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STABILITY PROBLEMS OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH ABSTRACT VOLTERRA OPERATOR

YIZENG LI

Introduction. In this paper we study stability problems for the functional differential equations involving abstract Volterra operators under second kind initial value

$$\begin{cases} \dot{x}(t) = (Vx)(t), & t > t_0, \\ x(t) = \phi(t), & t \in [0, t_0), \\ x(t_0) = x^0 \in \mathbf{R}^n, \end{cases} \tag{1.1}$$

where V is a continuous Volterra operator acting on $L^2_{\text{loc}}([0,\infty),\mathbf{R}^n)$, with $(V\theta)(t) \equiv \theta \in \mathbf{R}^n$, and $\phi \in L^2([0,t_0),\mathbf{R}^n)$, where θ is used to denote both the zero function and the zero vector throughout the article.

We first give the definitions of stability for the trivial solution (or zero solution, or equilibrium) of the system (1.1). Although there are many kinds of stability to be discussed, among them we emphasize five main stability concepts. They are *stability, uniform stability, asymptotic stability, uniformly asymptotic stability*, and *exponentially asymptotic stability*.

Then we shall present the necessary and sufficient conditions for the stabilities with regard to the trivial solution $x = \theta \in \mathbf{R}^n$ of the linear system

$$\begin{cases} \dot{x}(t) = (Lx)(t), & t > t_0, \\ x(t) = \phi(t), & t \in [0, t_0), \\ x(t_0) = x^0 \in \mathbf{R}^n, \end{cases} \tag{1.2}$$

where L is a linear continuous Volterra operator acting on $L^2_{\text{loc}}([0,\infty),\mathbf{R}^n)$ with $(L\theta)(t) \equiv \theta \in \mathbf{R}^n$, and $\phi \in L^2([0,t_0),\mathbf{R}^n)$. These are new contributions.

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Finally, we shall discuss some asymptotic behavior of the solutions to the nonhomogeneous system

$$
(1.3)\qquad \begin{cases} \dot{x}(t) = (Lx)(t) + (Fx)(t), & 0 < t_0 < t < T \le \infty, \\ x(t) = \phi(t), & 0 \le t \le t_0, \\ x(t_0) = x^0 \in \mathbf{R}^n, \end{cases}
$$

where L is a linear continuous Volterra operator acting on $L^2_{\text{loc}}([0,\infty), \mathbf{R}^n)$ with $(L\theta)(t) \equiv \theta \in \mathbf{R}^n$, $\phi \in L^2([0,t_0),\mathbf{R}^n)$. The nonlinear operator F has certain properties, the case when it is of Niemytzki type being the most useful.

Definitions. The definitions which we mention here, basically, can be found in many books, for instance, R. Driver [**6**], Wolfgang Hahn [**7**], T.A. Burton [**1**], C. Corduneanu [**3**], but not for equations with abstract Volterra operators.

Let $x(t;t_0,x^0,\phi)$ be a nonzero solution of (1.1) and $x(t) = \theta$ be the zero solution of it.

Definition 1 (stable). The zero solution $x(t) = \theta$ of (1.1) will be called *stable* if, for any $t_0 > 0$, every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$, such that $|x(t;t_0,x^0,\phi)| < \varepsilon$, for $t \ge t_0$, provided $|x^0| < \delta$, and $|\phi|_{L^2([0,t_0),\mathbf{R}^n)} < \delta$.

Definition 2 (uniformly stable). The zero solution $x(t) = \theta$ of (1.1) will be called *uniformly stable* if the number δ in Definition 1 can be chosen independently of t_0, that is, $\delta \equiv \delta(\varepsilon)$, a function of ε only.

Definition 3 (asymptotically stable). The zero solution $x(t) = \theta$ of (1.1) is said to be *asymptotically stable* if it is stable in the sense of Definition 1, and for each $t_0 > 0$, there exists $\gamma(t_0) > 0$ such that

$$
(1.4)\qquad \lim_{t\to\infty} |x(t;t_0,x^0,\phi)| = 0,
$$

for all $x(t;t_0,x^0,\phi)$ with $|x^0| < \gamma(t_0)$ and $|\phi|_{L^2([0,t_0),\mathbf{R}^n)} < \gamma(t_0)$.

In other words, the zero solution $x = \theta$ is asymptotically stable if it is stable, and for each $t_0 > 0$ there exists $\gamma(t_0) > 0$ and for any $\varepsilon > 0$ there exists $T(\varepsilon, t_0) > 0$ such that $|x(t; t_0, x^0, \phi)| < \varepsilon$, for $t \geq t_0 + T(\varepsilon, t_0)$, provided $|x^0| < \gamma(t_0)$ and $|\phi|_{L^2([0,t_0),\mathbf{R}^n)} < \gamma(t_0)$.

Definition 4 (uniformly asymptotically stable). The zero solution $x(t) = \theta$ of (1.1) will be called *uniformly asymptotically stable* if it is uniformly stable in the sense of Definition 2, and if there exists $\gamma > 0$, and for any $\varepsilon > 0$, there exists $T(\varepsilon) > 0$ such that

$$|x(t; t_0, x^0, \phi)| < \varepsilon, \quad \text{for } t \geq t_0 + T(\varepsilon), \tag{1.5}$$

when $|x^0| < \gamma$ and $|\phi|_{L^2([0,t_0),\mathbf{R}^n)} < \gamma$.

Definition 5 (exponentially asymptotically stable). The zero solution $x(t) = \theta$ of (1.1) is said to be *exponentially asymptotically stable* if there exists $N > 0$, $\alpha > 0$ and $\gamma > 0$, such that

$$|x(t; t_0, x^0, \phi)| \leq N(|x^0| + |\phi|_{L^2([0,t_0),\mathbf{R}^n)}) \exp(-\alpha(t - t_0)), \tag{1.6}$$

for $t \geq t_0$, provided $|x^0| < \gamma$ and $|\phi|_{L^2([0,t_0),\mathbf{R}^n)} < \gamma$.

Remark 1.1. It is not difficult to see that if $|x^0|$ and $|\phi|_{L^2([0,t_0),\mathbf{R}^n)}$ are small enough, respectively, then $(|x^0| + |\phi|_{L^2([0,t_0),\mathbf{R}^n)})$ will be sufficiently small, and vice versa. Hence, we may use $(|x^0| + |\phi|_{L^2([0,t_0),\mathbf{R}^n)}) < \gamma$, or δ, instead of $|x^0| < \gamma$, or δ, and $|\phi|_{L^2([0,t_0),\mathbf{R}^n)} < \gamma$, or δ, in the above definitions.

Remark 1.2. It is obvious that the exponential asymptotic stability implies all other kinds of stabilities mentioned above.

It has been proved in [**9**] that the solution of the system (1.2) has the form

$$x(t; t_0, x^0, \phi) = X(t, t_0)x^0 + \int_0^{t_0} \tilde{X}(t, s; t_0)\phi(s)\, ds, \quad \text{for } t \geq t_0, \tag{1.7}$$

where $X(t, t_0)$ and $\tilde{X}(t, s; t_0)$ are square matrices of order n, $\tilde{X}$ depends also on t_0, belongs to $L^2([0, t_0), \mathbf{R}^n)$ and $\tilde{X}(t_0, s; t_0) = \theta$, and the solution $x(t; t_0, x^0, \phi)$ is in $C([t_0, \infty), \mathbf{R}^n)$, or more precisely, $x(t; t_0, x^0, \phi)$ is a locally absolutely continuous function.

Since (1.7) is true for any $x(t_0) = x^0 \in \mathbf{R}^n$ and any $\phi \in L^2([0,t_0),\mathbf{R}^n)$, therefore, if $\phi(t) = \theta$ on $[0,t_0)$, it will be reduced to the form

$$x(t;t_0,x^0,\phi) = x(t;t_0,x^0,\theta) = X(t,t_0)x^0, \quad \text{for } t \ge t_0; \tag{1.8}$$

on the other hand, if $x^0 = \theta$, then (1.7) will be changed to the form

$$x(t;t_0,x^0,\phi) = x(t;t_0,\theta,\phi) = \int_0^{t_0} \tilde{X}(t,s;t_0)\phi(s)\,ds, \quad \text{for } t \ge t_0. \tag{1.9}$$

These facts give us some ideas, namely, instead of discussing the concepts of stability properties for (1.7), we may deal with (1.8) or (1.9), respectively, in some circumstances.

Stability of linear systems. We present the following theorems with regard to the stabilities for the zero solution $x = \theta$ of system (1.2).

Theorem 1.1. *The zero solution $x = \theta$ of system* (1.2) *is stable if and only if*

$$|X(t,t_0)| \le M(t_0), \qquad t \ge t_0, \tag{1.10}$$

and

$$\left(\int_0^{t_0} |\tilde{X}(t,s;t_0)|^2\,ds\right)^{1/2} \le M(t_0), \qquad t \ge t_0, \tag{1.11}$$

for some $M(t_0) > 0$.

Proof. (a). *Sufficiency.* Suppose that (1.10) and (1.11) hold; then for $t_0 > 0$ and each $\varepsilon > 0$, consider (1.7), and by Cauchy's inequality, we obtain

$$\begin{aligned}
|x(t;t_0,x^0,\phi)| &\le |X(t,t_0)x^0| + \left|\int_0^{t_0} \tilde{X}(t,s;t_0)\phi(s)\,ds\right| \\
&\le |X(t,t_0)|\,|x^0| + \left(\int_0^{t_0} |\tilde{X}(t,s;t_0)|^2\,ds\right)^{1/2} |\phi|_{L^2([0,t_0),\mathbf{R}^n)} \\
&\le M(t_0)(|x^0| + |\phi|_{L^2([0,t_0),\mathbf{R}^n)}) \\
&< \varepsilon, \quad \text{for } t \ge t_0,
\end{aligned}$$

provided

$$(|x^0| + |\phi|_{L^2([0,t_0),\mathbf{R}^n)}) < \delta(\varepsilon, t_0) = \frac{\varepsilon}{M(t_0)}.$$

Thus, (1.10) and (1.11) are sufficient for the stability of the zero solution of (1.2).

(b). *Necessity.* In order to prove the necessity of (1.10), we use the formula (1.8), that is, in case the initial function data $\phi(t) = \theta$ on $[0, t_0)$, and the solution of (1.2) has the form (1.8)

$$x(t; t_0, x^0, \theta) = X(t, t_0)x^0, \quad \text{for } t \geq t_0.$$

Now suppose that the zero solution $x = \theta$ of (1.2) is stable, then fix $t_0 > 0$, for $\varepsilon = 1$, there exists $\delta = \delta(1, t_0) = \delta(t_0) > 0$ such that

$$|x(t; t_0, x^0, \theta)| = |X(t, t_0)x^0| < 1, \quad \text{for } t \geq t_0 \geq 0, \tag{1.12}$$

provided $|x^0| \leq \delta(t_0)$. Condition (1.12) is equivalent to

$$|x(t; t_0, x^0, \theta)| = |X(t, t_0)x^0| < [\delta(t_0)]^{-1}, \quad \text{for } t \geq t_0 \geq 0, \tag{1.13}$$

provided $|x^0| \leq 1$, where $[\delta(t_0)]^{-1} = 1/\delta(t_0)$.

If we choose x^0 in (1.13) such that $|x^0| = 1$ and all the coordinates are zero, except that of rank m (or the mth coordinate), for $1 \leq m \leq n$, then we get

$$|\text{col}_m X(t, t_0)| < [\delta(t_0)]^{-1}, \quad \text{for } t \geq t_0 \geq 0. \tag{1.14}$$

The norm of the $n \times n$ matrix $X(t, t_0) = (a_{ij})$, $i, j = 1, 2, \ldots, n$, could be chosen as

(1.15)
$$|X(t, t_0)| = \left(\sum_{i,j=1}^{n} a_{ij}^2 \right)^{1/2} = \left(\sum_{i=1}^{n} a_{i1}^2 + \sum_{i=1}^{n} a_{i2}^2 + \cdots + \sum_{i=1}^{n} a_{in}^2 \right)^{1/2}.$$

Clearly, $(\sum_{i=1}^{n} a_{im}^2) = |\text{col}_m X(t, t_0)|^2 < ([\delta(t_0)]^{-1})^2$, for $m = 1, 2, \ldots, n$. (1.14) and (1.15) will lead to

$$\begin{aligned} |X(t, t_0)| &\leq (n([\delta(t_0)]^{-1})^2)^{1/2} \\ &= \sqrt{n} \cdot [\delta(t_0)]^{-1} \\ &= M(t_0), \quad \text{for } t \geq t_0 \geq 0, \end{aligned} \tag{1.16}$$

where n is the order of the matrix $X(t, t_0)$.

To prove the necessity of (1.11), we may let $x^0 = \theta \in \mathbf{R}^n$, and consider the formula (1.9).

$$x(t; t_0, \theta, \phi) = \int_0^{t_0} \tilde{X}(t, s; t_0)\phi(s)\, ds, \quad \text{for } t \geq t_0.$$

For a fixed $t > t_0 \geq 0$, let us define a functional on $L^2([0, t_0), \mathbf{R}^n)$ as:

$$L_t(\phi) = \int_0^{t_0} \tilde{X}(t, s; t_0)\phi(s)\, ds, \quad \text{for any } \phi \in L^2([0, t_0), \mathbf{R}^n). \tag{1.17}$$

Obviously, $L_t(\phi)$ is a linear functional with respect to ϕ. Moreover, $L_t(\phi)$ will be bounded if the zero solution $x = \theta$ of (1.2) is stable. In fact, by definition of stable, for any $t_0 > 0$, for $\varepsilon = 1$, there exists $\delta(1, t_0) = \delta(t_0) > 0$, such that

$$|x(t; t_0, \theta, \phi)| = \left| \int_0^{t_0} \tilde{X}(t, s; t_0)\phi(s)\, ds \right| < 1,$$

as soon as

$$|\phi|_{L^2([0,t_0),\mathbf{R}^n)} \leq \delta(t_0).$$

This condition is equivalent to

$$|x(t; t_0, \theta, \phi)| = \left| \int_0^{t_0} \tilde{X}(t, s; t_0)\phi(s)\, ds \right| \leq [\delta(t_0)]^{-1},$$

provided

$$|\phi|_{L^2([0,t_0),\mathbf{R}^n)} \leq 1.$$

Therefore, for any $\phi \in L^2([0, t_0), \mathbf{R}^n)$, we get

$$\begin{aligned} |L_t(\phi)| &= \left| \int_0^{t_0} \tilde{X}(t, s; t_0)\phi(s)\, ds \right| \\ &< [\delta(t_0)]^{-1} \cdot |\phi|_{L^2([0,t_0),\mathbf{R}^n)} < \infty. \end{aligned} \tag{1.18}$$

Thus, $L_t(\phi)$ is a linear, bounded functional on $L^2([0, t_0), \mathbf{R}^n)$.

By Riesz representation theorem, the norm of $L_t(\phi)$ is at most $[\delta(t_0)]^{-1}$, that is,

$$\left(\int_0^{t_0} |\tilde{X}(t,s;t_0)|^2\,ds\right)^{1/2} \leq [\delta(t_0)]^{-1} \leq M(t_0), \tag{1.19}$$

where

$$M(t_0) = \sqrt{n}[\delta(t_0)]^{-1}, \quad \text{for } t > t_0.$$

The inequality (1.19) leads immediately to (1.11)

$$\left(\int_0^{t_0} |\tilde{X}(t,s;t_0)|^2\,ds\right)^{1/2} \leq [\delta(t_0)]^{-1} \leq M(t_0), \quad \text{for } t \geq t_0,$$

since $\tilde{X}(t_0,s;t_0) = \theta$, almost everywhere.

The proof of Theorem 1.1 is then complete (for the original idea of the proof, see [**2, 3**]). □

Theorem 1.2. *The zero solution $x = \theta$ of system* (1.12) *is uniformly stable, if and only if*

$$|X(t,t_0)| \leq M, \quad \textit{for } t \geq t_0, \tag{1.20}$$

and

$$\left(\int_0^{t_0} |\tilde{X}(t,s;t_0)|^2\,ds\right)^{1/2} \leq M, \quad \textit{for } t \geq t_0, \tag{1.21}$$

for some $M > 0$, where M does not depend on t_0.

Proof. The proof of Theorem 1.2 is similar to that of Theorem 1.1, keeping in mind that here, $\delta = \delta(\varepsilon) > 0$ does not depend on t_0. □

Theorem 1.3. *The zero solution $x = \theta$ of system* (1.2) *is asymptotically stable if and only if*

$$\lim_{t\to\infty} |X(t,t_0)| = 0, \tag{1.22}$$

and

$$\lim_{t\to\infty}\left(\int_0^{t_0}|\tilde{X}(t,s;t_0)|^2\,ds\right)^{1/2}=0, \tag{1.23}$$

for any $t_0 \geq 0$.

Proof. (a). *Sufficiency.* Suppose that (1.22) and (1.23) are true; then for any $t_0 \geq 0$, for each $\varepsilon > 0$, there always exists $T(\varepsilon, t_0) > 0$ such that

$$|X(t,t_0)| < \varepsilon,$$

and

$$\left(\int_0^{t_0}|\tilde{X}(t,s;t_0)|^2\,ds\right)^{1/2} < \varepsilon, \quad \text{for } t \geq t_0 + T(\varepsilon, t_0).$$

On the other hand,

$$x(t;t_0,x^0,\theta) = X(t,t_0)x^0, \quad \text{for } t \geq t_0,$$

and

$$x(t;t_0,\theta,\phi) = \int_0^{t_0}\tilde{X}(t,s;t_0)\phi(s)\,ds, \quad \text{for } t \geq t_0,$$

are continuous; therefore, there always exist some $M_1(t_0) > 0$ such that

$$|X(t,t_0)| \leq M_1(t_0),$$

and

$$\left(\int_0^{t_0}|\tilde{X}(t,s;t_0)|^2\,ds\right)^{1/2} \leq M_1(t_0),$$
$$\text{for } t_0 \leq t \leq t_0 + T(\varepsilon, t_0),$$

as soon as $|x^0|$ and $|\phi|_{L^2([0,t_0),\mathbf{R}^n)}$ are bounded.

Let $M(t_0) = \max(\varepsilon, M_1(t_0))$, then

$$|X(t,t_0)| \leq M(t_0),$$

and

$$\left(\int_0^{t_0}|\tilde{X}(t,s;t_0)|^2\,ds\right)^{1/2} \leq M(t_0), \quad \text{for } t \geq t_0,$$

by Theorem 1.1, the zero solution $x = \theta$ of (1.2) is stable.

The estimate

$$\begin{aligned}|x(t;t_0,x^0,\phi)| &\le |X(t,t_0)|\,|x^0| \\ &\quad + \left(\int_0^{t_0} |\tilde{X}(t,s;t_0)|^2\,ds\right)^{1/2} |\phi|_{L^2([0,t_0),\mathbf{R}^n)} \\ &\le \varepsilon(|x^0| + |\phi|_{L^2([0,t_0),\mathbf{R}^n)}), \quad \text{for } t \ge t_0 + T(\varepsilon,t_0),\end{aligned}$$

ensures that

$$\lim_{t\to\infty} |x(t;t_0,x^0,\phi)| = 0,$$

provided

$$(|x^0| + |\phi|_{L^2([0,t_0),\mathbf{R}^n)}) < \gamma(t_0), \quad \text{where } \gamma(t_0) > 0.$$

Thus, the zero solution $x = \theta$ of (1.2) is asymptotically stable.

(b). *Necessity.* In order to prove the necessity of (1.22), we use the formula (1.8), that is, in case the initial function data $\phi(t) = \theta$ on $[0, t_0)$, and the solution of (2.2) has the form (1.8)

$$x(t;t_0,x^0,\theta) = X(t,t_0)x^0, \quad \text{for } t \ge t_0.$$

By Definition 3, for any $t_0 \ge 0$, there exist $\gamma(t_0) > 0$, and to each $\varepsilon > 0$, there is a $T(\varepsilon, t_0) > 0$, such that

$$|x(t;t_0,x^0,\theta)| = |X(t,t_0)x^0| < \varepsilon, \quad \text{for } t \ge t_0 + T(\varepsilon,t_0), \tag{1.24}$$

provided $|x^0| \le \gamma(t_0)$. Condition (1.24) is equivalent to

$$\begin{aligned}|x(t;t_0,x^0,\theta)| = |X(t,t_0)x^0| < [\gamma(t_0)]^{-1}\varepsilon,& \\ \text{for } t \ge t_0 + T(\varepsilon,t_0),&\end{aligned} \tag{1.25}$$

provided $|x^0| \le 1$. From (1.25), we can easily find the inequality (see the proof of the necessity of Theorem 1.1)

$$|X(t,t_0)| < \sqrt{n}[\gamma(t_0)]^{-1}\varepsilon, \tag{1.26}$$

for $t \geq t_0 + T(\sqrt{n}[\gamma(t_0)]^{-1}\varepsilon, t_0) = t_0 + T(\varepsilon_1, t_0)$, where $\varepsilon_1 = \sqrt{n}[\gamma(t_0)]^{-1}\varepsilon$.

Thus, $\lim_{t\to\infty} |X(t, t_0)| = 0$.

For the necessity of (1.23), we let $x^0 = \theta \in \mathbf{R}^n$ and consider (1.9)

$$x(t; t_0, \theta, \phi) = \int_0^{t_0} \tilde{X}(t, s; t_0)\phi(s)\, ds, \quad \text{for } t \geq t_0.$$

If the zero solution $x = \theta$ of (1.2) is asymptotically stable, then it is stable; hence

$$\left(\int_0^{t_0} |\tilde{X}(t, s; t_0)|^2\, ds\right)^{1/2} \leq M(t_0), \quad \text{for } t \geq t_0 \text{ and some } M(t_0) > 0.$$

Moreover, by Definition 3 (asymptotic stability), for any $t_0 > 0$, there exists $\gamma(t_0) > 0$, particularly satisfying $0 < \gamma(t_0) \leq 1$, and to every $\varepsilon > 0$, there corresponds a $T(\varepsilon, t_0) > 0$, such that

$$|x(t; t_0, \theta, \phi)| = \left| \int_0^{t_0} \tilde{X}(t, s; t_0)\phi(s)\, ds \right| < \varepsilon, \quad \text{for } t \geq t_0 + T(\varepsilon, t_0),$$

and all $x(t; t_0, \phi)$ with $|\phi|_{L^2([0,t_0),\mathbf{R}^n)} \leq \gamma(t_0) \leq 1$. Then, for any $\phi \in L^2([0, t_0), \mathbf{R}^n)$, we get the following estimate:

$$\begin{aligned} |x(t; t_0, \theta, \phi)| &= \left| \int_0^{t_0} \tilde{X}(t, s; t_0)\phi(s)\, ds \right| \\ &< \varepsilon |\phi|_{L^2([0,t_0),\mathbf{R}^n)}, \quad \text{for } t \geq t_0 + T(\varepsilon, t_0). \end{aligned}$$

Now, fixing $t \geq t_0 + T(\varepsilon, t_0) > t_0$, the mapping

$$\phi \to \int_0^{t_0} \tilde{X}(t, s; t_0)\phi(s)\, ds$$

is a linear, continuous mapping with respect to ϕ. Indeed, for any $\phi, \psi \in L^2([0, t_0), \mathbf{R}^n)$, we may obtain the following estimate:

$$\begin{aligned} &\left| \int_0^{t_0} \tilde{X}(t, s; t_0)\phi(s)\, ds - \int_0^{t_0} \tilde{X}(t, s; t_0)\psi(s)\, ds \right| \\ &\qquad \leq \left(\int_0^{t_0} |\tilde{X}(t, s; t_0)|^2\, ds \right)^{1/2} |\phi - \psi|_{L^2([0,t_0),\mathbf{R}^n)} \\ &\qquad \leq M(t_0)|\phi - \psi|_{L^2([0,t_0),\mathbf{R}^n)} \\ &\qquad \to 0, \quad \text{as } \phi \to \psi. \end{aligned}$$

The linearity part is obvious. Therefore, by Riesz representation theorem, the norm of the mapping is at most ε, that is,

$$\left(\int_0^{t_0} |\tilde{X}(t,s;t_0)|^2\,ds\right)^{1/2} \le \varepsilon, \quad \text{for } t \ge t_0 + T(\varepsilon, t_0).$$

Thus, $\lim_{t\to\infty}(\int_0^{t_0} |\tilde{X}(t,s;t_0)|^2\,ds)^{1/2} = 0$, is the necessary condition.

The proof of Theorem 1.3 is complete. □

Theorem 1.4. *The zero solution $x = \theta$ of system* (1.2) *is uniformly asymptotically stable if and only if for each $\varepsilon > 0$ there exists $T(\varepsilon) > 0$ such that, for any $t_0 \ge 0$,*

$$|X(t,t_0)| < \varepsilon, \quad \textit{for } t \ge t_0 + T(\varepsilon), \tag{1.27}$$

and

$$\left(\int_0^{t_0} |\tilde{X}(t,s;t_0)|^2\,ds\right)^{1/2} < \varepsilon, \quad \textit{for } t \ge t_0 + T(\varepsilon). \tag{1.28}$$

Proof. The proof of Theorem 1.4 is similar to that of Theorem 1.3, keeping in mind that here, $M(t_0) = M > 0$, $\gamma(t_0) = \gamma > 0$ and $T(\varepsilon, t_0) = T(\varepsilon) > 0$, do not depend on t_0. □

Theorem 1.5. *The zero solution $x = \theta$ of system* (1.2) *is exponentially asymptotically stable if and only if there exists $N > 0$ and $\alpha > 0$ such that*

$$|X(t,t_0)| \le N\exp(-\alpha(t-t_0)), \quad \textit{for } t \ge t_0 > 0, \tag{1.29}$$

and

$$\left(\int_0^{t_0} |\tilde{X}(t,s;t_0)|^2\,ds\right)^{1/2} \le N\exp(-\alpha(t-t_0)), \quad \textit{for } t \ge t_0 > 0. \tag{1.30}$$

Proof. (a). *Sufficiency.* Assume that (1.29) and (1.30) hold; then from (1.7),

$$\begin{aligned}|x(t;t_0,x^0,\phi)| &= \left|X(t,t_0)x^0 + \int_0^{t_0} \tilde{X}(t,s;t_0)\phi(s)\,ds\right| \\ &\le |X(t,t_0)|\,|x^0| \\ &\quad + \left(\int_0^{t_0} |\tilde{X}(t,s;t_0)|^2\,ds\right)^{1/2} |\phi|_{L^2([0,t_0),\mathbf{R}^n)} \\ &\le N\exp(-\alpha(t-t_0))(|x^0| + |\phi|_{L^2([0,t)0),\mathbf{R}^n)}) \\ &\qquad \text{for } t \ge t_0 > 0.\end{aligned}$$

Thus, the zero solution $x = \theta$ of (1.2) is exponentially asymptotically stable, provided the initial data satisfy

$$(|x^0| + |\phi|_{L^2([0,t_0),\mathbf{R}^n)}) \le \gamma, \quad \text{for some } \gamma > 0.$$

(b). *Necessity.* To prove the necessity of condition (1.29), we choose $\phi = \theta$ on $[0,t_0)$ and consider the solution: $x(t;t_0,x^0,\theta) = X(t,t_0)x^0$ for $t \ge t_0$. By Definition 5, there exists $N_1 > 0$, $\alpha_1 > 0$ and $\gamma > 0$, such that

$$\begin{aligned}|x(t;t_0,x^0,\theta)| &= |X(t,t_0)x^0| \\ &\le N_1\exp(-\alpha_1(t-t_0))|x^0|, \\ &\text{for } t \ge t_0 \text{ and } |x^0| \le \gamma.\end{aligned} \tag{1.31}$$

Condition (1.31) is equivalent to

$$\begin{aligned}|x(t;t_0,x^0,\theta)| &= |X(t,t_0)x^0| \\ &\le N_1\exp(-\alpha_1(t-t_0))\gamma^{-1}, \\ &\text{for } t \ge t_0 \text{ and } |x^0| \le 1.\end{aligned} \tag{1.32}$$

Consequently, we obtain that

$$\begin{aligned}|X(t,t_0)| &\le \sqrt{n}\gamma^{-1}N_1\exp(-\alpha_1(t-t_0)) \\ &= \tilde{N}\exp(-\alpha_1(t-t_0)), \quad \text{for } t \ge t_0,\end{aligned}$$

with $\tilde{N} = \sqrt{n}\gamma^{-1} N_1$.

Using the formula (1.9), we can prove the necessity of (1.30). Indeed, the definition of exponentially asymptotically stable shows that there exists $N_2 > 0$, $\alpha_2 > 0$ and $0 < \gamma \le 1$, such that

$$\begin{aligned}|x(t;t_0,\theta,\phi)| &= \left|\int_0^{t_0} \tilde{X}(t,s;t_0)\phi(s)\,ds\right| \\ &\le N_2 \exp(-\alpha(t-t_0))|\phi|_{L^2([0,t_0),\mathbf{R}^n)} \\ &\le N_2 \exp(-\alpha(t-t_0)), \\ &\text{for } t \ge t_0 \text{ and } |\phi|_{L^2([0,t_0),\mathbf{R}^n)} \le 1.\end{aligned}$$

Therefore, for any $\phi \in L^2([0,t_0),\mathbf{R}^n)$, we have

$$\left|\int_0^{t_0} \tilde{X}(t,s;t_0)\phi(s)\,ds\right| \le N_2 \exp(-\alpha(t-t_0))|\phi|_{L^2([0,t_0),\mathbf{R}^n)}.$$

The mapping $\phi \to \int_0^{t_0} \tilde{X}(t,s;t_0)\phi(s)\,ds$ is a linear continuous mapping, for fixed $t \ge t_0$, its norm satisfies the following inequality

$$\left(\int_0^{t_0} |\tilde{X}(t,s;t_0)|^2\,ds\right)^{1/2} \le N_2 \exp(-\alpha(t-t_0)), \quad \text{for } t \ge t_0 > 0.$$

The necessity is proven if we choose $N = \max(\tilde{N}, N_2)$.

Hence the proof of Theorem 1.5 is complete. □

Before we end this paper, let us consider the nonlinear perturbed system

$$(1.3) \qquad \begin{cases} \dot{x}(t) = (Lx)(t) + (Fx)(t) & t > t_0, \\ x(t) = \phi(t) & t \in [0,t_0), \\ x(t_0) = x^0 \in \mathbf{R}^n, \end{cases}$$

where L is a linear continuous Volterra operator acting on $L^2_{\text{loc}}([0,\infty), \mathbf{R}^n)$ with $(L\theta)(t) \equiv \theta \in \mathbf{R}^n$, $\phi \in L^2([0,t_0),\mathbf{R}^n)$, and the operator F has certain properties which will be specified below.

If $(F\theta)(t) \equiv \theta \in \mathbf{R}^n$, then system (1.3) possesses the zero solution. We are interested to know: if the zero solution of linear system (1.2) has a certain stability property, which assumption we should impose on the operator F, such that the zero solution of nonlinear system (1.3) has the same stability property (or a weaker one).

As mentioned earlier, the zero solution of the linear system (1.2) has the form

$$x(t;t_0,x^0,\phi) = X(t,t_0)x^0 + \int_0^{t_0} \tilde{X}(t,s;t_0)\phi(s)\,ds, \quad \text{for } t \geq t_0.$$

Consequently, for a fixed F, the zero solution of (1.3) satisfies the nonlinear equation

$$\begin{aligned} x(t;t_0,x^0,\phi) = X(t,t_0)x^0 &+ \int_0^{t_0} \tilde{X}(t,s;t_0)\phi(s)\,ds \\ &+ \int_{t_0}^{t} X(t,s)(Fx)(s)\,ds, \quad \text{for } t \geq t_0. \end{aligned} \tag{1.33}$$

If we assume that the zero solution of (1.2) is exponentially asymptotically stable, then, by Theorem 1.5, there exist numbers $N > 0$, $\alpha > 0$, such that

$$|X(t,t_0)| \leq N\exp(-\alpha(t-t_0)), \quad \text{for } t \geq t_0 > 0,$$

and

$$\left(\int_0^{t_0} |\tilde{X}(t,s;t_0)|^2\,ds\right)^{1/2} \leq N\exp(-\alpha(t-t_0)), \quad \text{for } t \geq t_0 > 0.$$

Estimating (1.33), we obtain

$$\begin{aligned} |x(t;t_0,x^0,\phi)| \leq{}& |X(t,t_0)|\,|x^0| \\ &+ \left(\int_0^{t_0} |\tilde{X}(t,s;t_0)|^2\,ds\right)^{1/2}\left(\int_0^{t_0} |\phi(s)|^2\,ds\right)^{1/2} \\ &+ \int_{t_0}^{t} |X(t,s)|\,|(Fx)(s)|\,ds \\ \leq{}& N\exp(-\alpha(t-t_0))(|x^0| + |\phi|_{L^2([0,t_0),\mathbf{R}^n)}) \\ &+ N\exp(-\alpha t)\int_{t_0}^{t} \exp(\alpha s)|(Fx)(s)|\,ds. \end{aligned} \tag{1.34}$$

If $(Fx)(t) = F(t, x(t))$, that is, the operator F is a Niemytzki operator, such that $|Fx| \leq r|x|$, where $r > 0$, then (1.34) becomes

$$|x(t; t_0, x^0, \phi)| \leq N \exp(-\alpha(t - t_0))(|x^0| + |\phi|_{L^2([0,t_0),\mathbf{R}^n)}) + Nr \exp(-\alpha t) \int_{t_0}^{t} \exp(\alpha s)|x(s)|\, ds.$$

Multiplying both sides of it by $\exp(\alpha t)$ and letting

$$u(t) = \exp(\alpha t)|x(t; t_0, x^0, \phi)|,$$

we have

$$u(t) \leq N \exp(\alpha t_0)(|x^0| + |\phi|_{L^2([0,t_0),\mathbf{R}^n)}) + Nr \int_{t_0}^{t} u(s)\, ds, \qquad t > t_0.$$

Applying a Gronwall type inequality, we obtain

$$u(t) \leq N \exp(\alpha t_0)(|x^0| + |\phi|_{L^2([0,t_0),\mathbf{R}^n)}) \exp(Nr(t - t_0)), \qquad t > t_0,$$

or

$$|x(t; t_0, x^0, \phi)| \leq N(|x^0| + |\phi|_{L^2([0,t_0),\mathbf{R}^n)}) \exp(-(\alpha - Nr)(t - t_0)), \quad t > t_0.$$

Clearly, if $\alpha - Nr > 0$ or $r < \alpha/N$, then the zero solution of nonlinear system (1.3) is also exponentially asymptotically stable.

Summing the above discussion, we obtain the following

Theorem 1.6. *Assume that*

i) *the zero solution of the linear system* (1.2) *is exponentially asymptotically stable*;

ii) *the operator F is a Niemytzki operator such that $|(Fx)| \leq r|x|$, with $0 < r < \alpha/N$, where the numbers $N > 0$ and $\alpha > 0$ are from the definition of the exponential asymptotic stability of the zero solution for system* (1.2).

Then the zero solution of the nonlinear system (1.3) *is also exponentially asymptotically stable.*

If the operator F is a nonlinear Volterra operator, the proof of Theorem 1.6 is not valid. In the paper by Tadayuki Hara, Toshiaki Yoneyama and Toshiki Itoh (see [8]: 'Asymptotic Stability Criteria for Nonlinear Volterra Integro-Differential Equations'), they consider the case where the operator F has the form

$$(Fx)(t) = \int_0^t G(t, s, x(s))\, ds, \quad \text{for } t > t_0,$$

such that

$$|G(t, s, x(s))| \leq c(t, s)|x(t)|,$$

where $c(t, s)$ is continuous for $t \geq s \geq 0$ and $|x| < H$ for some $H > 0$. They proved that if there exists a positive constant μ such that

$$\sup_{t\geq 0} \left(\int_0^t \exp(\mu(t-s))c(t-s)\, ds \right) < \alpha/N,$$

then the exponential asymptotic stability of the zero solution for (1.2) implies the same stability for the zero solution of the system (1.3).

Now, if the operator F is a Volterra type operator acting on $L^2_{\text{loc}}([0, \infty), \mathbf{R}^n)$, such that

$$|(Fx)(s)|_{L^2([0,t),\mathbf{R}^n)} \leq r|x(s)|_{L^2([0,t],\mathbf{R}^n)}, \tag{1.35}$$

for $t \geq 0$ and some suitable positive number $r > 0$, do we still have the same conclusions as in Theorem 1.6? At this moment, we are not very sure of it, the Gronwall inequality method we have used does not lead to a result.

In the recent paper [**4, 5**] (written by C. Corduneanu), the stability of the zero solution of the system (1.1), where the nonlinear Volterra operator V is continuous and defined on the function space $L_{\text{loc}}([0, \infty), \mathbf{R}^n)$ has been discussed by using the comparison method. Lyapunov functionals in the form $W = W(x)(t)$, $t \geq 0$, $W : L_{\text{loc}}([0, \infty), \mathbf{R}^n) \to \mathbf{R}$, assuming that W is also of Volterra type, have been used.

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Mathematics Department, The University of Texas at Arlington, Arlington, Texas 76019-0408