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ON THE INVERSION OF HIGHER ORDER WIENER-HOPF OPERATORS

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ABSTRACT. It is known that the Banach algebra generated by classical Wiener-Hopf operators on the half-line is an algebra with symbol. This concept yields, in particular, a Fredholm criterion and an index formula. In the present paper we introduce a different symbol for the finitely generated algebra. It is based on matricially coupling of operators and implies a representation of a generalized inverse in terms of matrix factorization. Some examples demonstrate how to use these results for a discussion of properties of the solution of singular equations.

1. Introduction. Our main objective is to construct generalized inverses for particular classes of operators which are somehow related to singular operators. This desire comes from mathematical physics where analytical formulas are needed to obtain direct information about the qualitative behavior of the solution of a linear operator equation, for instance asymptotic expansion, which cannot be obtained from the knowledge of a Fredholm pseudoinverse.

More precisely, if L^- denotes a generalized inverse of a bounded linear operator $L \in \mathcal{L}(X,Y)$ acting between Banach spaces, i.e. if

$$LL^-L = L \tag{1.1}$$

holds, then the equation

$$Lf = g \tag{1.2}$$

(for given $g \in Y$ and unknown $f \in X$) is solvable if and only if $LL^-g = g$ and the general solution in this case reads explicitly

$$f = L^-g + (I - L^-L)h, \qquad h \in X. \tag{1.3}$$

One of the most popular examples where L^- can be represented in closed analytical form is the Wiener-Hopf equation on the half-line

$$Wf(x) = \lambda f(x) + \int_0^\infty k(x-y)f(y)dy = g(x) \tag{1.4}$$

for $f, g \in L^p(\mathbf{R}_+)$, if the basic operator

$$A = \lambda I + k* = \mathcal{F}^{-1}\phi \cdot \mathcal{F} \tag{1.5}$$

admits a cross factorization $A = A_- C A_+$ due to a generalized factorization of the Fourier symbol $\phi = \phi_- D \phi_+$. This yields a generalized inverse

$$W^- = r_+ A_+^{-1} P C^{-1} P A_-^{-1} l_0 \tag{1.6}$$

where l_0, r_+ denote the operators of extension by zero from $\mathbf{R}_+$ to $\mathbf{R}$ and restriction from $\mathbf{R}$ on $\mathbf{R}_+$, respectively, and $P = l_0 r_+ = 1_+ \cdot$ [**4, 15, 17**].

An asymptotic expansion of $W^- g$ at zero is obtained from an expansion of ϕ_+^{-1} at infinity by the help of Abel type theorems for the Fourier transformation, provided the "physical data g are reasonable" (smooth and decreasing), see [**16**].

Here we shall extend these ideas to WHOs of higher order

$$L = \sum_{i=1}^{N} \prod_{j=1}^{M} W^{ij} \tag{1.7}$$

(including the systems case) which are related to matrix WHOs, see [**5, 6**], and define for this reason:

Definition 1.1. Let $\mathcal{K} \subset \mathcal{L}(X, Y)$ be a class of operators acting between Banach spaces. A measurable matrix function ϕ on $\mathbf{R}$ is called a GI-*symbol* for $L \in \mathcal{K}$, if

i) generalized invertibility of L can be expressed in terms of properties of ϕ, and

ii) provided L is generalized invertible, a generalized inverse L^- can be represented in terms of a factorization of ϕ (and global parameters of the class $\mathcal{K}$ but not of the particular operator L).

Remarks 1. In order to serve the purposes of asymptotic analysis, it is important that the formula for L^- has "closed analytical form". For instance the representation of generalized inverses of WHOs on the quarter-plane is possible by the help of infinite products of operators [**14**], but not very useful for asymptotic considerations.

2. In general, composition formulas for GI-symbols do not hold, since the "reverse order law" $(L_1L_2)^- = L_2^- L_1^-$ is not satisfied. Thus the word "symbol" does not have the meaning of "invertibility symbol" but of "Fourier symbol of an associated translation invariant operator."

2. Construction of an equivalence relation. Now we are going to relate operators with matrices of WHOs.

Let X be a Banach space and $\mathcal{M}$ a linear manifold in the algebra of all linear bounded operators acting in X, $\mathcal{L}(X)$, such that $\mathcal{M}$ contains the identity operator I. We let $\mathcal{K}$ denote the set of operators of the form

$$L = \sum_{i=1}^{N} \prod_{j=1}^{M_i} T^{ij}$$
$$N, M_i \in \mathbf{N}, \quad T^{ij} \in \mathcal{M}$$

Theorem 2. *The operator matrix*

$$L = \left[\sum_{i=1}^{N_{kl}} \prod_{j=1}^{M_i} T_{kl}^{ij} \right]_{k,l=1,\dots,n} \in \mathcal{K}^{n\times n} \tag{2.2}$$

$(n \in \mathbf{N})$ *is equivalent after extension to an operator matrix*

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} : X^n \oplus X^m \longrightarrow X^n \oplus X^m \tag{2.3}$$

where, denoting by $(T_v)_{kl}$ *the* kl*-element of* T_v, $v = 1, \dots, 4$,

$$(T_1)_{kl} = \begin{cases} 0 & \textit{if } M_{i(k,l)} \neq 1,\ \forall\, 1 \le i \le N_{kl} \\ T_{kl}^{i'1} & \textit{otherwise} \end{cases} \tag{2.4}$$

(*where* i' *is the smallest* i *so that* $M_{i(k,l)} = 1$),

$$(T_2)_{kl} \in \{0, I, T_{kl}^{ij}\}, \qquad (T_3)_{kl} \in \{0, -T_{kl}^{ij}\}, \tag{2.5}$$

T_4 is a lower triangular matrix with $(T_4)_{kl} \in \{0, I, -T_{kl}^{ij}\}$, $(T_4)_{kk} = I$, and

$$m = \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{i=1}^{N_{kl}} f(M_{i(k,l)})$$

with

$$f(M_{i(k,l)}) = \begin{cases} M_{i(k,l)} - 1 & \text{if } M_i \geq 2 \vee N_{kl} = 1 \vee \\ & (N_{kl} \neq 1 \wedge M_i = 1 \wedge M_{i'} \neq 1, \\ & \forall\, i' < i) \\ 1 & \text{if } M_i = 1 \wedge N_{kl} \geq 2 \wedge \exists i' < i : \\ & M_{i'} = 1 \end{cases} \tag{2.7}$$

Proof. This result is an iteration of the following operation

$$\begin{aligned} \varphi_r : \mathcal{K}^{(n+r)\times(n+r)} &\longrightarrow \mathcal{K}^{(n+r+1)\times(n+r+1)} \\ L_r &\longmapsto L_{r+1}, \qquad r \in \mathbf{N}_0 \end{aligned} \tag{2.8}$$

beginning with $L_0 = L$ and defined by

$$L_{r+1} = \left[\begin{array}{ccccccc|c} & & & & & & & 0 \\ & & & & & & & \vdots \\ & & & & & & & 0 \\ & & & \check{L}_r & & & & \alpha \\ & & & & & & & 0 \\ & & & & & & & \vdots \\ & & & & & & & 0 \\ \hline 0 & \cdots & 0 & \beta & 0 & \cdots & 0 & I \end{array}\right] \tag{2.9}$$

where, after a reordering of the terms in (2.2) such that $M_1 \leq M_2 \leq \cdots \leq M_N$ holds,

$$\alpha = \begin{cases} \prod_{j=1}^{M_{N_{kl}}-1} T_{kl}^{N_{kl}j} & \text{if } M_{N_{kl}} \geq 2 \\ I & \text{if } M_{N_{kl}} = 1 \end{cases} \tag{2.10}$$

indicates the term in the place $(k,\ n+r+1)$ of the matrix,

$$\beta = -T_{kl}^{N_{kl}M_{N_{kl}}} \tag{2.11}$$

indicates the term in the place $(n+r+1,\ l)$ and $\check{L}_r$ is equal to L_r with the exception of the kl-element that is transformed from L_r to $\check{L}_r$ by the rule

$$\sum_{i=1}^{N_{kl}} \prod_{j=1}^{M_i} T_{kl}^{ij} \longmapsto \begin{cases} \sum_{i=1}^{N_{kl}-1} \prod_{j=1}^{M_i} T_{kl}^{ij} & \text{if } N_{kl} \geq 2 \\ 0 & \text{if } N_{kl} = 1 \end{cases} \tag{2.12}$$

where the operator $\sum_{i=1}^{N_{kl}} \prod_{j=1}^{M_i} T_{kl}^{ij}$ is the kl-element of the L_r matrix.

This iteration (which begins with $\varphi_0(L) = L_1$) is processed every time when the L_r matrix has kl-elements which are not in $\mathcal{M}$.

For a unique determination of the procedure one may choose, for instance, a lexicographical order, i.e., first L_{11} is reduced completely, then L_{12} and so on.

Each step in the process yields an equation

$$\begin{bmatrix} L_r & 0 \\ 0 & I \end{bmatrix} = E_{r+1} L_{r+1} F_{r+1} \tag{2.13}$$

where

$$E_{r+1} = \left[\begin{array}{ccc|c} & & & 0 \\ & & & \vdots \\ & & & 0 \\ & I & & -\alpha \\ & & & 0 \\ & & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & I \end{array}\right] \tag{2.14}$$

and

$$F_{r+1} = \left[\begin{array}{ccccccc|c} & & & & & & & 0 \\ & & & I & & & & \vdots \\ & & & & & & & 0 \\ \hline 0 & \cdots & 0 & -\beta & 0 & \cdots & 0 & I \end{array}\right] \tag{2.15}$$

From that we obtain

$$\begin{bmatrix} L & 0 \\ 0 & I \end{bmatrix} = ETF : X^n \oplus X^m \longrightarrow X^n \oplus X^m \tag{2.16}$$

in which

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$$

has only elements of the form T^{ij}_{kl}, $-T^{ij}_{kl}$, I, 0 and T_1, T_2, T_3, T_4 fulfill the conditions (2.4) and (2.5). The matrix $E = E_1$, for $m = 1$ and $E = (E_1 \oplus I \oplus \cdots \oplus I) \cdots E_m$ for $m > 1$, where I is $m-1$ times repeated, is an upper triangular matrix

$$E = \begin{bmatrix} I & \tilde{E} \\ 0 & I \end{bmatrix} : X^n \oplus X^m \longrightarrow X^n \oplus X^m \tag{2.18}$$

where $\tilde{E}$ only contains elements of the type 0, I, $-T^{ij}_{kl}$, T^{ij}_{kl} or products of those.

The operator matrix $F = F_1$ if $m = 1$ and $F = F_m \cdots (F_1 \oplus I \oplus \cdots \oplus I)$ if $m > 1$ is a lower triangular matrix and has the form

$$F = \begin{bmatrix} I & 0 \\ -\tilde{T}_3 & \tilde{T}_4^{-1} \end{bmatrix} : X^n \oplus X^m \longrightarrow X^n \oplus X^m \tag{2.19}$$

where $\tilde{T}_3 = T_3 + \mathcal{T}_3$, $\mathcal{T}_3 = 0$ or $\mathcal{T}_3$ is a matrix whose elements are product of elements in $-T_3$ or $-T_4$, see (2.17); $\tilde{T}_4 = T_4 + \mathcal{T}_4$, $\mathcal{T}_4$ is the null matrix or have some elements different from zero that are product of elements in $-T_4$. Finally, the index m indicates the number of steps that are necessary to obtain the W operator matrix, i. e., the number of times that the iteration is processed, that is

$$m = \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{i=1}^{N_{kl}} f(M_{i(k,l)}) \tag{2.20}$$

where

$$f(M_{i(k,l)}) = \begin{cases} M_{i(k,l)} - 1 & \text{if } M_i \geq 2 \vee N_{kl} = 1 \vee \\ & N_{kl} \neq 1 \wedge M_i = 1 \wedge M_{i'} \neq 1, \\ & \forall\, i' < i) \\ 1 & \text{if } M_i = 1 \wedge N_{kl} \geq 2 \wedge \exists i' < i\text{:} \\ & M_{i'} = 1 \end{cases} \tag{2.21}$$

From (2.18) and (2.19) it follows that there exist inverses of E and F, respectively, given by

$$E^{-1} = \begin{bmatrix} I & -\tilde{E} \\ 0 & I \end{bmatrix} : X^n \oplus X^m \longrightarrow X^n \oplus X^m \tag{2.22}$$

and

$$F^{-1} = \begin{bmatrix} I & 0 \\ \tilde{T}_4\tilde{T}_3 & \tilde{T}_4 \end{bmatrix} : X^n \oplus X^m \longrightarrow X^n \oplus X^m, \tag{2.23}$$

respectively. This fact and equation (2.16) imply the statement. □

Example. The operator $T^{11}+T^{21}T^{22} \in \mathcal{K}$ is equivalent after extension to

$$T = \begin{bmatrix} T^{11} & T^{21} \\ -T^{22} & I \end{bmatrix} \in \mathcal{K}^{2\times 2}. \tag{2.24}$$

This can be proved by following the iteration presented in the last theorem. If we apply that iteration we have the transformation, from $\mathcal{K}$ to $\mathcal{K}^{2\times 2}$,

$$T^{11} + T^{21}T^{22} \longmapsto \begin{bmatrix} T^{11} & T^{21} \\ -T^{22} & I \end{bmatrix} \tag{2.25}$$

and so the correspondent to formula (2.16) is, in this case,

$$\begin{bmatrix} T^{11} + T^{21}T^{22} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & -T^{21} \\ 0 & I \end{bmatrix} \begin{bmatrix} T^{11} & T^{21} \\ -T^{22} & I \end{bmatrix} \begin{bmatrix} I & 0 \\ T^{22} & I \end{bmatrix} \tag{2.26}$$

which shows the initial statement.

Remark. We emphasize that the equation presented in (2.16) is a factorization into simpler structured factors, that is, a lower-upper-factorization, with invertible outer factors and the matrix W contains only the particular elements of $\mathcal{M}$ that we found in L. The number of steps can be eventually reduced, if we combine $\mathcal{M}$ elements in linear combinations during the process. This leads only to a different identification of terms in (2.2) and, therefore, it can be considered as a subcase of the present formulation.

3. Generalized inversion of L. An operator $W : X_2 \to X_2$, which is equivalent after extension to $L : X_1 \to X_1$, is also called an indicator of L, see [**1, 2**].

$$L \overset{*}{\sim} W \tag{3.1}$$

$$\begin{bmatrix} L & 0 \\ 0 & I_{Z_1} \end{bmatrix} = E \begin{bmatrix} W & 0 \\ 0 & I_{Z_2} \end{bmatrix} F \tag{3.2}$$

or

$$l_{Z_1}(L) = E l_{Z_2}(W) F$$

where $E : X_2 \oplus Z_2 \to X_1 \oplus Z_1$, $F : X_1 \oplus Z_1 \to X_2 \oplus Z_2$ are invertible. Here $\tilde{L} = l_{Z_1}(L)$ denotes the extended operator on $X_1 \oplus Z_1$ according to the left hand side of formula (3.2) and $r_{X_1}(\tilde{L})$ the corresponding restriction, which is also defined for any linear operator $\tilde{L} : \widetilde{X_1} \to \widetilde{X_1}$ and any complemented subspace X_1 of $\widetilde{X_1}$. In a different context the operator matrix $l_{Z_1}(L)$ is identified with $L \oplus I_{Z_1}$, see [**2**], and $r_{X_1}(\tilde{L})$ is called a general (or abstract) Wiener-Hopf operator, usually written as $T_P(A) = PA_{|\mathrm{im}\, P}$ where $A = \tilde{L}$ and P is the projector onto X_1 along Z_1, see [**17**]. Similarly, we use r_{X_2} and $l_{Z_2}(W)$, respectively, due to the second space components.

It is known that

Corollary 3.1. *L and W belong to the same regularity class* (*see* [**17**, p. 10]), *i.e., the images are simultaneously closed, complemented, of the same codimension or not, respectively, and the kernels also complemented, of the same dimension or not, respectively.*

Now, because of the construction realized in Theorem 2.1, we want to find relations between the generalized inverses (if they exist) of operators that are equivalent after extension.

Theorem 3.2. *Let L and W be equivalent after extension, see* (3.2); *then a generalized inverse W^- of W yields a generalized inverse of L*

and vice versa:

$$L^- = r_{X_1}(F^{-1} l_{Z_2}(W^-) E^{-1}) \tag{3.4}$$

$$W^- = r_{X_2}(F l_{Z_1}(L^-) E). \tag{3.5}$$

Proof. By definition we have for a linear operator L on X_1 or $\tilde{L}$ on $X_1 \oplus Z_1$, respectively,

$$r_{X_1}(l_{Z_1}(L)) = L \tag{3.6}$$

$$l_{Z_1}(r_{X_1}(\tilde{L})) = \tilde{L} \tag{3.7}$$

if and only if $\tilde{L} = l_{Z_1}(L)$ for some $L : X_1 \to X_1$ and, due to (3.2),

$$L = r_{X_1}(E l_{Z_2}(W) F). \tag{3.8}$$

Therefore $WW^-W = W$ implies

(3.9)

$$E l_{Z_2}(W) F \;\; F^{-1} l_{Z_2}(W^-) E^{-1} \;\; E l_{Z_2}(W) F = E l_{Z_2}(W) F$$

$$\begin{bmatrix} L & 0 \\ 0 & I_{Z_1} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} L & 0 \\ 0 & I_{Z_1} \end{bmatrix} = \begin{bmatrix} L & 0 \\ 0 & I_{Z_1} \end{bmatrix} \tag{3.10}$$

where the middle factor was just abbreviated, and, furthermore, we have

$$\begin{bmatrix} LAL & LB \\ CL & D \end{bmatrix} = \begin{bmatrix} L & 0 \\ 0 & I_{Z_1} \end{bmatrix} \tag{3.11}$$

i.e., $A = r_{X_1}(F^{-1} l_{Z_2}(W^-) E^{-1})$ represents a generalized inverse of L.

An analogue proceeding proves (3.5). □

Corollary 3.3. *If* $WW^-W = W$ *and* L *is given by* (3.2), *then projectors onto the image and the kernel of* L, *respectively, read*

$$P_{\operatorname{im} L} = r_{X_1}(E l_{Z_2}(W) l_{Z_2}(W^-) E^{-1}) \tag{3.12}$$

$$P_{\ker L} = r_{X_1}(F^{-1}(I - l_{Z_2}(W^-) l_{Z_2}(W)) F) \tag{3.13}$$

Proof. Although in general $r_{X_1}(W_1)r_{X_1}(W_2) \neq r_{X_1}(W_1W_2)$, we have here

$$\begin{aligned} LL^- &= r_{X_1}(El_{Z_2}(W)F)\; r_{X_1}(F^{-1}l_{Z_2}(W^-)E^{-1}) \\ &= r_{X_1}\left(\begin{bmatrix} L & 0 \\ 0 & I_{Z_1}\end{bmatrix}\right)\; r_{X_1}\left(\begin{bmatrix} A & B \\ C & D\end{bmatrix}\right) \\ &= LA \qquad (3.14) \\ &= r_{X_1}\left(\begin{bmatrix} LA & LB \\ C & D\end{bmatrix}\right) \\ &= r_{X_1}(El_{Z_2}(W)F\; F^{-1}l_{Z_2}(W^-)E^{-1}) \end{aligned}$$

which yields the first statement, and the second part is proved similarly. □

Corollary 3.4. *If L denotes the operator in* (1.7)*, and if W satisfies* (3.2) *and the symbol of W has partial indices $\kappa_1, \dots, \kappa_s$ then*

$$\dim\ker L = \kappa^- = \sum_{j=1}^{s} \kappa_j^- \qquad (3.15)$$

$$\operatorname{codim}\operatorname{im} L = \kappa^+ = \sum_{j=1}^{s} \kappa_j^+ \qquad (3.16)$$

where

$$\kappa_j^- = \frac{1}{2}(|\kappa_j| - \kappa_j) \qquad (3.17)$$

$$\kappa_j^+ = \frac{1}{2}(|\kappa_j| + \kappa_j), \qquad j = 1, \dots, s. \qquad (3.18)$$

Proof. It is an immediate consequence of (3.2), see [**15**].

Remark. The last information cannot be obtained from an invertibility symbol that describes inversion in the Calkin algebra, cf. [**7, 8**].

As a matter of fact, in these representations the defect numbers do not depend on the transformations E, F, restriction or extension, but the kernel itself and the image of L do. Therefore the Fourier symbol of W is still not a GI-symbol of L. To achieve this property we need a little more specialization of E and F.

Proposition 3.5. *If E^{-1} and F^{-1} have the particular form of* (2.22) *and* (2.23)*, respectively, and if*

$$l_{Z_1}(L) = EWF \tag{3.19}$$

then the formula for the generalized inverse of L, given in Theorem 3.2*, may be reduced to*

$$L^- = r_{X_1}(W^-). \tag{3.20}$$

Proof. Formula (3.20) is a direct consequence of the definition of r_{X_1} and the computation of $F^{-1}W^-E^{-1}$ with these particular operators E^{-1} and F^{-1}. □

Theorem 3.6. *If L denotes a higher order WHO and T is constructed by Theorem* 2.1*, then the* (*associated*) *Fourier symbol of T is a* GI*-symbol for L.*

Proof. Combine the last proposition with former results. □

Conversely, for some applications it might be interesting to weaken the assumptions on E and F:

Proposition 3.7. *If $L : X_1 \to X_1$, $W : X_2 \to X_2$, $W^- : X_2 \to X_2$, $E : X_2 \oplus Z_2 \to X_1 \oplus Z_1$, $E^- : X_1 \oplus Z_1 \to X_2 \oplus Z_2$, $F : X_1 \oplus Z_1 \to X_2 \oplus Z_2$, $F^- : X_2 \oplus Z_2 \to X_1 \oplus Z_1$ are operators such that*

$$\begin{bmatrix} L & 0 \\ 0 & I_{Z_1} \end{bmatrix} = E \begin{bmatrix} W & 0 \\ 0 & I_{Z_2} \end{bmatrix} F \tag{3.21}$$

$$E^-E = I, \qquad FF^- = I \tag{3.22}$$

$$WW^-W = W \tag{3.23}$$

then a generalized inverse of L is represented by

$$L^- = r_{X_1}(F^- l_{Z_2}(W^-)E^-). \tag{3.24}$$

Proof. Changing E^{-1} and F^{-1} to E^- and F^-, respectively, in the proof of Theorem 3.2 we obtain relations that prove the statement. □

Clearly if E and F are not invertible we cannot obtain, in general, W^- from L^- which can be proved by counterexamples.

Corollary 3.8. *Under the conditions of the last proposition, if*

$$\begin{aligned} E^- &= \begin{bmatrix} I & * \\ 0 & * \end{bmatrix} : X_1 \oplus Z_1 \longrightarrow X_2 \oplus Z_2, \\ F^- &= \begin{bmatrix} I & * \\ * & * \end{bmatrix} : X_2 \oplus Z_2 \longrightarrow X_1 \oplus Z_1, \end{aligned} \tag{3.25}$$

where $*$ denotes any bounded linear operator acting between the corresponding spaces, or

$$\begin{aligned} E^- &= \begin{bmatrix} I & * \\ * & * \end{bmatrix} : X_1 \oplus Z_1 \longrightarrow X_2 \oplus Z_2, \\ F^- &= \begin{bmatrix} I & 0 \\ * & * \end{bmatrix} : X_2 \oplus Z_2 \longrightarrow X_1 \oplus Z_1 \end{aligned} \tag{3.26}$$

then a generalized inverse of L is represented by

$$L^- = r_{X_1}(l_{Z_2}(W^-)). \tag{3.27}$$

Proof. It is a direct computation of (3.24) for these particular E^- and F^-. □

4. Examples. Various examples are directly obtained, if we interpret the generalized inverse W^- in (1.6) as the composition of operators that come from a generalized factorization of the Fourier symbol matrix function ϕ of A. Here we like to present a less immediate example.

We know that a general WHO

$$\tilde{W} = PA_{|PX} \tag{4.1}$$

where X is a Banach space, A, $P \in \mathcal{L}(X)$, $P^2 = P$, is equivalent after extension to a paired operator $PA + Q$. This in brief is

$$PAP + Q = (I - PAQ)(PA + Q) \tag{4.2}$$

or, by identification of $X = PX \oplus QX$ with vectors of two components in PX and QX, respectively,

$$\begin{bmatrix} \tilde{W} & 0 \\ 0 & I_{|QX} \end{bmatrix} = \begin{bmatrix} I_{|PX} & -PA_{|QX} \\ 0 & I_{|QX} \end{bmatrix} \begin{bmatrix} \tilde{W} & PA_{|QX} \\ 0 & I_{|QX} \end{bmatrix}, \tag{4.3}$$

i.e.,

$$\tilde{W} \overset{*}{\sim} PA + Q. \tag{4.4}$$

As a particular case let us now think of a WHO of second kind on a finite interval $\Omega = [0, 1]$, W_Ω say, which can be seen as a restriction from the half-line $\mathbf{R}_+$, i.e., $X = L^p(\mathbf{R}_+)$, $P = \chi_\Omega \cdot$, and A is the WHO on (1.4) denoted by W.

So the projector $Q = I - \chi_\Omega \cdot = \chi_{[1,\infty[} \cdot$ on $L^p(\mathbf{R}_+)$ is itself a higher order WHO, namely

$$Q = W_1 W_{-1} \tag{4.5}$$

where W_h has the Fourier symbol $\phi_h(\xi) = e^{ih\xi}$ due to an h-shift. This yields

$$W_\Omega \overset{*}{\sim} PW + Q \tag{4.6}$$

$$= W - W_1 W_{-1} W + W_1 W_{-1} \tag{4.7}$$

$$\overset{*}{\sim} \begin{bmatrix} W & W_1 \\ W_{-1}(W - I) & I \end{bmatrix} \tag{4.8}$$

$$\sim \begin{bmatrix} W_{-1} & 0 \\ W & W_1 \end{bmatrix} \tag{4.9}$$

with a Fourier symbol, see [**10**],

$$\phi(\xi) = \begin{bmatrix} e^{-i\xi} & 0 \\ \varphi(\xi) & e^{i\xi} \end{bmatrix}, \qquad \xi \in \mathbf{R} \tag{4.10}$$

where $\varphi = \lambda + Fk$ and k is an extension of the convolution kernel of W_Ω from $[0,1]$ to $\mathbf{R}$.

The last relations are explicitly given by

$$\begin{aligned} &\begin{bmatrix} W - W_1 W_{-1}(W-I) & 0 \\ 0 & I \end{bmatrix} \\ &\quad = \begin{bmatrix} I & -W_1 \\ 0 & I \end{bmatrix} \begin{bmatrix} W & W_1 \\ W_{-1}(W-I) & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -W_{-1}(W-I) & I \end{bmatrix} \end{aligned} \tag{4.11}$$

and

$$\begin{bmatrix} W & W_1 \\ W_{-1}(W-I) & I \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & W_{-1} \end{bmatrix} \begin{bmatrix} W_{-1} & 0 \\ W & W_1 \end{bmatrix} \tag{4.12}$$

Proposition 4.1. i) *The Fourier symbol*

$$\psi(\xi) = \begin{bmatrix} \varphi(\xi) & e^{i\xi} \\ e^{-i\xi}(\varphi(\xi)-1) & 1 \end{bmatrix} \tag{4.13}$$

is a GI-*symbol for* W_Ω.

ii) ϕ *is not a* GI-*symbol for* W_Ω.

Proof. i) The relation (4.11) is obtained by Theorem 2.1 so that, see Theorem 3.6, ψ is a GI-symbol for $PW + Q$. Thus, Corollary 3.8 and relation (4.3) yield the first statement.

ii) Combining (4.11) and (4.12), we have

$$\begin{aligned} &\begin{bmatrix} W - W_1 W_{-1}(W-I) & 0 \\ 0 & I \end{bmatrix} \\ &\quad = \begin{bmatrix} W_1 & I - W_1 W_{-1} \\ -I & W_{-1} \end{bmatrix} \mathcal{W} \begin{bmatrix} I & 0 \\ -W_{-1}(W-I) & I \end{bmatrix} \end{aligned} \tag{4.14}$$

where

$$\mathcal{W} = \begin{bmatrix} W_{-1} & 0 \\ W & W_1 \end{bmatrix}. \tag{4.15}$$

Thus, if

$$\mathcal{W}^- = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \tag{4.16}$$

is a generalized inverse of $\mathcal{W}$, Theorem 3.2 shows that

$$L^- = r_X\left(\begin{bmatrix} I & 0 \\ W_{-1}(W-I) & I \end{bmatrix} \mathcal{W}^- \begin{bmatrix} W_{-1} & 0 \\ I & W_1 \end{bmatrix}\right) \tag{4.17}$$

$$= r_X\left(\mathcal{W}^- \begin{bmatrix} W_{-1} & 0 \\ I & W_1 \end{bmatrix}\right) \tag{4.18}$$

$$= \mathcal{A}W_{-1} + \mathcal{B} \tag{4.19}$$

is a generalized inverse of $L = PW + Q$.

Thus, in general, $r_X(\mathcal{W}^-) = \mathcal{A}$ is not a generalized inverse of L (consider, for instance, the possible case $\mathcal{A} = W_1$, $\mathcal{B} = 0$ and as a consequence $L = LQL \neq LW_1L$). From (4.3) and (4.6) we also have that, in general, $r_X(\mathcal{W}^-)$ is not a generalized inverse of W_Ω. □

The last proposition gives one of the reasons why it is useful to construct equivalence after extension relations based on the iteration presented in Theorem 2.1.

For the practical use of some of the relations presented here we need to test the generalized factorization of the symbols ψ and ϕ. For this purpose, if these symbols are in the class of (semi-) almost periodic matrix functions (which has various applications in mathematical physics) we refer to [**9, 10, 11** and **12**] where a particular case of generalized factorization, the P-factorization, is defined and for some classes a generalized factorization can be effectively constructed.

A different approach to the convolution operator on a finite interval, based on relations with a singular integral operator acting on a particular space of functions, is presented in [**3**] and [**13**].

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