delta$ and $\{w_j\}\{\cos\theta_j\}$ are the Gauss-Legendre weights and nodes on [-1, 1], where $g(\theta,\phi) = g(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is the representation of g in spherical polar coordinates on D.

Then approximate

$$\langle f, \hat{Y}_j \rangle \simeq Q_M(f\hat{Y}_j) \tag{5.9}$$

and

$$\langle K\hat{Y}_k, \hat{Y}_j \rangle \simeq Q_M \times Q_M(\kappa \hat{Y}_k \hat{Y}_j). \tag{5.10}$$

This gives $v_N \simeq u_N$ where

$$v_N = \sum_{k=1}^{N} b_k \hat{Y}_k \tag{5.11}$$

and $\{b_k\}_{k=1}^N$ satisfy

$$\begin{aligned} b_j = \sum_{k=1}^{N} b_k \left[\sum_{m=1}^{M} \sum_{l=1}^{M} w_l w_m \kappa(\mathbf{P}_m, \mathbf{Q}_l) \hat{Y}_k(\mathbf{Q}_l) \hat{Y}_j(\mathbf{P}_m) \right] \\ + \sum_{l=1}^{M} w_l f(\mathbf{Q}_l) \hat{Y}_j(\mathbf{Q}_l), \end{aligned} \tag{5.12}$$

and $\{\mathbf{P}_m\}$ and $\{\mathbf{Q}_l\}$ represent the same set of points.

Letting P be the orthogonal projection onto X_n,

$$K_N u = \sum_{l=1}^{M} w_l \kappa(\mathbf{P}, \mathbf{Q}_l) u(\mathbf{Q}_l) \tag{5.13}$$

and

$$\pi_N u = \sum_{k=1}^{N} Q_M(u\hat{Y}_k)\hat{Y}_k, \tag{5.14}$$

v_N satisfies

$$v_N = P_N K v_N + R_N v_N + P_N f + r_N$$

where

$$R_N v_N = [P_N K - \pi_N K_N] v_N = \sum_{k=1}^{N} E_k(\kappa \hat{Y}_k v_N) \hat{Y}_k \tag{5.15}$$

and

$$r_N = -[P_N f - \pi_N f] = \sum_{k=1}^{N} e_k(f \hat{Y}_k) \hat{Y}_k. \tag{5.16}$$

As in the one-dimensional cases, define

$$\begin{aligned} \|R_N\|_N &= lub\{\|R_N w_N\|, w_N \in X_n, \|w_N\| = 1\} \\ &= lub_{\{\|w_N\|=1\}} \left[\sum_{k=1}^{N} E_k^2(\kappa \hat{Y}_k w_N) \right]^{1/2}, \end{aligned} \tag{5.17}$$

and

$$\|r_N\| = \left[\sum_{k=1}^{N} e_k(f \hat{Y}_k) \right]^{1/2}. \tag{5.18}$$

Then we can show that $v_N \to u$ provided that

$$\|R_N\|_N \to 0 \quad \text{and} \quad \|r_N\| \to 0. \tag{5.19}$$

For this we need to bound the quadrature errors $E_k(\kappa \hat{Y}_k w_N)$ and $e_k(f \hat{Y}_k)$. This is done just as in Theorems 3.1 and 3.2 with a theorem of Ragozin replacing Jackson's theorem to bound the uniform approximation error of f and κ by spherical polynomials of degree $\leq n$. To avoid being repetitive, we will just state the relevant results and give a brief outline of the proofs.

Theorem 5.1. *Let $f \in C^r(D)$, $r > 1$, and let Q_M be a quadrature rule satisfying* (5.7)–(5.8). *Let $w_N \in X_n$ with $\|w_N\| = 1$. Then the integration error*

$$e(f w_N) = \langle f, w_N \rangle - Q_M(f w_N) \tag{5.20}$$

satisfies

$$|e| \leq cn^{-r}, \qquad r > 1, \tag{5.21}$$

where c depends only on f and not on n.

Proof. Arguing as in Lemma 4.2 of [**12**] using (5.7)–(5.8) we find that

$$|e| \leq c \inf_{g \in X_n} \|f - g\| \tag{5.22}$$

where c depends only on f and not on n. By a theorem of Ragozin [**19**] there exists a function $h \in X_n$ such that

$$\|f - h\|_\infty \leq cn^{-r}\|f\|_r \tag{5.23}$$

where $\|f\|_r$ is the C^r norm of f given in [**19**]. Hence $\inf_{g \in X_n} \|f-g\|_\infty \leq \|f - h\|_\infty \leq cn^{-r}\|f\|_r$ and using this in (5.22) gives (5.21). □

Corollary 5.1. $\|r_N\| \leq cn^{-r+1}$, $r > 1$.

Proof. Use (5.23) in (5.16). □

Theorem 5.2. *Let $\kappa(\mathbf{P}, \mathbf{Q}) \in C^r(D \times D)$, $r > 1$, and let Q_M satisfy (5.7)–(5.8). If $w_N \in X_n$ and $z_N \in X_n$ with $\|w_N\| = \|z_N\| = 1$, the integration error*

$$E(\kappa w_N z_N) = \langle K w_N, z_N \rangle - Q_M \times Q_M(\kappa w_N z_N) \tag{5.24}$$

satisfies

$$|E| \leq cn^{-r}, \qquad r > 1, \tag{5.25}$$

where c depends only on κ and not on n.

Proof. Repeat the argument in Theorem 3.2 using Theorem 5.1 in place of Theorem 3.1 and using Ragozin's theorem in place of Jackson's theorem. □

Corollary 5.2. *$|E_k| \leq cn^{-r}$ independent of w_N.*

Proof. Letting $z_N = \hat{Y}_k$, $1 \leq k \leq N$ in $E(\kappa w_N \hat{Y}_k)$, then (5.25) gives $|E| \leq cn^{-r}$ independent of w_N. □

Corollary 5.3. *$\|R_N\|_N \leq cn^{-r+1}$.*

Proof. Using Corollary 5.2 ,

$$lub_{\{\|w_N\|=1\}}\bigg[\sum_{k=1}^{N} E_k^2(\kappa w_N \hat{Y}_k)\bigg]^{1/2} \leq cnn^{-r} = cn^{-r+1}. \qquad \square$$

Using Theorems 5.1–5.2 one establishes the convergence of v_N.

Theorem 5.3. *Let $f \in C^r(D)$ $\kappa \in C^r(D \times D)$, $r > 1$, and $\{Q_M\}$ be a sequence of quadrature rules satisfying* (5.7)–(5.8). *Then $v_N \to u$ in $L^2(D)$ and*

$$\|u - v_N\| \leq cn^{-r+1}. \tag{5.26}$$

Proof. From Corollaries (5.1)–(5.3) $\|r_N\| \to 0$ and $\|R_N\|_N \to 0$. Hence, it follows from

$$\|u - v_N\| \leq c[\|u - u_N\| + \|R_N\|_N + \|r_N\|] \leq cn^{-r+1} \tag{5.27}$$

and Corollaries (5.1)–(5.3) and (5.5) that $v_N \to u$ and $\|u - v_N\| \leq cn^{-r+1}$. □

Again, as in the one-dimensional cases, this appears to be the best bound that can be obtained by perturbation methods. For another approach to this problem, see [**7**].

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