JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 8, Number 2, Spring 1996

# PASSIVE BOUNDARY DAMPING OF VISCOELASTIC STRUCTURES

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ABSTRACT. We consider a linear viscoelastic structure controlled and observed by finitely many collocated actuators and sensors. We discuss properties of the open loop transfer function and the corresponding impulse response. Moreover we give a decomposition formula for the solutions of the closed loop problem with passive boundary damping in terms of solutions with energy conserving boundary conditions.

**1. Introduction.** This is a technical paper on the role of damping boundary conditions for the motion of viscoelastic structures. The mechanical systems that will fit into this framework are made up of one or several viscoelastic parts and their motion satisfies the following conditions:

- The deflections are small enough to justify a linear constitutive law.

- If large rotations of the system as a whole are considered, the system obeys symmetry relations which allow the angular velocities to enter the equations linearly. (Usually violated if the center of rotation itself may move.)

- Finitely many sensors and actuators, located at the boundary, control and observe the motion of the system. Sensors and actuators are collocated.

- The mass of the sensors and actuators is small enough to be ignored.

Received by the editors on May 20, 1995, and in revised form on December 18, 1995.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 45K05, 73K50, 73F99.
*Key words and phrases.* Linear viscoelasticity, boundary feedback control, abstract Volterra equations, operator valued relaxation modulus.
Research partially funded by FWF (Austria) and NSF (USA) within the US-Austrian Cooperative Research Program, Grants P4722-PHY and INT-8922490, respectively. Work by W.D. done within the Spezialforschungsbereich F 003, Optimierung und Kontrolle. Work by K.B.H. and R.L.W. partially supported by the Air Force Office of Scientific Research under grant AFOSR-91-8003.

By now two aspects of such systems have been thoroughly investigated:

The dynamics of linear viscoelastic structures with energy conserving boundary conditions is well understood. This concerns not only well-posedness of the initial value problem, but also the dependence of smoothing and damping behavior on the properties of the relaxation modulus. A vast literature on special problems exists. In addition, a general theory is available that covers a broad variety of different structures. Very roughly, the essence of these investigations is that the qualitative behavior of the system is almost exclusively governed by the relaxation modulus, i.e., by the material, while the geometric structure of the system, expressed by the form of the differential operators, determines only the precise location of poles.

On the other hand, special systems have been investigated with respect to boundary feedback damping and the limitations of damping due to slow creep modes or to its nonrobustness with respect to time delay. These investigations are mostly based on explicit calculations of transfer functions, and are therefore confined to comparably simple systems such as rods or beams. The geometric configuration of the system (or its model) plays a much more important role in this context. There are, for instance, deep qualitative differences between a Timoshenko beam and an Euler-Bernoulli beam.

In order to avoid misunderstandings, let us explain briefly what we mean by the expression "geometric configuration" as opposed to material properties, without any claim that our terminology is more than an appeal to assign an intuitive meaning to our abstract theorems. Here, "geometry" refers not only to the physical object modelled but also to the structure of the model chosen. It may be surprising to read that we find a geometric difference between an Euler-Bernoulli beam and a Timoshenko beam, which is basically a system of coupled wave equations. Both equations may refer to the same physical beam. However, the Euler-Bernoulli model is derived from the assumption that the beam performs only motions such that the cross sections orthogonal to the axis in the undeformed configurations stay orthogonal to the (bent) axis in deformed configuration. In particular, deformations described by an Euler-Bernoulli model are restricted to be shear free. The Timoshenko model allows for some shear, still much more restricted than a three dimensional model. Considering the degrees of freedom of the admissible motions, we see that in some sense the geometry of these models is strikingly different. This is reflected in the different way "strain" is computed from the displacements and the way "divergence" of the stress is evaluated, in order to compute the resultant forces in Newton's equation.

The objective of the present paper is three-fold. We propose a general setting that covers a variety of viscoelastic systems controlled at the boundary. The transfer function of the open loop control problem in this generality is characterized. In particular, we show how strong the singularity of the relaxation kernel at 0 has to be in order that the transfer function corresponds to a locally integrable impulse response function in the time domain. Finally, we show that state space settings for the system with damped boundary conditions inherit much of the qualitative behavior from the corresponding systems with energy conserving conditions. This can be seen from the observation that in the frequency domain the solution to the damped problem is a convex combination of solutions to some undamped problems. Of course, most frequency domain estimates for problems with energy conserving boundary conditions could be redone for passive damping conditions with some efforts. However, this paper states a general principle why such estimates have to carry over from the energy conserving case to the situation of passive damping.

The paper is organized as follows: Section 2 presents the abstract framework and explains by an example how to apply it to a viscoelastic structure. It is also shown how the standard examples of a rod and an Euler-Bernoulli beam fit into the setting. Section 3 states our results and discusses some of their relations to existing literature. The proofs are deferred to Section 4 containing the functional analysis part and Section 5 containing the linear algebra part.

## 2. The setting.

**2.1. The abstract viscoelastic equations.** We consider the following abstract integrodifferential system describing the motion of a viscoelastic solid body.

**System 2.1.**

$$R\frac{d}{dt}v(t) = D\sigma(t),$$
$$\sigma(t) = \int_{-\infty}^{t} A(t-s)\tilde{D}v(s)\,ds$$

with additional feedback conditions relating

$$p(t) = P\sigma(t) \quad \text{and} \quad q(t) = \tilde{P}v(t).$$

Here $v$ denotes the velocity field, and $\sigma$ denotes the stress field. The first equation is Newton's equation of momentum. $D$ is a suitable differential operator, $R$ represents the mass density of the material. The second equation is the constitutive equation of the material. $\tilde{D}$ is a differential operator that relates the displacement to linear strain. $A(t)$ describes the stress relaxation modulus of the material. The operator $P$ relates the stress field to a vector $p$ consisting of finitely many control forces $p_1, \dots, p_n$, while $\tilde{P}$ relates the velocity field to finitely many velocities $q_1, \dots, q_n$, observed at the position of the actuators which exert the control forces. (Instead of forces and velocities one may as well consider torques and angular velocities.)

Throughout this paper we require the following assumptions. Our first hypothesis concerns the operators describing the geometric configuration of our system.

**Hypothesis 2.1.** *$X$ and $Y$ are Hilbert spaces, $D : X \supset \operatorname{dom}(D) \to Y$ and $\tilde{D} : Y \supset \operatorname{dom}(\tilde{D}) \to X$ are unbounded linear operators. $P : \operatorname{dom}(D) \to \mathbf{R}^n$ and $\tilde{P} : \operatorname{dom}(\tilde{D}) \to \mathbf{R}^n$ are linear operators, relatively bounded with respect to $D$ or $\tilde{D}$, respectively. For all $\sigma \in \operatorname{dom}(D)$ and all $v \in \operatorname{dom}(\tilde{D})$ the following identity holds*:

$$\langle D\sigma, v\rangle + \langle \sigma, \tilde{D}v\rangle = \langle P\sigma, \tilde{P}v\rangle. \tag{2.1}$$

*Here, $\langle\ ,\ \rangle$ denotes the inner product in each Hilbert space. Moreover, if $D_1, \tilde{D}_1, P_1, \tilde{P}_1$ are extensions of $D, \tilde{D}, P, \tilde{P}$, respectively, such that*

$$\langle D_1\sigma, v\rangle + \langle \sigma, \tilde{D}_1 v\rangle = \langle P_1\sigma, \tilde{P}_1 v\rangle$$

*holds for all* $\sigma \in \operatorname{dom}(D_1)$ *and all* $v \in \operatorname{dom}(\tilde{D}_1)$*, then* $D = D_1$*,* $\tilde{D} = \tilde{D}_1$*,* $P = P_1$*,* $\tilde{P} = \tilde{P}_1$*. In particular,* $P$ *and* $\tilde{P}$ *are surjective onto* $\mathbf{R}^n$*.*

We assume that the density is bounded and bounded away from 0:

**Hypothesis 2.2.** *$R$ is a bounded, self-adjoint linear operator on $Y$ which is positive definite and admits a continuous inverse.*

The last hypothesis concerns the constitutive equation of the material. In this paper we make no claims on the well-posedness of System 2.1. Therefore we give only minimal hypotheses on the Laplace transform of the relaxation kernel $A(t)$. For some results it is sufficient to have them only in the right half-plane; sometimes we will require a domain $\mathcal{U}$ which contains a sector with an angle opening larger than $\pi$.

**Hypothesis 2.3.** *For $t \in (0, \infty)$ let $A(t)$ be a bounded linear operator on $X$. For fixed $x \in X$ the function $t \to A(t)x$ admits a Laplace transform $\hat{A}(s)x$ for $\Re(s) > 0$. The operator $\hat{A}(s)$ can be extended analytically to $s \in \mathcal{U}$, where $\mathcal{U} \subset \mathbf{C} \setminus (-\infty, 0]$ is a domain containing the open right half-plane.*

*For each fixed $s \in \mathcal{U}$, the operator $\hat{A}(s)$ is continuously invertible, and there exist constants $M(s) > 0$ and $\theta(s) \in (0, \pi)$ such that for all $x \in X \setminus \{0\}$*

$$\left| \left\langle x, \frac{1}{s}\hat{A}(s)x \right\rangle \right| \geq M(s)\|x\|^2,$$

$$\arg\left( \left\langle x, \frac{1}{s}\hat{A}(s)x \right\rangle \right) \begin{cases} \in (-\theta(s), 0] & \textit{if } \Im(s) > 0, \\ = 0 & \textit{if } \Im(s) = 0, \\ \in [0, \theta(s)) & \textit{if } \Im(s) < 0. \end{cases}$$

*If $s > 0$ is real, then $\hat{A}(s)$ is self-adjoint and positive definite.*

A large class of kernels frequently considered in viscoelastic problems, including kernels of fractional derivative type, fits into the hypothesis above.

**Proposition 2.1.** *For $t \in (0,\infty)$ let $A(t)$ be a bounded linear operator on $X$. Let $A$ be completely monotone in the following sense: There exists an operator valued function $G$ and a nondecreasing function $\nu : [0,\infty) \to [0,\infty)$ such that for all $x \in X$*

$$A(t)x = \int_0^\infty e^{-\zeta t} G(\zeta) x \, d\nu(\zeta).$$

*For each $\zeta > 0$, the linear operator $G(\zeta)$ is bounded, self-adjoint and positive semi-definite. For each $x \in X$ and each $t > 0$, the function $e^{-\zeta t} G(\zeta)x$ is Bochner integrable on $[0,\infty)$ with respect to the measure $d\nu$. Moreover, we assume that $\|A\|$ is integrable on $[0,1]$, and that for some $t_0 > 0$ the operator $A(t_0)$ is continuously invertible. Then $A$ satisfies Hypothesis 2.3 with $\mathcal{U}$ being the open right half-plane.*

*Proof.* The proof can be given by straightforward computations, using the formula

$$\hat{A}(s) = \int_0^\infty \frac{1}{\zeta + s} G(\zeta) \, d\nu(\zeta).$$

(For details on this type of completely monotonic operator-valued kernel see [**7**].) □

We obtain uniform estimates on domains $\mathcal{U}_1$ where the conditions on the phase angle of $\hat{A}(s)$ hold uniformly:

**Hypothesis 2.4.** *$\mathcal{U}_1 \subset \mathcal{U}$ and there exist $\delta \in (0,\pi)$ and $\theta \in (0,\pi)$ such that for all $s \in \mathcal{U}_1$ and all $x \in X \setminus \{0\}$*

$$|\arg(s)| < \delta \quad \textit{and} \quad \left| \arg\left( \left\langle x, \frac{1}{s}\hat{A}(s)x \right\rangle \right) \right| < \theta.$$

**2.2. An example from three-dimensional viscoelasticity.** To explain the meaning of the abstract equations, let us start with the following example.

**Example 2.1.** Consider a viscoelastic solid, filling a domain $\Omega \subset \mathbf{R}^3$ in stress-free reference configuration. Several rigid patches are glued to

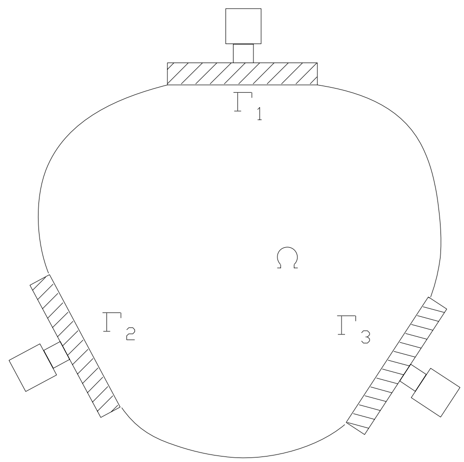


FIGURE 1. A body controlled by finitely many actuators.

the body and cover the disjoint parts $\Gamma_1, \dots, \Gamma_n$ of the boundary. The remaining part $\Gamma_0$ of the boundary is free. These patches can be moved, with one degree of freedom each, in the directions $w_1, \dots, w_n$, and their velocities $q_1, \dots, q_n$ are observed by sensors. Simultaneously they can exert forces $p_1, \dots, p_n$ to control the motion of the body. We consider only small deformations, so that a linear constitutive law is justified.

For technical reasons we assume that $\Omega$ has a piecewise smooth boundary. Let $t$ denote time and $x = (x_1, x_2, x_3)$ the (Lagrangian) space coordinates. The vector $v(t,x)$ denotes the velocity field, the symmetric $3 \times 3$-tensor $\sigma(t,x)$ denotes the stress field. The positive, scalar valued function $\rho(x)$ denotes the density (in reference configuration). Newton's law of momentum reads then

$$\rho(x) \frac{\partial}{\partial t} v_i(t,x) = \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \sigma_{ij}(t,x).$$

Let $\dot{\varepsilon}(t,x)$ denote the time derivative of the (linear) strain tensor:

$$\dot{\varepsilon}_{ij}(t,x) = \frac{1}{2}\left[\frac{\partial}{\partial x_i} v_j(t,x) + \frac{\partial}{\partial x_j} v_i(t,x)\right].$$

Assume that the material satisfies a linear viscoelastic constitutive equation

$$\sigma_{ij}(t,x) = \int_{-\infty}^{t} \sum_{k,l=1}^{3} a_{ijkl}(t-s,x)\dot{\varepsilon}_{kl}(s,x)\,ds$$

with a relaxation modulus $a(t,x) = (a_{ijkl}(t,x))$ which is a fourth order tensor depending on space and time and satisfies the usual symmetry conditions

$$a_{ijkl} = a_{jikl} = a_{ijlk} = a_{klij}.$$

The boundary condition for the free boundary is

$$\sum_{j=0}^{3} \sigma_{ij}(t,x)n_j(x) = 0 \quad \text{for } i = 1,2,3,\ x \in \Gamma_0,$$

where $n(x)$ is the outward unit normal vector at $x$. At the patches $\Gamma_m$, $m = 1, \cdots, n$, the velocities are confined to the directions $w_m$, and the stresses integrate up to the control forces:

$$v_i(t,x) = q_m(t)w_{mi} \quad \text{with some } q_m(t) \in \mathbf{R},$$
$$\int_{\Gamma_m} \sum_{i,j=1}^{3} w_{mi}\sigma_{ij}(t,x)n_j(x)\,dS = p_m(t).$$

Depending on the relations between the observed velocities $q_m$ and the control forces $p_m$ various feedback controls may be modelled. The simplest situations are given by energy conserving boundary conditions, either fixing the patches ($q_m = 0$) or switching off the control forces ($p_m = 0$), or by passive damping such as simple dashpots at the patches ($p_m = -k_m q_m$).

We introduce function spaces to obtain an abstract formulation of these equations. We consider

$$v(t) = v(t,\cdot) \in Y = L(\Omega, \mathbf{R}^3)$$

and

$$\sigma(t) = \sigma(t,\cdot) \in X = \{\sigma \in L(\Omega, \mathbf{R}^{3\times 3}) \mid \sigma \text{ symmetric}\}.$$

We rewrite the law of momentum

$$R\frac{d}{dt}v(t) = D\sigma(t),$$

where the operator $R$ means multiplication by $\rho$, and $D$ is the divergence. The domain of $D$ includes all stress-free boundary conditions:

$$\begin{aligned}\operatorname{dom}(D) &= \Big\{\sigma \in X \mid \operatorname{div}(\sigma) \in X,\ \sum_{j=1}^{3} \sigma_{ij}(x)n_j(x) = 0 \\ &\qquad\qquad \text{for } i = 1, \dots, 3,\ x \in \Gamma_0\Big\}, \\ (D\sigma)_i(x) &= \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\sigma_{ij}(x).\end{aligned}$$

The derivative of the strain tensor is given by

$$\dot{\varepsilon}(t) = \tilde{D}v(t).$$

The restrictions on the degree of freedom for the velocity field are expressed by the domain of $\tilde{D}$:

$$\begin{aligned}\operatorname{dom}(\tilde{D}) &= \{v \in W^{1,2}(\Omega, \mathbf{R}^3) \mid \exists q_1, \dots, q_n \in \mathbf{R} : \\ &\qquad\qquad v(x) = q_m w_m \ \text{ for } x \in \Gamma_m\}, \\ (\tilde{D}v)_{ij}(x) &= \frac{1}{2}\left[\frac{\partial}{\partial x_i}v_j(x) + \frac{\partial}{\partial x_j}v_i(x)\right].\end{aligned}$$

If $A(t)$ denotes the stress relaxation tensor

$$(A(t)\varepsilon)_{ij}(x) = \sum_{k,l=1}^{3} a_{ijkl}(t,x)\varepsilon_{kl}(x),$$

the constitutive equation reads

$$\sigma(t) = \int_{-\infty}^{t} A(t-s)\tilde{D}v(s)\,ds.$$

The observed velocities are described by the operator $\tilde{P} : \operatorname{dom}(\tilde{D}) \to \mathbf{R}^n$, defined by

$$(\tilde{P}v)_m = q_m$$

with the values $q_m$ from the definition of $\operatorname{dom}(\tilde{D})$. The control forces are described by the operator $P : \operatorname{dom}(D) \to \mathbf{R}^n$

$$p_m = (P(\sigma))_m = \int_{\Gamma_m} \sum_{i,j=1}^{3} w_{mi}\sigma_{ij}(x)n_j(x)\,dS.$$

To show that the model example above fits in the abstract setting we prove:

**Proposition 2.2.** *The operators $D$, $\tilde{D}$, $P$, and $\tilde{P}$ defined for Example* 2.1 *satisfy Hypothesis* 2.1.

*Proof.* Let $v \in \operatorname{dom}(\tilde{D})$ and $\sigma \in \operatorname{dom}(D)$. Using the symmetry of $\sigma$ and Green's formula we compute

$$\begin{aligned}
\langle D\sigma, v\rangle + \langle \sigma, \tilde{D}v\rangle &= \sum_{i=1}^{3}\int_{\Omega}\sum_{j=1}^{3} v_i(x)\frac{\partial}{\partial x_j}\sigma_{ij}(x)\,dx \\
&\quad + \sum_{i,j=1}^{3}\int_{\Omega}\sigma_{ij}(x)\frac{1}{2}\left[\frac{\partial}{\partial x_j}v_i(x) + \frac{\partial}{\partial x_i}v_j(x)\right]dx \\
&= \sum_{i,j=1}^{3}\int_{\Omega}\left[v_i(x)\frac{\partial}{\partial x_j}\sigma_{ij}(x) + \sigma_{ij}(x)\frac{\partial}{\partial x_j}v_i(x)\right]dx \\
&= \sum_{i,j=1}^{3}\sum_{m=0}^{n}\int_{\Gamma_m} v_i(x)\sigma_{ij}(x)n_j(x)\,dS \\
&= 0 + \sum_{m=1}^{n}\sum_{i,j=1}^{3}\int_{\Gamma_m} q_m w_{mi}\sigma_{ij}(x)n_j(x)\,dS \\
&= \sum_{m=1}^{n} q_m p_m = \langle P\sigma, \tilde{P}v\rangle.
\end{aligned}$$

Suppose that $D_1$, $\tilde{D}_1$, $P_1$, and $\tilde{P}_1$ are extensions of $D$, etc., satisfying (2.1). Let $\tau$ be a test function in $\mathcal{C}^\infty(\Omega, \mathbf{R}^{3\times 3})$ and define $\sigma =$

$1/2[\tau + \tau^*]$. Then $\sigma \in \mathrm{dom}\,(D)$. Let $\delta = \tilde{D}_1 v$ for some $v \in \mathrm{dom}\,(\tilde{D}_1)$. Notice that $\delta$ is symmetric by definition of the space $X$. Then we have

$$\begin{aligned}
0 &= \langle \delta, \sigma \rangle + \langle v, D\sigma \rangle \\
&= \sum_{i,j=1}^{3} \int_\Omega \delta_{ij}\sigma_{ij}\,dx + \sum_{i=1}^{3} \int_\Omega v_i \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\sigma_{ij}\,dx \\
&= \sum_{i,j=1}^{3} \int_\Omega \delta_{ij}\tau_{ij}\,dx \\
&\quad + \sum_{i,j=1}^{3} \int_\Omega v_i \frac{1}{2}\left[\frac{\partial}{\partial x_j}\tau_{ij} + \frac{\partial}{\partial x_j}\tau_{ji}\right] dx \\
&= \sum_{i,j=1}^{3} \int_\Omega \delta_{ij}\tau_{ij}\,dx + \sum_{i,j=1}^{3} \int_\Omega \frac{1}{2} v_i \frac{\partial}{\partial x_j}\tau_{ij}\,dx \\
&\quad + \sum_{j,i=1}^{3} \int_\Omega \frac{1}{2} v_j \frac{\partial}{\partial x_i}\tau_{ij}\,dx.
\end{aligned}$$

Since this holds for any test function $\tau$, we have

$$\tilde{D}_1 v = \delta = \frac{1}{2}[\nabla(v) + (\nabla(v))^*]$$

in the sense of distributions. On the other hand, if $\sigma \in \mathrm{dom}\,(D_1)$, a similar manipulation with test functions $v \in \mathcal{C}^\infty(\Omega, \mathbf{R}^3)$ with compact support in the interior of $\Omega$, yields that $D_1\sigma$ is the divergence of $\sigma$ in the sense of distributions.

We can now use Green's formula again and obtain for any $v \in \mathrm{dom}\,(\tilde{D}_1)$ and $\sigma \in \mathrm{dom}\,(D_1)$

$$\begin{aligned}
\langle \tilde{P}_1 v, P_1\sigma \rangle &= \langle \tilde{D}_1 v, \sigma \rangle + \langle v, D_1\sigma \rangle \\
&= \sum_{m=0}^{n} \int_{\Gamma_m} \sum_{i,j=1}^{3} v_i(x)\sigma_{ij}(x)n_j(x)\,dS.
\end{aligned}$$

First let $v \in \mathrm{dom}\,(\tilde{D}_1)$ and $\sigma \in \mathrm{dom}\,(D)$. Then $P_1\sigma = P\sigma$. Let

$q = \tilde{P}_1 v$.

$$\sum_{m=1}^{n} q_i \int_{\Gamma_m} w_{mi}\sigma_{ij}(x)n_j(x)\,dS = \langle \tilde{P}_1 v, P_1\sigma\rangle$$
$$= \sum_{m=0}^{n}\int_{\Gamma_m}\sum_{i,j=1}^{3} v_i(x)\sigma_{ij}(x)n_j(x)\,dS.$$

Since this holds for all $\sigma \in \operatorname{dom}(D)$, a density argument implies that

$$v_i(x) = \begin{cases} q_i w_{mi} & \text{for } x \in \Gamma_m, \\ 0 & \text{for } x \in \Gamma_0. \end{cases}$$

Thus $v \in \operatorname{dom}(\tilde{D})$ and $\tilde{P}_1 v = \tilde{P} v$. With $\sigma \in \operatorname{dom}(D_1)$ and arbitrary $v \in \operatorname{dom}(\tilde{D})$, a similar procedure yields that $\sigma \in \operatorname{dom}(D)$ and $P_1\sigma = P\sigma$. Therefore $D$, $\tilde{D}$, $P$, $\tilde{P}$ admit no proper extensions satisfying (2.1). □

**2.3. Meaning of Equation 2.1.** Let us give a physical interpretation of Equation 2.1. Suppose the material is linearly elastic, which means that the relaxation modulus is constant $A(t) = E$ with a positive definite operator $E$. If the displacement is denoted by $u(t) = \int_{-\infty}^{t} v(t)\,dt$, then $\tilde{D}u(t)$ is the (linearized) strain tensor. The constitutive law reads $\sigma(t) = E\tilde{D}u(t)$.

At time $t$ the kinetic energy of the system is $(1/2)\langle v(t), Rv(t)\rangle$, and the potential (strain) energy is $(1/2)\langle \tilde{D}u(t), E\tilde{D}u(t)\rangle$. The time derivative of the total energy of the system is then

$$\frac{d}{dt}\left(\frac{1}{2}\langle v(t), Rv(t)\rangle + \frac{1}{2}\langle \tilde{D}u(t), E\tilde{D}u(t)\rangle\right)$$
$$= \left\langle v(t), R\frac{d}{dt}v(t)\right\rangle + \left\langle \frac{d}{dt}\tilde{D}u(t), E\tilde{D}u(t)\right\rangle$$
$$= \langle v(t), D\sigma(t)\rangle + \langle \tilde{D}v(t), E\sigma(t)\rangle$$
$$= \langle \tilde{P}v(t), P\sigma(t)\rangle.$$

Thus $\langle q(t), p(t)\rangle$ is the energy fed into the system by the control forces $p = P\sigma(t)$, when the observed velocities are $q = \tilde{P}v(t)$.

**2.4. Other examples.** We mention two standard examples which fit in our framework and have been subject to thorough investigation by several authors.

**Example 2.2.** A cylindrical rod is made from a viscoelastic material. No longitudinal motion is considered, but we consider torsional vibrations of the rod. It is fixed at one end and subject to a control torque $p$ at the other end. The angular velocity $q$ at the controlled end is observed.

The model leads to a viscoelastic modification of the wave equation (cf. [**16**, p. 47, **12**]). Let $t$ again denote the time, $x$ the space coordinate measuring the distance from the fixed end. Let $r$ be the radius of the rod, $l$ the length of the rod, $\rho$ its density and $a(t)$ its shear relaxation modulus. At the cross section with space coordinate $x$, the angular velocity of the torsion is $v(t,x)$. The shear stresses at this cross section integrate up to a torque $\sigma(t,x)$. Some easy computations yield the following formulations for the equation of momentum, the constitutive law, the energy conserving fixed end boundary condition and the control and observation. (The equations are understood for $t \geq 0$ and $x \in [0,l]$.)

$$\begin{aligned}
\rho\frac{r^4\pi}{2}\frac{\partial}{\partial t}v(t,x) &= \frac{\partial}{\partial x}\sigma(t,x), \\
\sigma(t,x) &= \frac{r^4\pi}{2}\int_{-\infty}^{t} a(t-s)\frac{\partial}{\partial x}v(s,x)\,ds, \\
v(t,0) = 0, \qquad & p(t) = \sigma(t,l), \qquad q(t) = v(t,l).
\end{aligned}$$

The spaces and operators in the abstract setting are then given by

$$\begin{aligned}
X = Y &= L^2([0,l],\mathbf{R}), \\
\operatorname{dom}(D) &= W^{1,2}([0,l],\mathbf{R}), \\
(D\sigma)(x) &= \frac{d}{dx}\sigma(x), \\
\operatorname{dom}(\tilde{D}) &= \Big\{ v \in W^{1,2}([0,l],\mathbf{R}) \mid v(0) = 0 \Big\},
\end{aligned}$$

$$(\tilde{D}v)(x) = \frac{d}{dx}v(x),$$
$$(Rv)(x) = \rho\frac{r^4\pi}{2}v(x),$$
$$(A(t)\varepsilon)(x) = \frac{r^4\pi}{2}a(t)\varepsilon(x),$$
$$P\sigma = \sigma(l), \qquad \tilde{P}v = v(l).$$

It is easy to check that with this notation the viscoelastic wave equation is transformed into System 2.1 and that Hypothesis 2.1 is satisfied.

**Example 2.3.** A cantilevered beam is subject to transversal deflection. The deflections are assumed to be small so that we may resort to a linear model. At the free end, control is implemented by a moment $p_1$ and a shear force $p_2$. The corresponding angular velocity $q_1$ and velocity $q_2$ are observed.

This model has been discussed in detail in [**8**]. More general boundary damping designs for the elastic Euler-Bernoulli beam are discussed, e.g., in [**2**]. The system may be modelled by a viscoelastic Euler-Bernoulli beam (cf. [**16**, p. 48]). (A Timoshenko beam would fit in our setting as well.) Let $t$ again denote time, let $x$ be the space coordinate measuring distance from the fixed end, $l$ the length of the beam, $A$ the cross sectional area of the beam, $I$ the cross sectional moment of inertia, $\rho$ the mass density of the beam, and $a(t)$ be Young's modulus. Let $v(t,x)$ denote the transversal velocity at the cross section with coordinate $x$, and $\sigma(t,x)$ be the bending moment due to shear stress. Then the Euler-Bernoulli beam equations read

$$A\rho\frac{\partial}{\partial t}v(t,x) = -\frac{\partial^2}{\partial x^2}\sigma(t,x),$$
$$\sigma(t,x) = I\int_{-\infty}^{t} a(t-s)\frac{\partial^2}{\partial x^2}v(s,x)\,ds,$$
$$v(t,0) = 0, \qquad \frac{\partial}{\partial x}v(t,0) = 0,$$
$$p_1(t) = \sigma(t,l), \qquad p_2(t) = -\frac{\partial}{\partial x}\sigma(t,l),$$
$$q_1(t,l) = \frac{\partial}{\partial x}v(t,l), \qquad q_2(t) = v(t,l).$$

These equations fit in the framework of System 2.1 with the following spaces and operators:

$$
\begin{aligned}
X = Y &= L^2([0,l],\mathbf{R}),\\
\operatorname{dom}(D) &= W^{2,2}([0,l],\mathbf{R}), \quad (D\sigma)(x) = -\frac{d^2}{dx^2}\sigma(x),\\
\operatorname{dom}(\tilde{D}) &= \left\{ v \in W^{2,2}([0,l],\mathbf{R}) \mid v(0)=0, \frac{d}{dx}v(0)=0 \right\},\\
(\tilde{D}v)(x) &= \frac{d^2}{dx^2}v(x),\\
(Rv)(x) &= A\rho v(x),\\
(A(t)\varepsilon)(x) &= Ia(t)\varepsilon(x)\\
P\sigma &= \begin{pmatrix} \sigma(l) \\ -(d/dx)\sigma(l) \end{pmatrix}, \qquad \tilde{P}v = \begin{pmatrix} (d/dx)v(l) \\ v(l) \end{pmatrix}.
\end{aligned}
$$

**3. The main results.** Most of our results will be stated in the frequency domain. Therefore we take formal Laplace transforms in System 2.1. (The Laplace transform of a function $f$ will be denoted by $\hat{f}$.)

**System 3.1.**

$$
\begin{aligned}
sR\hat{v}(s) &= D\hat{\sigma}(s) + Rv_0,\\
\hat{\sigma}(s) &= \hat{A}(s)\tilde{D}\hat{v}(s) + \sigma_0(s),\\
P\hat{\sigma}(s) &= \hat{p}(s), \qquad \tilde{P}\hat{v}(s) = \hat{q}(s).
\end{aligned}
$$

Here $v_0$ and $\sigma_0$ describe the effects of the history of the deformation before $t=0$:

$$
v_0 = v(0), \qquad \sigma_0(s) = \int_0^\infty \int_{-\infty}^0 e^{-st} A(t-\tau)\tilde{D}v(\tau)\, d\tau\, dt.
$$

Of course, at this point questions about well-posedness and a justification of the formal transform arise. We do not consider such problems in this paper, since the considerable technicalities involved would only obscure the main points of our work. Well-posedness of integral equations can be handled within the framework of resolvent operators described in [**19**] where the reader finds special chapters devoted to the equations of linear viscoelasticity. It can also be handled in semigroup settings, as in [**7, 9**] and [**20**]. In any case, Hypothesis 2.3 has to be sharpened.

**3.1. The open loop problem.** We consider now the open loop problem relating the control forces $p_1, \dots, p_n$ to the observed velocities $q_1, \dots, q_n$. We are interested in the properties of the transfer matrix.

**Theorem 3.1.** (i) *For each* $s \in \mathcal{U}$ *and each* $z \in \mathbf{C}^n$ *there exists exactly one* $\hat{v} \in \operatorname{dom}(\tilde{D})$ *and one* $\hat{\sigma} \in \operatorname{dom}(D)$ *solving System* 3.1 *with* $\hat{p}(s) = z$, $v_0 = 0$, *and* $\sigma_0 = 0$. *We obtain a matrix valued function* $S$ *such that* $S(s)\hat{p}(s) = \hat{q}(s)$, *where the last equation of System* 3.1 *defines* $\hat{q}(s)$.

(ii) $S$ *is analytic in* $\mathcal{U}$.

(iii) $\langle z, S(s)z\rangle \notin (-\infty, 0]$ *for all* $s \in \mathcal{U}$ *and all* $z \in \mathbf{C}^n \setminus \{0\}$. *In particular,* $S(s)$ *is nonsingular.*

(iv) *If* $\mathcal{U}_1 \subset \mathcal{U}$ *satisfies Hypothesis* 2.4 *then* $|\arg(\langle z, S(s)z\rangle)| < \max(\theta, \delta)$ *for all* $s \in \mathcal{U}_1$ *and* $z \in \mathbf{C}^n \setminus \{0\}$. *In particular, if* $\delta \le \pi/2$ *and* $\theta \le \pi/2$, *then* $\Re(\langle z, S(s)z\rangle) > 0$.

(v) *If* $s \in (0, \infty)$, *then* $S(s)$ *is self-adjoint and positive definite.*

In the time domain, one would formally write the output as a convolution (Stieltjes) integral of the input with a measure describing the impulse response of the system. This is possible if and only if continuous input functions yield continuous output. However, $S(s)$ is not always the Laplace transform of a measure. Strong viscoelastic damping has a smoothing effect, so that with additional conditions the impulse response is a function of time.

The following theorem guarantees the existence of a Laplace transformable impulse response function under conditions that require a tradeoff between the geometric configuration of the system and the material properties. This should be seen in contrast to many results on smoothing properties concerning dynamics of systems with homogeneous boundary conditions, which depend only on the material properties, in particular the singularity of the relaxation kernel at 0.

**Theorem 3.2.** *Suppose that* $\mathcal{U}$ *contains a sector* $\Sigma = \{s \in \mathbf{C} \mid |s| > r, |\arg(s)| < \delta\}$ *with some* $r > 0$ *and* $\delta \in (\pi/2, \pi)$. *Let*

$$\lim_{s\to\infty, s\in\Sigma} \left\| \frac{1}{s}\hat{A}(s) \right\| = 0.$$

*Moreover, suppose that there are constants* $M_1 > 0$, $\gamma > 1$, *and* $\theta \in (\pi/2, \pi)$ *such that for all* $s \in \Sigma$ *and for all* $x \in X$ *the following inequalities hold*:

$$\left| \arg\left( \left\langle x, \frac{1}{s}\hat{A}(s)x \right\rangle \right) \right| \leq \theta < \pi,$$
$$\left| \left\langle x, \frac{1}{s}\hat{A}(s)x \right\rangle \right| \geq M_1 |s|^{-\gamma} \|x\|^2.$$

*Let* $L$ *be the self-adjoint, positive definite operator defined by*

$$\operatorname{dom}(L) = \{x \in \operatorname{dom}(\tilde{D}R^{-1}D) \mid Px = 0\}$$
$$Lx = -\tilde{D}R^{-1}Dx,$$

(*see Lemma* 4.1) *and assume that for some* $\beta \geq 0$ *the solutions* $\omega_j$ *to the elliptic boundary value problem*

$$\beta\omega_j - \tilde{D}R^{-1}D\omega_j = 0, \qquad P\omega_j = e_j$$

*are contained in* $\operatorname{dom}(L^\alpha)$ *with some* $1/2 > \alpha > (\gamma - 1)/(2\gamma)$. ($e_j$ *is the* $j$*th unit vector in* $\mathbf{C}^n$.) *Then the transfer matrix* $S(s)$ *is the Laplace transform of a matrix valued function which is integrable on compact intervals and grows at most exponentially when* $t \to \infty$.

We remark that the conditions of this theorem are only satisfied by relaxation kernels $A(t)$ with a strong singularity at $t = 0$. This type of kernel gives rise to solutions which are infinitely smooth in time even for rough initial data. If the semigroup setting from [**7**] is applicable, the conditions of Theorem 3.2 imply (by [**7**, Theorem 3.1]) that the solution semigroup is analytic.

One of the most serious limitations of models of damped structures that ignore the inertial forces of actuators and sensors is their nonrobustness with respect to small time delays in the negative feedback. In certain systems, a small delay in feedback can even cause loss of well-posedness, in the sense that the closed loop transfer function has poles with arbitrary large real parts. At least three facts counteract this destructive effect in real life systems: First, all sensors and actuators have some positive mass. This changes the model behavior drastically, in the sense that neither essential destabilization nor stabilization beyond the essential growth rate is possible. Moreover, all feedback loops will have some finite band width. Finally, materials themselves have some inherent damping. Here we show how viscoelastic damping counteracts destabilization due to delay. Of course, the destabilization phenomena disappear in the context of passive damping, e.g. by friction, where no delay is expected. (More information on the destabilization phenomena can be found, e.g., in [**3, 4, 5, 8, 10, 17, 18**].)

If we close the loop by a delayed feedback

$$P\sigma(t) = -K\tilde{P}v(t-\tau) + f(t)$$

the closed loop transfer function relating the observed velocity $\tilde{P}v$ to the external control force $f$ is

$$(1 + S(s)Ke^{-s\tau})^{-1}S(s). \tag{3.1}$$

We can prove

**Theorem 3.3.** *If the hypotheses of Theorem* 3.2 *are satisfied, then for any matrix* $K$ *and any* $\tau > 0$*, the closed loop transfer function given by* (3.1) *is analytic in some right half-plane* $\Re(s) > \omega$.

Theorem 3.2 requires an assumption about the spatial smoothness of the solutions to an elliptic boundary value problem. This assumption can be checked using the boundary data of the eigenvectors of $L$.

**Theorem 3.4.** *Let* $L$ *be the self-adjoint, positive definite operator defined by*

$$\begin{aligned} \mathrm{dom}\,(L) &= \{x \in \mathrm{dom}\,(\tilde{D}R^{-1}D) \mid Px = 0\} \\ Lx &= -\tilde{D}R^{-1}Dx. \end{aligned}$$

*Let $\alpha, \beta \geq 0$, and let $\omega_j$ be the solutions to the elliptic boundary value problem*

$$\beta\omega_j - \tilde{D}R^{-1}D\omega_j = 0, \qquad P\omega_j = e_j.$$

*Assume that the embedding of* $\operatorname{dom}(L)$ *into $X$ is compact, so that $L$ has discrete point spectrum. For $k = 1, 2, 3, \dots$ let $\lambda_k$ and $\phi_k$ denote the eigenvalues and the normalized eigenvectors of $L$. Then the solutions $\omega_j$ are contained in* $\operatorname{dom}(L^\alpha)$ *if and only if*

$$\sum_{k=1}^{\infty} \lambda_k^{2\alpha-2} \|\tilde{P}R^{-1}D\phi_k\|^2 < \infty.$$

For Example 2.2, one computes (details are left to the reader) that every $\alpha < 1/2$ satisfies the condition of Theorem 3.4, so the conditions on $A(t) = a(t)$ in Theorem 3.2 with any $\gamma > 1$ ensure that $S(s)$ is a transform. An even weaker smoothing condition ($a'(0+) = -\infty$) is shown to be sufficient in [**13**], by direct calculation of $S(s)$.

The destabilizing effect of delays in the moment feedback for an Euler-Bernoulli beam with a fractional derivative constitutive law of order $\nu$ has been investigated in [**8**] and [**10**]. There it was shown that loss of well-posedness takes place for $\nu < 2/3$, and that the control is robust with respect to time delays if $\nu > 2/3$. With respect to this background, the example below shows that the hypotheses of Theorem 3.2 are sharp. (The use of the Euler-Bernoulli model for describing phenomena happening at high frequencies, however, remains questionable.)

**Example 3.1.** We consider the viscoelastic Euler-Bernoulli beam from Example 2.3, where the shear control force is switched off and the only control is performed by the bending moment. Then the hypotheses of Theorem 3.2 hold for $\alpha < 1/8$ and $\gamma < 4/3$. In particular we impose a fractional derivative constitutive law of order $\nu \in (0, 1)$:

$$A(t) = E_1 + E_2 e^{-\delta t} \frac{1}{\Gamma(1-\nu)} t^{-\nu}.$$

Then for $\nu > 2/3$, the open loop problem admits a locally integrable transfer function.

**3.2. Dynamics of the stabilized system.** Let $J \subset N = \{1, \ldots, n\}$. We consider first the boundary conditions which are obtained by fixing some observers (which restricts the degrees of freedom of the system) and leaving the others uncontrolled:

$$\begin{cases} q_j = 0 & \text{if } j \in J, \\ p_j = 0 & \text{if } j \notin J, \end{cases} \tag{3.2}$$

where $J \subset N = \{1, \ldots, n\}$ is the index set of the fixed boundary conditions.

We will refer to (3.2) as energy conserving conditions, since they do not allow transmission of energy across the boundary, so that all damping of the system is due to the internal damping by viscoelasticity. With the boundary conditions (3.2), System 2.1 can be written in the form

$$\begin{aligned} \frac{d}{dt} Rv(t) &= D_J \sigma(t), \\ \sigma(t) &= \int_{-\infty}^{t} A(t-s) \tilde{D}_J v(s)\, ds. \end{aligned}$$

where $D_J$ and $\tilde{D}_J$ are suitable restrictions of $D$ and $\tilde{D}$:

$$\begin{aligned} \operatorname{dom}(D_J) &= \{\sigma \in \operatorname{dom}(D) \mid (P\sigma)_j = 0 \text{ for } j \notin J\}, \\ \operatorname{dom}(\tilde{D}_J) &= \{v \in \operatorname{dom}(\tilde{D}) \mid (\tilde{P}v)_j = 0 \text{ for } j \in J\}. \end{aligned} \tag{3.3}$$

Notice that $D_N = D$ and $\tilde{D}_\varnothing = \tilde{D}$. Since $D_J$ and $-\tilde{D}_J$ are adjoint to each other (see Lemma 4.1), this system can be treated using well-known settings (e.g., [**7, 19**]). If the relaxation modulus is scalar, one can use the spectral decomposition of the self-adjoint operator $-\tilde{D}_J R^{-1} D_J$ to decompose the system into scalar equations (see, e.g., [**1**]). In the last 15 years much information has been gained about the viscoelastic problem with energy conserving boundary conditions.

We now replace the boundary conditions 3.2 by the simplest possible damping boundary conditions:

$$p(t) = -Kq(t). \tag{3.4}$$

**Hypothesis 3.1.** $K = \operatorname{diag}(k_1, \ldots, k_n)$ *is a positive definite diagonal matrix.*

This type of damping condition can be regarded as passive damping by friction. In our pilot example 2.1 it would be achieved by dashpots connected to each patch. The results obtained for diagonal $K$ can be easily extended to general positive semidefinite matrices, since a unitary coordinate transform in the range of $P$ and $\tilde{P}$ will diagonalize $K$.

In order to consider the damped problem, one could adapt the energy methods mentioned above, which would require somewhat more involved estimates but in principle seems to be possible. However, in [**13**] a method was introduced which allows one to transfer results about System 2.1 with energy conserving boundary conditions to the same problem with damping. There a damping problem for a rod (Example 2.2) was investigated. The authors observed that the solution operator to the damped system in frequency domain is a convex combination of the solution operators to two systems with energy conserving boundary conditions. This allows one to transfer information on systems with no boundary damping to the damped system. In this section we show that the same technique is also applicable when more than one degree of freedom is controlled.

Although we can give an explicit formula (see Proposition 5.2) in terms of $K$ and the open loop transfer matrix $S(s)$, the computation of the coefficients is very tedious if not impossible as soon as one proceeds beyond very simple examples. However, we can show that the coefficients are uniformly bounded in every domain satisfying Hypothesis 2.4, in particular in the open right half-plane. Therefore the theorem below has a simple consequence: *Whatever can be said about systems with energy conserving boundary conditions because of some uniform bounds for the Laplace transform of the solutions holds as well for the system with passive damping.* This pertains for instance to [**7**, Theorem 3.1] on the smoothing effect of singular kernels and [**6**, Theorem 2.9], [**9**, Theorem 3.1] on the essential growth rate of the solution semigroups, which have been proved for energy conserving boundary conditions but can be rephrased literally for the passive damping boundary conditions 3.4.

**Theorem 3.5.** *Let $K$ be as in Hypothesis* 3.1. *There exist complex valued analytic functions $\lambda_J : \mathcal{U} \to \mathbf{C}$ with the following properties*:

(i) *Let $v_0 \in Y$ and $\sigma_0$ in $X$ be given. For all $J \subset N$ let $\hat{v}_J : \mathcal{U} \to Y$ and $\hat{\sigma}_J : \mathcal{U} \to X$ solve System* 3.1 *subject to the boundary condition*

(3.2). *Then*

$$\hat{v}(s) = \sum_{J\subset N} \lambda_J(s)\hat{v}_J(s) \quad \textit{and} \quad \hat{\sigma}(s) = \sum_{J\subset N} \lambda_J(s)\hat{\sigma}_J(s)$$

*solve System* 3.1 *with the damping boundary condition* (3.4).

(ii) $\sum_{J\subset N} \lambda_J(s) = 1$.

(iii) *If* $\mathcal{U}_1 \subset \mathcal{U}$ *satisfies Hypothesis* 2.4, *then there exists some constant* $M_2$ *such that* $|\lambda_J(s)| \leq M_2$ *for all* $s \in \mathcal{U}_1$ *and all* $J \subset N$. *In particular if* $\delta \leq \pi/2$ *and* $\theta \leq \pi/2$, *then* $M_2 = 1$ *will work.*

(iv) *If* $s \in (0, \infty)$, *then* $\lambda_J(s) \geq 0$.

It is not crucial for the decomposition method that the feedback gains $k_m$ are independent of $s$. We expect that a similar technique may be useful in the case of feedback control with dynamic compensators, if the right conditions on the phase of $k_m(s)$ are specified. However, the diagonalizability of $K$ is of central importance, and it is, of course, a severe restriction on the applicability of our decomposition technique.

If the impulse response of the open loop problem is a function, Theorem 3.5 has an interpretation in the time domain:

**Theorem 3.6.** *Let the conditions of Theorem* 3.5 *hold, and let* $S(s)$ *defined in Theorem* 3.1 *be the Laplace transform of a matrix valued function. Then for* $J \subset N$ *there are Laplace transformable functions* $\mu_J$ *and scalars* $\nu_J$ *such that the following assertions hold*:

(i) *For all* $J \subset N$ *let* $v_J : \mathbf{R} \to Y$ *and* $\sigma_J : \mathbf{R} \to X$ *be Laplace transformable functions solving System* 2.1 *subject to the boundary condition* (3.2). *Then*

$$v(t) = \sum_{J\subset N} \left(\nu_J v_J(t) + \int_0^t \mu_J(t-s) v_J(s)\,ds\right),$$
$$\sigma(t) = \sum_{J\subset N} \left(\nu_J \sigma_J(t) + \int_0^t \mu_J(t-s) \sigma_J(s)\,ds\right)$$

*solve System* 2.1 *with the damping boundary condition* (3.4).

(ii) $\sum_{J\subset N} \nu_J = 1$ *and* $\sum_{J\subset N} \mu_J(t) = 0$.

## 4. Proofs.

**4.1. Proof of Theorem 3.1.** Throughout this chapter, $e_j$ will always denote the $j$th unit vector in $\mathbf{C}^n$.

**Lemma 4.1.** *The operators $D_J$ and $-\tilde{D}_J$ defined in* (3.3) *are adjoint to each other. The operator $L = -\tilde{D}R^{-1}D_\varnothing$ (which is the operator $L$ defined in Theorem* 3.2) *is self-adjoint and positive semi-definite.*

*Proof.* Let $v \in \mathrm{dom}\,(\tilde{D}_J)$, $\sigma \in \mathrm{dom}\,(D_J)$. Then

$$\langle \tilde{D}_J v, \sigma\rangle + \langle v, D_J\sigma\rangle = \langle \tilde{P}v, P\sigma\rangle = 0.$$

Therefore $-\tilde{D}_J \subset D_J^*$. Now let $v \in \mathrm{dom}\,(D_J^*)$, $D_J^* v = \tau$. For $\sigma \in \mathrm{dom}\,(D)$ the linear functional

$$f(\sigma) := \langle \tau, \sigma\rangle + \langle v, D\sigma\rangle$$

is defined and vanishes for $\sigma \in \mathrm{dom}\,(D_J)$. In particular, $f$ vanishes on $\mathrm{dom}\,(D) \cap \ker(P)$. Therefore there exists a vector $z \in \mathbf{C}^n$ such that for all $\sigma \in \mathrm{dom}\,(D)$

$$f(\sigma) = \langle \tau, \sigma\rangle + \langle v, D\sigma\rangle = \langle z, P\sigma\rangle.$$

Since $\tilde{D}$ and $\tilde{P}$ do not allow proper extensions satisfying (2.1), we infer that $\tilde{D}v = \tau$ and $\tilde{P}v = z$. Since $f$ vanishes for all $\sigma \in \mathrm{dom}\,(D_J)$, we infer that $\langle e_j, z\rangle = 0$ for $j \in J$. Consequently, $v \in \mathrm{dom}\,(\tilde{D}_J)$. Since $-\tilde{D} = D_\varnothing^*$ and $R^{-1}$ is self-adjoint, positive definite, bounded and continuously invertible, we have that

$$-\tilde{D}R^{-1}D_\varnothing = D_\varnothing^*(R^{-1/2})^* R^{-1/2} D_\varnothing$$

is self-adjoint and positive semidefinite. □

**Lemma 4.2.** *For each $s \in \mathcal{U}$, the operator $(1 + (1/s)\hat{A}(s)L)$ admits a continuous inverse.*

*Proof.* Let $x, y \in X$ such that $x + (1/s)\hat{A}(s)Lx = y$. Taking inner products with $Lx$, we obtain

$$\|Lx\|\,\|y\| \geq |\langle Lx, y\rangle| = \left|\langle Lx, x\rangle + \left\langle Lx, \frac{1}{s}\hat{A}(s)Lx\right\rangle\right|.$$

Take $\theta = \theta(s)$ and $M = M(s)$ from Hypothesis 2.3. Since $\langle Lx, x\rangle \geq 0$ we obtain

$$\|Lx\|\,\|y\| \geq \sin(\theta)\left|\left\langle Lx, \frac{1}{s}\hat{A}(s)Lx\right\rangle\right| \geq M\sin(\theta)\|Lx\|^2.$$

Therefore there exists some constant $C$ such that $\|Lx\| \leq C\|y\|$. Since $(1/s)\hat{A}(s)$ is a bounded operator, we can now infer easily that the operator

$$\left(1+\frac{1}{s}\hat{A}(s)L\right)^{-1} : \mathrm{range}\left(1+\frac{1}{s}\hat{A}(s)L\right) \longrightarrow \mathrm{dom}\,(L)$$

is continuous. Since $\hat{A}(s)$ is invertible, the operator $(1+(1/s)\hat{A}(s)L)$ is closed, and by the considerations above its range must be closed. We show finally that its range is dense. Suppose that some $z$ is orthogonal to the whole range. Then for all $y \in \mathrm{dom}\,(L)$ we have $\langle z, y + (1/s)\hat{A}(s)Ly\rangle = 0$ which implies that $x = ((1/s)\hat{A}(s))^* z \in \mathrm{dom}\,(L)$ and $Lx = -z$, thus

$$\left(1+\left(\frac{1}{s}\hat{A}(s)\right)^* L\right)x = 0.$$

A similar estimate as above (just with the adjoint operators) implies that $x = 0$, whence $z = 0$. □

*Proof of Theorem* 3.1 (i), (ii). Given $z \in \mathbf{C}^n$, we pick some $\sigma_1 \in \mathrm{dom}\,(\tilde{D}R^{-1}D)$ with $P\sigma_1 = z$. Putting $\hat{\sigma} = \sigma_1 + \tau$, we have $P\tau = 0$, so that $\tau \in \mathrm{dom}\,(D_\varnothing)$ and we have to solve

$$\begin{aligned} sR\hat{v} &= D_\varnothing \tau + D\sigma_1, \\ \tau &= -\sigma_1 + \hat{A}(s)\tilde{D}\hat{v}. \end{aligned}$$

Thus $\tau$ solves

$$\tau = -\sigma_1 + \frac{1}{s}\hat{A}(s)\tilde{D}R^{-1}D_\varnothing\tau + \frac{1}{s}\hat{A}(s)\tilde{D}R^{-1}D\sigma_1,$$

i.e.,

$$\left(1+\frac{1}{s}\hat{A}(s)L\right)\tau = -\left(1-\frac{1}{s}\hat{A}(s)\tilde{D}R^{-1}D\right)\sigma_1.$$

By Lemma 4.2 there exists a unique solution $\tau$, from which $\hat{\sigma}$ and $\hat{v}$ can be computed:

$$\tau = -\left(1 + \frac{1}{s}\hat{A}(s)L\right)^{-1}\left(1 - \frac{1}{s}\hat{A}(s)\tilde{D}R^{-1}D\right)\sigma_1, \tag{4.1}$$
$$\hat{\sigma} = \sigma_1 + \tau, \qquad \hat{v} = \frac{1}{s}R^{-1}D\hat{\sigma}.$$

The analytic dependence on $s$ follows easily from the analyticity of $(1/s)\hat{A}(s)$. □

*Proof of Theorem* 3.1 (iii), (iv). Let $z \in \mathbf{C}^n$ and let $\hat{\sigma}$ and $\hat{v}$ be defined according to Theorem 3.1. Then we have

$$\begin{aligned}\langle S(s)z, z\rangle &= \langle \tilde{P}\hat{v}, P\hat{\sigma}\rangle = \langle \hat{v}, D\hat{\sigma}\rangle + \langle \tilde{D}\hat{v}, \hat{\sigma}\rangle \\ &= \langle \hat{v}, sR\hat{v}\rangle + \langle \tilde{D}\hat{v}, \hat{A}(s)\tilde{D}\hat{v}\rangle \\ &= s\left(\langle \hat{v}, R\hat{v}\rangle + \left\langle \tilde{D}\hat{v}, \frac{1}{s}\hat{A}(s)\tilde{D}\hat{v}\right\rangle\right).\end{aligned}$$

Since $\langle \hat{v}, R\hat{v}\rangle \geq 0$ while $\langle \tilde{D}\hat{v}, (1/s)\hat{A}(s)\tilde{D}\hat{v}\rangle \notin (-\infty, 0)$, their sum cannot be zero unless $\hat{v} = 0$ which implies $z = 0$. Moreover,

$$\begin{aligned}\left|\arg\left(\langle \hat{v}, R\hat{v}\rangle + \left\langle \tilde{D}\hat{v}, \frac{1}{s}\hat{A}(s)\tilde{D}\hat{v}\right\rangle\right)\right| &\leq \left|\arg\left(\left\langle \tilde{D}\hat{v}, \frac{1}{s}\hat{A}(s)\tilde{D}\hat{v}\right\rangle\right)\right| \\ &\leq \theta(s).\end{aligned}$$

By Hypothesis 2.3, the arguments of $\langle \tilde{D}\hat{v}, (1/s)\hat{A}(s)\hat{v}(s)\rangle$ and $s$ have opposite signs. Therefore

$$|\arg(\langle z, S(s)z\rangle)| \leq \max\{\theta(s), |\arg(s)|\}.$$

These estimates hold pointwise for any $s \in \mathcal{U}$ and uniformly for $s \in \mathcal{U}_1$ if $\mathcal{U}_1$ satisfies Hypothesis 2.4. □

*Proof of Theorem* 3.1 (v). Once the symmetry is proved, the definiteness follows from (iii). To prove the symmetry, let $s > 0$ and $\hat{v}_j$, $\hat{\sigma}_j$ be defined according to Theorem 3.1 for $z = e_j$. Then the $(i,j)$-entry of the transfer matrix $S(s)$ is

$$\begin{aligned} S_{ij}(s) &= \langle e_i, S(s)e_j\rangle = \langle P\hat{\sigma}_i, \tilde{P}\hat{v}_j\rangle = \langle D\hat{\sigma}_i, \hat{v}_j\rangle + \langle \hat{\sigma}_i, \tilde{D}\hat{v}_j\rangle \\ &= \langle sR\hat{v}_i, \hat{v}_j\rangle + \langle \hat{A}(s)\tilde{D}\hat{v}_i, \tilde{D}\hat{v}_j\rangle = \langle \hat{v}_i, sR\hat{v}_j\rangle + \langle \tilde{D}\hat{v}_i, \hat{A}(s)\tilde{D}\hat{v}_j\rangle \\ &= \langle \hat{v}_i, D\hat{\sigma}_j\rangle + \langle \tilde{D}\hat{v}_i, \hat{\sigma}_j\rangle = \langle \tilde{P}\hat{v}_i, P\hat{\sigma}_j\rangle = \langle S(s)e_i, e_j\rangle \\ &= S_{ji}(s). \qquad \square \end{aligned}$$

**4.2. Proof of Theorems 3.2 and 3.3.** We assume in this subsection that the assumptions of Theorem 3.2 are satisfied.

**Lemma 4.3.** *There exists a constant $r_1 > r$ such that for all*

$$s \in \Sigma_1 = \{s \in \Sigma \mid \ |s| \geq r_1\}$$

*and all $y \in X$ the following assertions are true*:

(i) $\|(\beta/s)\hat{A}(s)\| < 1$,

(ii) $(1 - (\beta/s)\hat{A}(s))$ *is continuously invertible*,

(iii) $\|(1/s)\hat{A}(s)y\|^2 \leq (1/(2\beta))|\langle y, (1/s)\hat{A}(s)y\rangle|$.

*Proof.* Since $\lim_{|s|\to\infty, s\in\Sigma} \|(1/s)\hat{A}(s)\| = 0$, the first assertion is obvious and the second assertion follows by Neumann series.

To prove (iii) we decompose

$$(1/s)\hat{A}(s) = U(s) + iV(s)$$

with self-adjoint bounded operators $U$ and $V$. Since the argument of $\langle y, (1/s)\hat{A}(s)y\rangle$ is either positive or negative for all $y$ (depending on $\Im(s)$), the operator $V(s)$ is either positive or negative semi-definite. In the sequel we give all estimates for positive semi-definite $V(s)$; the negative case is handled exactly the same way. Since $|\arg(\langle y, (1/s)\hat{A}(s)y\rangle)| \leq \theta$, we have for $\kappa = 1/|\sin(\theta)|$:

$$\langle y, (U(s) + \kappa V(s))y\rangle = \Re\left\langle y, \frac{1}{s}\hat{A}(s)y\right\rangle + \kappa\Im\left\langle y, \frac{1}{s}\hat{A}(s)y\right\rangle \geq 0.$$

Therefore $(U(s)+\kappa V(s))$ is positive semidefinite. Evidently, $\|U(s)\|$ and $\|V(s)\|$ are bounded by $\|(1/s)\hat{A}(s)\|$. Therefore, for any $\varepsilon>0$ we may pick $r_1$ sufficiently large such that for $s\in\Sigma_1$

$$\|U(s)\pm\kappa V(s)\|\le\varepsilon \quad\text{and}\quad \|V(s)\|\le\varepsilon.$$

Spectral resolution implies that

$$\begin{aligned}\|V(s)y\|^2 &\le \|V(s)^{1/2}\|^2\,\|V(s)^{1/2}y\|^2\\ &= \|V(s)\|\,|\langle y, V(s)y\rangle|\\ &\le \varepsilon\left|\left\langle y, \frac{1}{s}\hat{A}(s)y\right\rangle\right|.\end{aligned}$$

Similarly

$$\begin{aligned}\|(U(s)+\kappa V(s))y\|^2 &\le \varepsilon|\langle y,(U(s)+\kappa V(s))y\rangle|\\ &\le (1+\kappa)\varepsilon\left|\left\langle y, \frac{1}{s}\hat{A}(s)y\right\rangle\right|.\end{aligned}$$

Then

$$\begin{aligned}\left\|\frac{1}{s}\hat{A}(s)y\right\|^2 &= \|(U(s)+\kappa V(s))y+(i-\kappa)V(s)y\|^2\\ &\le 2\|(U(s)+\kappa V(s))y\|^2+2(1+\kappa^2)\|V(s)y\|^2\\ &\le 2\varepsilon[(1+\kappa)+(1+\kappa^2)]\left|\left\langle y, \frac{1}{s}\hat{A}(s)y\right\rangle\right|.\end{aligned}$$

Choosing $\varepsilon$ sufficiently small, we obtain assertion (iii). □

For $s\in\Sigma_1$ we can now define the operator

$$\begin{aligned}A_\beta(s) &= \frac{1}{s}\hat{A}(s)\left(1-\frac{\beta}{s}\hat{A}(s)\right)^{-1}\\ &= \frac{1}{\beta}\left[\left(1-\frac{\beta}{s}\hat{A}(s)\right)^{-1}-1\right].\end{aligned}$$

(For $\beta=0$, we have $A_0(s)=(1/s)\hat{A}(s)$.) For shorthand we define also

$$L_\beta=\beta+L.$$

The following lemma shows that $A_\beta(s)$ and $(1/s)\hat{A}(s)$ satisfy similar conditions:

**Lemma 4.4.** *There exist constants $\theta_\beta \in (0,\pi)$ and $M_\beta > 0$ such that for all $s \in \Sigma_1$ and all $x \in X \setminus \{0\}$ the following inequalities hold*:

$$\arg(\langle x, A_\beta x\rangle) \begin{cases} \in (0,\theta_\beta) & \text{if } \Im(s) < 0, \\ = 0 & \text{if } \Im(s) = 0, \\ \in (-\theta_\beta, 0) & \text{if } \Im(s) > 0, \end{cases}$$
$$|\langle x, A_\beta x\rangle| \geq M_\beta |s|^{-\gamma}\|x\|^2.$$

*Proof.* Let $x \in X$ and $y = (1-(\beta/s)\hat{A}(s))^{-1}x$. Then $A_\beta x = (1/s)\hat{A}(s)y$. Thus

$$\begin{aligned} \langle x, A_\beta x\rangle &= \left\langle y - \frac{\beta}{s}\hat{A}(s)y, \frac{1}{s}\hat{A}(s)y\right\rangle \\ &= \left\langle y, \frac{1}{s}\hat{A}(s)y\right\rangle - \beta\left\|\frac{1}{s}\hat{A}(s)y\right\|^2. \end{aligned} \tag{4.2}$$

We see immediately that

$$\Im(\langle x, A_\beta x\rangle) = \Im\left(\left\langle y, \frac{1}{s}\hat{A}(s)y\right\rangle\right)$$

has the opposite sign of $\Im(s)$. The argument of $\langle y, (1/s)\hat{A}(s)y\rangle$ is bounded away from $\pi$ uniformly for $s \in \Sigma_1$. By Lemma 4.3,

$$\beta\left\|\frac{1}{s}\hat{A}(s)y\right\|^2 \leq \frac{1}{2}\left|\left\langle y, \frac{1}{s}\hat{A}(s)y\right\rangle\right|.$$

Equation (4.2) implies then that the argument of $\langle x, A_\beta x\rangle$ is bounded away from $\pi$ uniformly for $s \in \Sigma_1$, and that

$$\left|\left\langle y, \frac{1}{s}\hat{A}(s)y\right\rangle\right| \leq 2|\langle x, A_\beta x\rangle|.$$

Since $x = (1 - (\beta/s)\hat{A}(s))y$ and $\|(\beta/s)\hat{A}(s)\| < 1$ we can estimate

$$\frac{M_1}{8}|s|^{-\gamma}\|x\|^2 \leq \frac{1}{2}M_1|s|^{-\gamma}\|y\|^2 \leq \frac{1}{2}|\langle y, \frac{1}{s}\hat{A}(s)y\rangle| \leq |\langle x, A_\beta x\rangle|. \qquad \square$$

**Lemma 4.5.** *The entries of the transfer matrix are given by*

$$\begin{aligned} S_{ij}(s) = &\frac{1}{s}\langle D\omega_i, R^{-1}D\omega_j\rangle + \frac{\beta}{s}\langle \omega_i, \omega_j\rangle \\ &+ \frac{1}{s}\langle \omega_i, L_\beta(1 + A_\beta(s)L_\beta)^{-1}\omega_j\rangle. \end{aligned}$$

*Proof.* The entry $S_{ij}(s)$ is obtained by $S_{ij}(s) = \langle e_i, \tilde{P}\hat{v}\rangle$, where $\hat{v}$ and $\hat{\sigma}$ are given by Theorem 3.1 with the boundary condition $P\hat{\sigma} = z = e_j$. We utilize (4.1) with $\sigma_1 := \omega_j$. (Recall also that $D_\varnothing^* = -\tilde{D}$.)

$$\begin{aligned} S_{ij}(s) &= \langle e_i, \tilde{P}\hat{v}\rangle = \langle P\omega_i, \tilde{P}\hat{v}\rangle = \langle D\omega_i, \hat{v}\rangle + \langle \omega_i, \tilde{D}\hat{v}\rangle \\ &= \frac{1}{s}\langle D\omega_i, R^{-1}D\omega_j + R^{-1}D_\varnothing\tau\rangle \\ &\quad + \frac{1}{s}\langle \omega_i, \tilde{D}R^{-1}D\omega_j + \tilde{D}R^{-1}D_\varnothing\tau\rangle \\ &= \frac{1}{s}\langle D\omega_i, R^{-1}D\omega_j\rangle - \frac{1}{s}\langle \tilde{D}R^{-1}D\omega_i, \tau\rangle \\ &\quad + \frac{1}{s}\langle \omega_i, \beta\omega_j\rangle + \frac{1}{s}\langle \omega_i, \tilde{D}R^{-1}D_\varnothing\tau\rangle \\ &= \frac{1}{s}\langle D\omega_i, R^{-1}D\omega_j\rangle + \frac{\beta}{s}\langle \omega_i, \omega_j\rangle - \frac{1}{s}\langle \omega_i, \beta\tau + L\tau\rangle. \end{aligned}$$

Now,

$$\begin{aligned} -(\beta + L)\tau &= L_\beta\left(1 + \frac{1}{s}\hat{A}(s)L\right)^{-1}\left(1 - \frac{\beta}{s}\hat{A}(s)\right)\omega_j \\ &= L_\beta\left(1 - \frac{\beta}{s}\hat{A}(s) + \frac{1}{s}\hat{A}(s)L_\beta\right)^{-1}\left(1 - \frac{\beta}{s}\hat{A}(s)\right)\omega_j \\ &= L_\beta\left(1 + \frac{1}{s}\hat{A}(s)\left(1 - \frac{\beta}{s}\hat{A}(s)\right)^{-1}L_\beta\right)^{-1}\omega_j \\ &= L_\beta\left(1 + A_\beta(s)L_\beta\right)^{-1}\omega_j. \qquad \square \end{aligned}$$

**Lemma 4.6.** *There exists a constant $M_3$ such that for all $s \in \Sigma_1$*

$$|\langle \omega_i, L_\beta (1 + A_\beta(s) L_\beta)^{-1} \omega_j \rangle| \le M_3 |s|^{\gamma(1-2\alpha)}.$$

*Proof.* Since $\omega_i \in \operatorname{dom}(L^\alpha) = \operatorname{dom}(L_\beta^\alpha)$, it is sufficient to show the following estimate for a suitable constant $M_4$:

$$\|L_\beta^{1-\alpha}(1 + A_\beta(s)L_\beta)^{-1}\omega_j\| \le M_4 |s|^{\gamma(1-2\alpha)} \|L_\beta^\alpha \omega_j\|. \tag{4.3}$$

We define a vector $x$ and scalars $\zeta$, $\eta$, depending on $s$, by

$$\begin{aligned} x &= (1 + A_\beta(s)L_\beta)^{-1}\omega_j, \\ \zeta &= \langle x, L_\beta x\rangle = \|L_\beta^{1/2} x\|^2 > 0, \\ \eta &= \langle L_\beta x, A_\beta(s) L_\beta x\rangle \in \mathbf{C}. \end{aligned}$$

Evidently,

$$L_\beta^\alpha \omega_j = L_\beta^\alpha (1 + A_\beta(s) L_\beta) x.$$

Inequality (4.3) is then rewritten as

$$\|L_\beta^{1-\alpha} x\| \le M_4 |s|^{\gamma(1-2\alpha)} \|L_\beta^\alpha (1 + A_\beta(s)L_\beta)x\|. \tag{4.4}$$

We will prove the following sufficient condition for (4.4):

$$\|L_\beta^{1-\alpha} x\|^2 \le M_4 |s|^{\gamma(1-2\alpha)} |\langle L_\beta^{1-\alpha} x, L_\beta^\alpha (1 + A_\beta(s)L_\beta)x\rangle|$$

which can be rewritten as

$$\begin{aligned} \|L_\beta^{1-\alpha} x\|^2 &\le M_4 |s|^{\gamma(1-2\alpha)} |\langle x, L_\beta x\rangle + \langle L_\beta x, A_\beta(s) L_\beta x\rangle| \\ &= M_4 |s|^{\gamma(1-2\alpha)} |\zeta + \eta|. \end{aligned} \tag{4.5}$$

Lemma 4.4 implies that

$$\|L_\beta x\|^2 \le M_\beta^{-1} |s|^\gamma |\langle L_\beta x, A_\beta(s) L_\beta x\rangle| = M_\beta^{-1} |s|^\gamma \, |\eta|.$$

By interpolation there exists a constant $M_5$ such that

$$\begin{aligned} \|L_\beta^{1-\alpha} x\|^2 &\le M_5 \|L_\beta x\|^{2-4\alpha} \|L_\beta^{1/2} x\|^{4\alpha} \\ &\le M_5 M_\beta^{2\alpha - 1} |s|^{\gamma(1-2\alpha)} \, |\eta|^{1-2\alpha} \, \zeta^{2\alpha}. \end{aligned} \tag{4.6}$$

On the other hand, since $|\arg(\eta)|$ is bounded away from $\pi$ uniformly for $s \in \Sigma_1$, there exists a constant $M_6$ such that

$$|\zeta + \eta| \geq M_6(\zeta + |\eta|). \tag{4.7}$$

Combining (4.6) and (4.7) and dividing by $\zeta|s|^{\gamma(1-2\alpha)}$ we see that the following is a sufficient condition for (4.5):

$$M_5 M_\beta^{2\alpha-1}\left(\frac{|\eta|}{\zeta}\right)^{1-2\alpha} \leq M_4 M_6\left[1 + \left(\frac{|\eta|}{\zeta}\right)\right]. \tag{4.8}$$

Since the righthand side of this equation is bounded away from zero and grows faster than the lefthand side as $|\eta|/\zeta \to \infty$, there is a suitable constant $M_4$ such that (4.8) is satisfied. □

*Proof of Theorem* 3.2. We show that the entry $S_{ij}(s)$ of the transfer matrix is the Laplace transform of a locally integrable function $h_{ij}(t)$. According to Lemma 4.5 $h_{ij}$ should be of the form

$$h_{ij}(t) = \langle D\omega_i, R^{-1}D\omega_j\rangle + \beta\langle\omega_i, \omega_j\rangle + k_{ij}(t),$$

where the Laplace transform of $k_{ij}$ is

$$\hat{k}_{ij}(s) = \frac{1}{s}\langle\omega_i, L_\beta(1 + A_\beta(s)L_\beta)^{-1}\omega_j\rangle.$$

We will show that such a function $k_{ij}$ exists and that for small $t$

$$|k_{ij}(t)| \leq M_7 t^{-1+\varepsilon}$$

with some constants $\varepsilon > 0$ and $M_7 > 0$, while $k_{ij}(t)$ grows at most exponentially when $t \to \infty$.

Our proof is based on the complex inversion formula

$$k_{ij}(t) = \frac{1}{2\pi i}\int_\Gamma e^{ts}\hat{k}_{ij}(s)\,ds$$

with a contour consisting of two rays pointing into the negative half plane and an arc around the origin $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$, where

$$\begin{aligned}
&\Gamma_1(\tau) = -\tau\overline{z}, \quad \tau \in (-\infty, -\tau_0],\\
&\Gamma_2(\tau) = \tau_0 e^{i\tau}, \quad \tau \in (-\arg(z), \arg(z)),\\
&\Gamma_3(\tau) = \tau z, \quad \tau \in [\tau_0, \infty).
\end{aligned}$$

Here $z \in \mathbf{C}$ is taken with $|z| = 1$, $\arg(z) \in (\pi/2, \theta)$ and $\tau_0 > r_1$ so that all estimates derived in the lemmas above hold for $s$ on the paths $\Gamma_1$ and $\Gamma_3$.

Evidently the contribution of the arc $\Gamma_2$ to the contour integral is uniformly bounded for $t$ in compact intervals and grows at most exponentially when $t \to \infty$. Therefore we have only to worry about the contributions of the rays. Of course, $\Gamma_1$ and $\Gamma_3$ can be treated the same way. We give the estimate for $\Gamma_3$. Utilizing Lemma 4.6 we see that for $s \in \Gamma_3$ the following estimate holds:

$$|\hat{k}_{ij}(s)| \leq M_3 |s|^{\gamma(1-2\alpha)-1} = M_3 |s|^{-\varepsilon}$$

with $\varepsilon = 1 - \gamma(1-2\alpha) \in (0,1)$. We infer the estimate

$$\begin{aligned} t^{1-\varepsilon}\left|\frac{1}{2\pi i}\int_{\Gamma_3} e^{st}\hat{k}_{ij}(s)\,ds\right| &\leq M_3 t^{1-\varepsilon}\int_{\tau_0}^{\infty} e^{\tau t \Re(z)}\tau^{-\varepsilon}\,d\tau \\ &\leq M_3 \int_0^{\infty} e^{\tau t \Re(z)}(\tau t)^{-\varepsilon}\,t d\tau \\ &= M_3 \int_0^{\infty} e^{u\Re(z)} u^{-\varepsilon}\,du \\ &= M_7 < \infty. \end{aligned}$$

Therefore the inversion integral converges and $|k_{ij}(t)| \leq M_7 t^{\varepsilon-1}$. □

*Proof of Theorem* 3.3. Since $S$ is the Laplace transform of a locally integrable function with exponential growth, it is bounded in some right half-plane $\{s \in \mathbf{C} \mid \Re(s) \geq \omega\}$. As a consequence $\|e^{-s\tau}KS(s)\|$ is small if $\Re(s)$ is large. Therefore the inverse $(1 + e^{-s\tau}KS(s))^{-1}$ exists and is analytic. □

### 4.3. Proofs of Theorem 3.4 and Example 3.1.

*Proof of Theorem* 3.4. We set up the Fourier series

$$\omega_j = \sum_{k=1}^{\infty} \mu_{jk}\phi_k \quad \text{with} \quad \mu_{jk} = \langle \phi_k, \omega_j \rangle.$$

Then $\omega_j$ is contained in $\operatorname{dom} L^\alpha$ if and only if

$$\sum_{k=1}^{\infty} \lambda_k^{2\alpha} |\mu_{jk}|^2 < \infty.$$

However, $\mu_{jk}$ can be expressed in terms of the $j$th entry of $\tilde{P}R^{-1}D\phi_k$. (We use $\tilde{D} = -D_\varnothing^*$.)

$$\begin{aligned}
(\tilde{P}R^{-1}D\phi_k)_j &= \langle \tilde{P}R^{-1}D\phi_k, P\omega_j \rangle \\
&= \langle \tilde{D}R^{-1}D\phi_k, \omega_j \rangle + \langle R^{-1}D\phi_k, D\omega_j \rangle \\
&= \langle -L\phi_k, \omega_j \rangle - \langle \phi_k, \tilde{D}R^{-1}D\omega_j \rangle \\
&= (-\lambda_k - \beta)\langle \phi_k, \omega_j \rangle.
\end{aligned}$$

Therefore $\omega_j \in \operatorname{dom} L^\alpha$ if and only if

$$\sum_{k=1}^{\infty} \lambda_k^{2\alpha} (\lambda_k + \beta)^{-2} |(\tilde{P}R^{-1}D\phi_k)_j|^2 < \infty. \qquad \square$$

*Proof of Example* 3.1. For simplicity we rescale all physical constants ($\rho$, $l$, $I$, etc.) to 1. Once $\alpha$ is found, the estimates for $\gamma$ and $\nu$ follow immediately from

$$\alpha > \frac{\gamma+1}{2\gamma} \quad \text{and} \quad \hat{A}(s) = \frac{E_1}{s} + \frac{E_2}{(s+\delta)^{1-\nu}}.$$

The operator $L$ is given by

$$L\phi = -\frac{\partial^4}{\partial x^4}\phi,$$

$$\operatorname{dom}(L) = \big\{\phi \in W^{4,2}([0,l], \mathbf{R}) \mid \phi(1) = 0, \phi_x(1) = 0, \phi_{xx}(0) = 0, \phi_{xxx}(0) = 0\big\}.$$

A straightforward computation yields the eigenvalues and eigenvectors:

$$\begin{aligned}
\lambda_k &= \xi_k^4,\\
\phi_k(x) &= c_k[(\cosh(\xi_k x) + \cos(\xi_k x))\\
&\qquad - \beta_k(\sinh(\xi_k x) + \sin(\xi_k x))],\\
\beta_k &= \frac{\sinh(\xi_k) - \sin(\xi_k)}{\cosh(\xi_k) + \cos(\xi_k)},\\
(R^{-1}D\phi_k)(x) &= c_k\xi_k^2[(\cosh(\xi_k x) - \cos(\xi_k x))\\
&\qquad - \beta_k(\sinh(\xi_k x) - \sin(\xi_k x))],\\
\tilde{P}R^{-1}D\phi_k &= c_k\xi_k^3[(\sinh(\xi_k) + \sin(\xi_k))\\
&\qquad - \beta_k(\cosh(\xi_k) - \cos(\xi_k))].
\end{aligned}$$

Here $c_k$ has to be chosen such that $\|\phi_k\| = 1$, and $\xi_k$ is the $k$th positive root of

$$\cosh(\xi_k)\cos(\xi_k) = -1.$$

The estimates in [**10**, p. 80], with $\omega_k = \xi_k^2$, and with $\phi_k$ replaced by $\sqrt{2}\phi_k$, and $\tilde{P}R^{-1}D\phi_k$ replaced by $\sqrt{2}\xi_k^2\tilde{\phi}_k$, give the asymptotic behavior for $k \to \infty$:

$$\xi_k \sim (k - \frac{1}{2})\pi, \qquad c_k \sim 1, \qquad \beta_k \sim 1, \qquad |\tilde{P}R^{-1}D\phi_k| \sim 2\xi_k^3.$$

Theorem 3.4 says that $w_j \in \operatorname{dom}(L^\alpha)$ if and only if

$$\infty > \sum_{k=1}^{\infty} (\xi_k^4)^{2\alpha-2}\xi_k^6 \sim \sum_{k=1}^{\infty} k^{8\alpha-2}.$$

This is true if and only if $\alpha < 1/8$. $\square$

**4.4. Proof of Theorems 3.5 and 3.6.** The proofs of these theorems requires some lengthy linear algebra, which we defer to Section 5.

*Proof of Theorem* 3.5. Let $v_0 \in Y$ and $\sigma_0$ in $X$ be given. For all $J \subset N$ let $\hat{v}_J : \mathcal{U} \to Y$ and $\hat{\sigma}_J : \mathcal{U} \to X$ solve System 3.1 subject to the boundary condition (3.2). We put $\hat{p}_J(s) = P\hat{\sigma}_J(s)$ and $\hat{q}_J(s) = \tilde{P}\hat{v}_J(s)$.

We start from the formulas

$$\hat{v}(s) = \sum_{J\subset N} \lambda_J(s)\hat{v}_J(s)$$
$$\hat{\sigma}(s) = \sum_{J\subset N} \lambda_J(s)\hat{\sigma}_J(s)$$
$$\sum_{J\subset N} \lambda_J(s) = 1.$$

Since for fixed $s$ the vectors $\hat{v}(s)$ and $\hat{\sigma}(s)$ are linear combinations of solutions to the inhomogeneous linear System 3.1, and since the coefficients of the linear combination sum up to 1, $\hat{v}$ and $\hat{\sigma}$ themselves are solutions to System 3.1.

We have only to check the boundary condition (3.4):

$$P\hat{\sigma}(s) = -K\tilde{P}\hat{v}(s), \tag{4.9}$$

i.e.,

$$\sum_{J\subset N} \lambda_J(s)\hat{p}_J(s) = -K\sum_{J\subset N} \lambda_J(s)\hat{q}_J(s).$$

Our knowledge about the vectors $\hat{p}_J(s)$ and $\hat{q}_J(s)$ includes the boundary conditions (3.2). Moreover, for any two subsets $J, L \subset N$ the differences $\hat{v}_J - \hat{v}_L$ and $\hat{\sigma}_J - \hat{\sigma}_L$ satisfy the homogeneous System 3.1, where $v_0$ and $\sigma_0$ are replaced by 0. Therefore their boundary data are linked by the open loop transfer matrix:

$$(\hat{q}_J - \hat{q}_L) = S(s)(\hat{p}_J - \hat{p}_L).$$

Summing up, the vectors $\hat{p}_J$ and $\hat{q}_J$ fit exactly into Hypothesis 5.3.

We can therefore take the functions $\lambda_J$ from Proposition 5.1 to guarantee (4.9). All estimates claimed in Theorem 3.5 are given in Proposition 5.1. An explicit formula for the coefficients is given in Proposition 5.2. □

*Proof of Theorem* 3.6. For $\omega > 0$, let $V_\omega$ be the algebra of functions $f : [0,\infty) \to \mathbf{C}$ such that $e^{-\omega t}f$ is integrable, with the convolution as product and with the delta distribution $\delta$ added formally as a unit element. We define the half-plane $\mathcal{U}_\omega = \{s \in \mathbf{C} \mid \Re(s) \geq \omega\}$. Then the maximal ideal space of $V_\omega$ is $\mathcal{U}_\omega \cup \{\infty\}$ [**11**, Section 17], and the Gelfand transform of an element is simply its Laplace transform.

Theorem 3.2 states that the entries of the transfer matrix are Laplace transforms of functions $h_{ij} \in V_\omega$ for sufficiently large $\omega$. Notice that the Gelfand transform at $\infty$ satisfies $S_{ij}(\infty) = 0$, since the impulse response $h_{ij}$ is a function and does not contain a $\delta$ measure.

For sufficiently large $\omega$ the half-plane $\mathcal{U}_\omega$ is contained in the sector $\Sigma$. Therefore $\mathcal{U}_\omega$ satisfies Hypothesis 2.4, and by Theorem 3.2 the argument of $\langle z, S(s)z\rangle$ is bounded away from $\pi$ uniformly for $z \in \mathbf{C}^n \setminus \{0\}$ and $s \in \mathcal{U}_\omega$. Of course, $\langle z, K^{-1}z\rangle > 0$ for all $z \neq 0$. Therefore there exists some constant $M_8$ such that

$$\begin{aligned} |\langle z, (S(s) + K^{-1})z\rangle| &= |\langle z, S(s)z\rangle + \langle z, K^{-1}z\rangle| \\ &> M_8\langle z, K^{-1}z\rangle. \end{aligned} \tag{4.10}$$

The same estimate holds for $s = \infty$ if we take $M_8 < 1$.

Let $\mathcal{M} = \{T \in \mathbf{C}^{n\times n} \mid (4.10) \text{ holds for } T\}$. Then $\mathcal{M}$ is open and because of (4.10) the determinant $\det(T + K^{-1})$ is bounded away from 0 uniformly for $T \in \mathcal{M}$. The transfer matrix $S(s)$ takes its values in $\mathcal{M}$ whenever $s \in \mathcal{U}_\omega \cup \{\infty\}$. The coefficients $\lambda_J(s)$ are obtained from the entries of $S(s)$ via Proposition 5.2 by a rational function, which is analytic whenever the denominator $\det(S(s) + K^{-1})$ is nonzero. In particular it is analytic if $S(s) \in \mathcal{M}$. Therefore $\lambda_J$ is a locally analytic function of the entries $S_{ij}$. We infer from [**11**, Section 13, Theorem 1] that $\lambda_J$ is the Gelfand transform (i.e., the Laplace transform) of an element $\nu_J\delta + \mu_J \in V_\omega$.

From $\hat{v}(s) = \sum_{J\subset N} \lambda_J(s)\hat{v}_J(s)$ and $\hat{\sigma}(s) = \sum_{J\subset N} \lambda_J(s)\hat{\sigma}_J(s)$ we infer now by the convolution theorem for Laplace transforms

$$\begin{aligned} v(t) &= \sum_{J\subset N} \left(\nu_J v_J(t) + \int_0^t \mu_J(t-s)v_J(s)\,ds\right), \\ \sigma(t) &= \sum_{J\subset N} \left(\nu_J \sigma_J(t) + \int_0^t \mu_J(t-s)\sigma_J(s)\,ds\right). \end{aligned}$$

(More information on the use of local analyticity to determine integrability of solutions to integral equations can be found in [**15**].) □

**5. Matrix lemmas.** This section contains the linear algebra part of the proof of Theorem 3.5. We sum up what we need to know about the transfer matrix from Theorem 3.1:

**Hypothesis 5.1.** *Let $\mathcal{U} \subset \mathbf{C}$ be a domain (not necessarily bounded). For $s \in \mathcal{U}$, $S(s)$ is a complex $n \times n$-matrix, depending analytically on $s$, with the following properties*:

$$\langle z, S(s)z\rangle \notin (-\infty, 0] \quad \text{for all } s \in \mathcal{U} \quad \text{and all } z \in \mathbf{C}^n \setminus \{0\}. \tag{5.1}$$

*In particular, $S(s)$ is nonsingular. We define*

$$\theta_1(s) = \sup\{|\arg(\langle z, S(s)z\rangle)| \mid z \in \mathbf{C}^n \setminus \{0\}\} < \pi.$$

*If $\mathcal{U}_1 \subset \mathcal{U}$ satisfies Hypothesis* 2.4, *then $\theta_1(s) \leq \max\{\delta, \theta\}$ for all $s \in \mathcal{U}_1$. If $s \in (0, \infty)$, then $S(s)$ is self-adjoint and positive definite.*

**Hypothesis 5.2.** *$k_1, \ldots, k_n$ are positive real numbers.*

We will consider vectors $\hat{p}_J$ and $\hat{q}_J$ with the following properties:

**Hypothesis 5.3.** *For each $s \in \mathcal{U}$ and each subset $J \subset N := \{1, \ldots, n\}$, the vectors*

$$\hat{p}_J(s) = \begin{pmatrix} \hat{p}_{J1}(s) \\ \vdots \\ \hat{p}_{Jn}(s) \end{pmatrix}, \quad \text{and} \quad \hat{q}_J(s) = \begin{pmatrix} \hat{q}_{J1}(s) \\ \vdots \\ \hat{q}_{Jn}(s) \end{pmatrix}$$

*satisfy*

$$\hat{p}_{Ji} = 0 \quad \text{for } i \notin J, \tag{5.2}$$

$$\hat{q}_{Ji} = 0 \quad \text{for } i \in J, \tag{5.3}$$

$$\hat{q}_J - \hat{q}_L = S(s)(\hat{p}_J - \hat{p}_L) \quad \text{for } J, L \subset N. \tag{5.4}$$

The proof of Theorem 3.5 depends on the following

**Proposition 5.1.** *For each subset $J \subset N = \{1, \cdots, n\}$ there exists a function $\lambda_J(s)$, analytic in $\mathcal{U}$, such that for any $\hat{p}_J(s)$ and $\hat{q}_J(s)$ satisfying Hypothesis* 5.3 *and for all $s \in \mathcal{U}$ and all $i \in N$ the following identity holds*:

$$\sum_{J \subset N} \lambda_J(s) \hat{p}_{Ji}(s) = -k_i \sum_{J \subset N} \lambda_J(s) \hat{q}_{Ji}(s). \tag{5.5}$$

The functions $\lambda_J$ have the following properties:

$$\lambda_J(s) \in \mathbf{R} \quad \text{and} \quad \lambda_J(s) > 0 \quad \text{for all } s \in \mathbf{R} \tag{5.6}$$

$$\sum_{J \subset N} \lambda_J(s) = 1 \quad \text{for all } s \in \mathcal{U}. \tag{5.7}$$

Moreover there exists a constant $M_2(\theta_1)$ depending only on $\theta_1(s)$ such that

$$|\lambda_J(s)| \le M_2(\theta_1(s)) \quad \text{for all } s \in \mathcal{U}. \tag{5.8}$$

In particular, $M_2(\pi/2) = 1$.

In order to give an explicit formula for $\lambda_J$ we define:

**Definition 5.1.** For $J \subset N$ let $E_J$ be the *idempotent diagonal* $n \times n$-matrix with coefficients

$$c_{ij} = \begin{cases} 1 & \text{if } i = j \in J \\ 0 & \text{else.} \end{cases}$$

$E_\varnothing = 0$. The identity matrix $E_N$ will usually be denoted by 1.

**Definition 5.2.** For $s \in \mathcal{U}$, $J \subset N$ we define

$$\begin{aligned} d_J(s) &= \det(E_{N \setminus J} + E_J S(s) E_J), \\ Y_J(s) &= E_J(E_{N \setminus J} + E_J S(s) E_J)^{-1}, \\ X_J(s) &= 1 - S(s) Y_J(s), \\ Y(s) &= (S(s) + 1)^{-1}. \end{aligned}$$

**Proposition 5.2.** *Let the conditions of Proposition* 5.1 *be satisfied. Then the coefficients* $\lambda_J$ *introduced in Proposition* 5.1 *are given by*

$$\lambda_J(s) = \frac{d_J(s)}{\det(S(s) + K^{-1})} \prod_{j \notin J} k_j^{-1}.$$

This section is devoted to proving Propositions 5.1 and 5.2 by a sequence of lemmas.

**Proposition 5.3.** *Without loss of generality we may assume that* $K$ *is the unit matrix.*

*Proof.* Thanks to an anonymous referee for this simple but efficient observation. In order to justify it, we replace $S$ by $K^{1/2}S(s)K^{1/2}$, $P$ by $K^{-1/2}P$, and $\tilde{P}$ by $K^{1/2}\tilde{P}$. Some straightforward calculations show that the statements of the propositions and hypotheses now reduce precisely to the case $K = 1$. □

**Definition 5.3.** For simplicity we introduce the following notation: The estimate

$$A(s) \leq M_\theta B(s)$$

means that the estimate holds for $s \in \mathcal{U}$ with some constant $M_\theta(s)$ depending only on $\theta_1(s)$. Moreover $M_\theta = 1$ if $\theta_1(s) \leq \pi/2$.

**Lemma 5.1.** *Let* $s \in \mathcal{U}$.

(i) $(E_{N\setminus J} + E_J S(s) E_J)$ *is nonsingular. Thus* $d_J(s) \neq 0$ *and* $Y_J(s)$ *exists.*

(ii) *The block matrix*

$$\begin{pmatrix} 1 & -S(s) \\ E_J & E_{N\setminus J} \end{pmatrix}$$

*is nonsingular with inverse*

$$\begin{pmatrix} X_J(s) & U_J(s) \\ -Y_J(s) & V_J(s) \end{pmatrix},$$

*with suitable matrices* $U_J(s)$ *and* $V_J(s)$.

(iii) *The block matrix*

$$\begin{pmatrix} 1 & -S(s) \\ 1 & 1 \end{pmatrix}$$

*is nonsingular with inverse*

$$\begin{pmatrix} Y(s) & 1-Y(s) \\ -Y(s) & Y(s) \end{pmatrix}.$$

*Proof. Part* (i). We will omit the argument $(s)$ throughout the proof. Suppose that $(E_{N\setminus J} + E_J S E_J)z = 0$. Then

$$E_{N\setminus J} z = E_{N\setminus J}(E_{N\setminus J} + E_J S E_J)z = 0$$

and

$$\langle E_J z, S E_J z\rangle = \langle z, E_J S E_J z\rangle = \langle z, E_J(E_{N\setminus J} + E_J S E_J)z\rangle = 0,$$

so that by assumption (5.1) $E_J z = 0$. Thus $z = E_{N\setminus J} z + E_J z = 0$. This proves (i).

*Part* (ii). To show that $\begin{pmatrix} 1 & -S \\ E_J & E_{N\setminus J} \end{pmatrix}$ is invertible we have to show that $E_{N\setminus J} + E_J S$ is invertible (see [**14**, Section 0.8.5]). Suppose $(E_{N\setminus J} + E_J S)z = 0$. Then $E_{N\setminus J} z = E_{N\setminus J}(E_{N\setminus J} + E_J S)z = 0$, thus $z = E_J z$. This implies

$$(E_{N\setminus J} + E_J S E_J)z = (E_{N\setminus J} + E_J S)z = 0,$$

and by part (i) we infer that $z = 0$.

To show that the first columns of the inverse consist of $\begin{pmatrix} X_J \\ -Y_J \end{pmatrix}$, we need only to check the product

$$\begin{pmatrix} 1 & -S \\ E_J & E_{N\setminus J} \end{pmatrix} \begin{pmatrix} X_J \\ -Y_J \end{pmatrix} = \begin{pmatrix} 1 - SY_J + SY_J \\ E_J - E_J S Y_J - E_{N\setminus J} Y_J \end{pmatrix}.$$

The first component is evidently 1. The second component is

$$E_J - E_J S Y_J - E_{N\setminus J} Y_J = 1 - (E_{N\setminus J} + E_J S E_J)(E_{N\setminus J} + Y_J) = 0.$$

Thus

$$\begin{pmatrix} 1 & -S \\ E_J & E_{N\setminus J} \end{pmatrix} \begin{pmatrix} X_J \\ -Y_J \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

*Part* (iii). The last assertion is checked by direct computation. □

**Lemma 5.2.** *For some* $s \in \mathcal{U}$ *let* $\lambda_J \in \mathbf{C}$ *satisfy*

$$\sum_{J\subset N} \lambda_J = 1 \quad \text{and} \quad \sum_{J\subset N} \lambda_J Y_J(s) = Y(s).$$

*Let* $\hat{p}_J(s)$ *and* $\hat{q}_J(s)$ *satisfy Hypothesis* 5.3. *Then the numbers* $\lambda_J$ *satisfy* (5.5) *in Proposition* 5.1.

*Proof.* Hypothesis 5.3 can be rewritten in matrix form: $\mathcal{M}\mathcal{P} = 0$ with

$$\mathcal{M} = \begin{pmatrix} 1 & -S & \cdots & 0 & 0 & \cdots & -1 & S \\ E_N & E_\varnothing & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -S & \cdots & -1 & S \\ 0 & 0 & \cdots & E_J & E_{N\setminus J} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & E_\varnothing & E_N \end{pmatrix},$$

$$\mathcal{P} = \begin{pmatrix} \hat{q}_N \\ \hat{p}_N \\ \vdots \\ \hat{q}_J \\ \hat{p}_J \\ \vdots \\ \hat{q}_\varnothing \\ \hat{p}_\varnothing \end{pmatrix}.$$

Equation (5.5) can be rewritten in the form $\mathcal{H}\mathcal{P} = 0$ with

$$\mathcal{H} = (\,\lambda_N 1 \quad \lambda_N 1 \quad \cdots \quad \lambda_J 1 \quad \lambda_J 1 \quad \cdots \quad \lambda_\varnothing 1 \quad \lambda_\varnothing 1\,).$$

Thus, we need to show that the kernels satisfy $\ker(\mathcal{M}) \subset \ker(\mathcal{H})$, or, equivalently, that the block row $\mathcal{H}$ is linearly dependent on the rows of $\mathcal{M}$.

We perform row reduction in $\binom{\mathcal{M}}{\mathcal{H}}$, utilizing Part (ii) from Lemma 5.1, and obtain the block matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & -X_N & X_N S \\ 0 & 1 & \cdots & 0 & 0 & \cdots & Y_N & -Y_N S \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & -X_J & X_J S \\ 0 & 0 & \cdots & 0 & 1 & \cdots & Y_J & -Y_J S \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & R & 0 \end{pmatrix}$$

with

$$R = \lambda_0 1 + \sum_{J \neq \varnothing} \lambda_J (X_J - Y_J) = \sum_{J \subset N} \lambda_J \,(\,1 \quad 1\,) \begin{pmatrix} X_J \\ -Y_J \end{pmatrix}.$$

Therefore the last row depends on the upper rows if and only if $R = 0$. Since by assumption (5.7) holds and

$$(\,1 \quad -S\,) \begin{pmatrix} X_J \\ -Y_J \end{pmatrix} = 1,$$

this is equivalent to

$$\sum_{J \subset N} \lambda_J \begin{pmatrix} 1 & -S \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X_J \\ -Y_J \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Multiplying by the inverse we obtain the equivalent formulation

$$\sum_{J \subset N} \lambda_J \begin{pmatrix} X_J \\ -Y_J \end{pmatrix} = \begin{pmatrix} Y \\ -Y \end{pmatrix}.$$

Finally, since $X_J = 1 - SY_J$ and $Y = 1 - SY$, and again by (5.7), this is equivalent to $\sum_{J\subset N} \lambda_J Y_J = Y$. ◻

**Lemma 5.3.** *Let* $s \in \mathcal{U}$. *Let* $S(s)$ *satisfy Hypothesis* 5.1. *Let* $d_J, Y_J$ *be defined according to Definition* 5.2. *Then the following assertions hold*:

$$\sum_{J\subset N} d_J(s) = \det(S(s)+1), \tag{5.9}$$

$$|\det(S(s))| \le M_\theta |\det(S(s)+1)|, \tag{5.10}$$

$$|d_J(s)| \le M_\theta |\det(S(s)+1)|. \tag{5.11}$$

*Proof.* Equation (5.9) is a special case of [**21**, formula (23)].

Let $\kappa_1, \dots, \kappa_n$ be the eigenvalues of $S(s)$, with eigenvectors $x_1, \dots, x_n$. Since $|\arg(\langle x_i, S(s)x_i\rangle)| \le \theta_1$ we infer that $|\arg(\kappa_i)| \le \theta_1$. This implies that $|\kappa_i + 1| \ge \sin(\theta_1)|\kappa_i|$. Therefore

$$\begin{aligned} |\det(S(s)+1)| &= \prod_{i=1}^n |\kappa_i + 1| \ge \sin(\theta_1)^n \prod_{i=1}^n |\kappa_i| \\ &= \sin(\theta_1)^n |\det S(s)|. \end{aligned}$$

This proves (5.10).

We rearrange the indices such that $J = \{1, \dots, m\}$ and write $S+1$ as a block matrix:

$$S + 1 = \begin{pmatrix} S_{11}+1 & S_{12} \\ S_{21} & S_{22}+1 \end{pmatrix}.$$

Then $d_J = \det(S_{11})$. From (5.10) we infer that $|\det S_{11}| \le M(\theta)$ $|\det(S_{11}+1)|$. Using ([**14**, Section 0.8.5]) we see that

$$\begin{aligned} |\det(S+1)| &= |\det(S_{11}+1)\det[(S_{22}+1) - S_{21}(S_{11}+1)^{-1}S_{12}]| \\ &\ge \frac{d_J}{M(\theta)} |\det[(S_{22}+1) - S_{21}(S_{11}+1)^{-1}S_{12}]|. \end{aligned}$$

Therefore we have to show that

$$\frac{1}{|\det[(S_{22}+1) - S_{21}(S_{11}+1)^{-1}S_{12}]|} \le M(\theta).$$

A straightforward computation shows for $x \in \mathbf{C}^{n-m}$

$$\langle x, [(S_{22}+1) - S_{21}(S_{11}+1)^{-1}S_{12}]x\rangle = \|x\|^2 + \|(S_{11}+1)^{-1}S_{12}x\|^2 + \xi$$

with

$$\xi = \left\langle \begin{pmatrix} -(S_{11}+1)^{-1}S_{12}x \\ x \end{pmatrix}, S\begin{pmatrix} -(S_{11}+1)^{-1}S_{12}x \\ x \end{pmatrix} \right\rangle,$$

so that $|\arg(\xi)| \le \theta_1$. Therefore

$$|\langle x, [(S_{22}+1) - S_{21}(S_{11}+1)^{-1}S_{12}]x\rangle| \ge \sin(\theta_1)\|x\|^2.$$

As a consequence, we infer that all eigenvalues $\gamma_i$ of $[(S_{22}+1)-S_{21}(S_{11}+1)^{-1}S_{12}]$ satisfy $|\gamma_i| \ge \sin(\theta_1)$. Hence

$$|\det[(S_{22}+1) - S_{21}(S_{11}+1)^{-1}S_{12}]| \ge \sin(\theta_1)^n. \qquad \square$$

**Lemma 5.4.** *Let $s \in \mathcal{U}$. Let $S(s)$ satisfy Hypothesis* 5.1, *and let $d_J, Y_J$ be defined according to Definition* 5.2. *Then*

$$\sum_{J\subset N} d_J(s)Y_J(s) = \det(S(s)+1)Y. \tag{5.12}$$

*Proof.* Let $M \subset N$. We show by induction with respect to $m = \#M$:

$$\operatorname{adj}(S+E_M) = \det(S+E_M)(S+E_M)^{-1} = \sum_{J\supset N\setminus M} d_J Y_J. \tag{5.13}$$

Here $\operatorname{adj}(A)$ denotes the adjugate matrix to $A$. For $M = \varnothing$, Equation (5.13) is trivial. Suppose (5.13) holds for $\#M = m$. Let $L \subset N$ be such that $\#L = m+1$. Without loss of generality we assume that $1 \in L$. Put $M = L \setminus \{1\}$.

Separating the first row and column we rewrite

$$(S+E_L) = \begin{pmatrix} \sigma+1 & c^T \\ b & \tilde{S}+\tilde{E}_M \end{pmatrix},$$

$$(S+E_M) = \begin{pmatrix} \sigma & c^T \\ b & \tilde S + \tilde E_M \end{pmatrix}.$$

A direct computation shows that

$$\begin{aligned} &(S+E_M)^{-1} \\ &= \begin{pmatrix} \alpha^{-1} & -\alpha^{-1} c^T (\tilde S + \tilde E_M)^{-1} \\ -\alpha^{-1}(\tilde S + \tilde E_M)^{-1} b & (\tilde S + \tilde E_M)^{-1} - \alpha^{-1}(\tilde S + \tilde E_M)^{-1} b c^T (\tilde S + \tilde E_M)^{-1} \end{pmatrix} \end{aligned}$$

with $\alpha = \sigma - c^T(\tilde S + \tilde E_M)^{-1} b$. From [**14**, Section 0.8.5] we infer that $\det(S+E_M) = \alpha \det(\tilde S + \tilde E_M)$. Therefore,

$$\begin{aligned} &\operatorname{adj}(S+E_M) \\ &= \begin{pmatrix} \det(\tilde S + \tilde E_M) & -c^T \operatorname{adj}(\tilde S + \tilde E_M) \\ \operatorname{adj}(\tilde S + \tilde E_M) b & \alpha \operatorname{adj}(\tilde S + \tilde E_M) - \operatorname{adj}(\tilde S + \tilde E_M) b c^T (\tilde S + \tilde E_M)^{-1} \end{pmatrix} \end{aligned}$$

The same computation for $L$ instead of $M$ yields

$$\begin{aligned} &\operatorname{adj}(S+E_L) \\ &= \begin{pmatrix} \det(\tilde S + \tilde E_M) & -c^T \operatorname{adj}(\tilde S + \tilde E_M) \\ \operatorname{adj}(\tilde S + \tilde E_M) b & (\alpha+1)\operatorname{adj}(\tilde S + \tilde E_M) - \operatorname{adj}(\tilde S + \tilde E_M) b c^T (\tilde S + \tilde E_M)^{-1} \end{pmatrix} \end{aligned}$$

Therefore

$$\operatorname{adj}(S+E_L) = \operatorname{adj}(S+E_M) + \begin{pmatrix} 0 & 0 \\ 0 & \operatorname{adj}(\tilde S + \tilde E_M) \end{pmatrix}.$$

The induction hypothesis may be applied to the adjugates of $S+E_M$ and $\tilde S + \tilde E_M$. Therefore

$$\begin{aligned} \operatorname{adj}(S+E_L) &= \sum_{J \supset N\setminus M} d_J Y_J + \sum_{\tilde J \subset N\setminus\{1\},\, \tilde J \supset N\setminus\{1\}\setminus M} \begin{pmatrix} 0 & 0 \\ 0 & d_J \tilde Y_{\tilde J} \end{pmatrix} \\ &= \sum_{J \supset N\setminus L,\, 1\in J} d_J Y_J + \sum_{J \supset N\setminus L,\, 1\notin J} d_J Y_J \\ &= \sum_{J\supset N\setminus L} d_J Y_J. \qquad \square \end{aligned}$$

We are now in the position to give the

*Proof of Propositions* 5.1 *and* 5.2. We put

$$\lambda_J(s) = \frac{1}{\det(S(s)+1)} d_J(s).$$

Evidently $\lambda$ depends analytically on $s \in \mathcal{U}$. By Lemma 5.4 we infer (5.7) and (5.12), which in turn implies (5.5) by Lemma 5.2. Equation (5.11) implies (5.8). Condition (5.1) carries over to $E_{N\setminus J} + E_J S E_J$; therefore, all eigenvalues of $E_{N\setminus J} + E_J S E_J$ have positive real part. If $s \in \mathbf{R}$, the matrix $E_{N\setminus J} + E_J S(s) E_J$ is self-adjoint. Therefore the determinant $d_J$ is positive. This implies (5.6). □

**Acknowledgment.** The authors wish to thank an anonymous referee for his helpful comments which lead to a shorter and more readable exposition of the linear algebra used in this paper.

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