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INTEGRAL EQUATIONS CONVERTIBLE TO FIXED POINT EQUATIONS OF ORDER-PRESERVING OPERATORS

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1. Introduction. In this paper we study integral equations of Hammerstein type and show that under certain conditions such a problem is equivalent to a fixed point problem for a certain compact order-preserving operator. This conversion is quite interesting since fundamental work has been done for the study of fixed point equations with compact order-preserving operators, see [**1,2,3,5,6**]. Consequently, upon having such a conversion, one can establish for the integral equations various results based on the corresponding results for fixed point equations for compact order preserving operator. For instance, the minimal and maximal solutions, and the numerical iterative techniques, due to Krasnoselskii [**6**], the existence of solutions involving discontinuous functions due to Amann [**3**], and the existence of multiple solutions, see Amann [**1**].

More specifically, we consider problems of the form

$$\varphi(x) = \int_\Omega k(x,y) f(y,\varphi(y))\,dy \tag{1}$$

where Ω is some bounded closed domain in $\mathbf{R}^n$ with Lebesgue measure $\mu(\Omega)$. Here $f:\Omega\times\mathbf{R}\to\mathbf{R}$ need not be continuous in its second variable, but we assume $k:\Omega\times\Omega\to\mathbf{R}^+$ is continuous (this can be relaxed). A solution of (1) is understood to be an element of $X = C(\Omega,\mathbf{R})$.

Let $P = C(\Omega,\mathbf{R}^+)$. Then (X,P) is an ordered Banach space. The operator $B : X \to X$ is said to be order-preserving or increasing if $Bx \geq By$ for $x \geq y$. It is clear that, if B is linear, this is equivalent to assuming B is positive on P. Define $f_m:\Omega\times\mathbf{R}\to\mathbf{R}$ by $f_m(x,u) = f(x,u) + mu$ for real m. Our ultimate goal in this paper is to show that Equation (1) is equivalent to

$$\varphi(x) = \int_\Omega k^0(x,y) f_m(y,\varphi(y))\,dy + \int_\Omega k^{00}(x,y)\varphi(y)\,dy \tag{2}$$

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for some k^0 and k^{00}, continuous and nonnegative on $\Omega \times \Omega$. It is clear that the righthand side of (2) is a compact and order-preserving operator if $f_m(x,u)$ is increasing in u.

The paper is organized as follows. In Section 2 we prove the equivalence results between Equations (1) and (2). In Section 3 some applications of these results are provided. We conclude this paper with some additional remarks in Section 4.

2. Equivalence conversions. Let $K : X \to X$ be the linear compact operator defined by $K\varphi = \int_\Omega k(x,y)\varphi(y)\,dy$, and let $(F\varphi)(x) = f(x,\varphi(x))$ be the Nemytskii operator. Similarly, $(F_m\varphi)(x) = f_m(x,\varphi(x))$, $\sigma(K)$ denotes the spectrum of K, and $r(K)$ the Gelfand radius of the spectrum.

Theorem 1. *Assume that* $-1 \notin \sigma(K)$ *and* $(I+K)^{-m}K$ *is order-preserving for some positive integer* m. *Then*

$$\varphi = KF\varphi \tag{3}$$

is equivalent to

$$\varphi = K^0F_m\varphi + K^{00}\varphi \tag{4}$$

where $K^0, K^{00} : X \to X$ *are linear, compact and order-preserving. If, in addition,* $r(K) < 1$, *then* K^0 *and* K^{00} *may be represented in terms of nonnegative continuous kernels* k^0 *and* k^{00}, *respectively, i.e.,*

$$K^0\varphi = \int_\Omega k^0(x,y)\varphi(y)\,dy, \qquad K^{00}\varphi = \int_\Omega k^{00}(x,y)\varphi(y)\,dy. \tag{5}$$

Proof. $-1 \notin \sigma(K)$ implies that $(I+K)^{-1} : X \to X$ is bounded and linear. Hence, from (3),

$$\varphi = KF_1\varphi - K\varphi$$

so that

$$\begin{aligned}\varphi &= (I+K)^{-1}KF_1\varphi = (I+K)^{-1}KF_2\varphi - (I+K)^{-1}K\varphi\\ &= (I+K)^{-1}KF_2\varphi - K\varphi + [I-(I+K)^{-1}]K\varphi\\ \varphi &= (I+K)^{-2}KF_2\varphi + (I+K)^{-2}[(I+K)-I]K\varphi.\end{aligned}$$

Similarly, we have

$$\varphi = (I+K)^{-3}KF_3\varphi + (I+K)^{-3}[(I+K)^2 - I]K\varphi + (I+K)^{-3}[(I+K) - I]K\varphi.$$

Inductively, we obtain

$$\varphi = (I+K)^{-m}KF_m\varphi + (I+K)^{-m}\sum_{n=1}^{m-1}[(I+K)^n - I]K\varphi. \tag{6}$$

Therefore, (6) becomes (4) if we define

$$K^0 = (I+K)^{-m}K$$

and

$$K^{00} = (I+K)^{-m}\sum_{n=1}^{m-1}[(I+K)^n - I]K.$$

Obviously, K^0 and K^{00} are order-preserving, compact and linear.

If, additionally, $r(K) < 1$, then, see [**9**], $(I+K)^{-1}$ can be expressed explicitly as

$$(I+K)^{-1} = \sum_{n=0}^{\infty}(-K)^n$$

which converges in the norm of operators. The proof will be finished if we can show that $(I+K)^{-1}$ is generated by a continuous kernel, since then $(I+K)^{-m}K = [(I+K)^{-1}]^m K$ is also generated by such a kernel, and so are K^0 and K^{00}.

Let $k_n : \Omega \times \Omega \to \mathbf{R}^+$ be the kernel generating the operator K^n, namely, $k_1(x,y) = k(x,y)$ and $k_{n+1}(x,y) = \int_\Omega k_n(x,\eta)k(\eta,y)\,d\eta$ for $n \geq 1$. Thus, formally, $(I+K)^{-1}$ is generated by

$$\theta(x,y) = \sum_{n=0}^{\infty}(-1)^n k_n(x,y). \tag{7}$$

Since $r(K) < 1$, we have

$$\sum_{n=0}^{\infty}|K^n| < \infty. \tag{8}$$

Also, for $n \geq 2$ and $(x, y) \in \Omega \times \Omega$,

$$k_n(x, y) = [K^{n-1}k(\cdot, y)](x) \leq |K^{n-1}| \cdot |k(\cdot, y)| \leq C|K^{n-1}| \tag{9}$$

where $C = \max\{k(x, y)|(x, y) \in \Omega \times \Omega\}$.

By (8) and (9), the series (7) is absolutely and uniformly convergent. Therefore, $(I + K)^{-1}$ is generated by the continuous function $\sum_{n=0}^{\infty}(-1)^n k_n(x, y)$. Consequently, K^0 is generated by

$$k^0(x, y) = \int \theta(x, y_1) \int \Gamma(y_1, y_2) \cdots \int \theta(y_m, \eta) \int k(\eta, y)\, d\eta\, dy_m dy_{m-1} \cdots dy_1$$

and similarly, K^{00} is generated by some continuous function k^{00}. Furthermore, k^0 and k^{00} are also nonnegative since, by assumption, $K^0 = (I + K)^{-m}K$ is increasing and therefore so is $K^{00} = (I + K)^{-m} \sum_{n=1}^{m-1}[(I + K)^n - I]$ as $\sum_{n=1}^{m-1}[(I + K)^n - I]$ is increasing. The proof is thus complete. □

Remark 1. In case $r(K) < 1$, direct calculations show that if $(I - K)^m K$ is increasing, so is $(I + K)^{-m}K$.

We recall that a *total* cone P in a Banach space Y is a cone which satisfies $\overline{P - P} = Y$ and a *normal* cone is a cone which satisfies $||x + y|| \geq \delta > 0$ for all $x, y \in Y$ with $||x|| = ||y|| = 1$.

Before we state the next result, we quote the following lemma (see [**7**]).

Lemma 1 (Krein, Rutman). *Let Y be a Banach space, $P \subset Y$ a total cone, and let $T : Y \to Y$ be compact, linear and positive on P with $r(T) > 0$. Then $r(T)$ is an eigenvalue with a positive eigenvector.*

Theorem 2. *Assume that for some $m > 0$ and $\varepsilon > 0$,*

$$k(x, y) \geq (m + \varepsilon)k_2(x, y) \tag{10}$$

on $\Omega \times \Omega$ *with* $k_2(x,y) := \int_\Omega k(x,\eta)k(\eta,y)\,d\eta$. *Then Equation* (3) *is equivalent to*

$$\varphi = (I + mK)^{-1}KF_m\varphi \tag{11}$$

where $(I + mK)^{-1}K$ *is generated by some nonnegative continuous kernel.*

Proof. Let $\tilde{K} = (m+\varepsilon)K$. Then the assumptions imply

$$\tilde{K} \geq \tilde{K}^2.$$

Assume for the moment that $r(\tilde{K}) > 0$. By Lemma 1,

$$\tilde{K}\varphi = r(\tilde{K})\varphi \tag{12}$$

for some $\varphi \in P\backslash\{0\}$. Hence, by (2)

$$\tilde{K}^2\varphi = [r(\tilde{K})]^2\varphi \leq \tilde{K}\varphi = r(\tilde{K})\varphi.$$

Thus we have $r(\tilde{K}) \leq 1$. Therefore, if we let $\overline{K} = mK$, then we always have $r(\overline{K}) < 1$.

Write $\varphi = KF\varphi$ as $\varphi = \overline{K}((1/m)F\varphi)$. Equation (3) is then equivalent to

$$\varphi = (I + \overline{K})^{-1}\overline{K}((1/m)F\varphi + \varphi) \tag{13}$$

where, since $r(\overline{K}) < 1$, again see [**9**],

$$(I + \overline{K})^{-1}\overline{K} = \sum_{n=0}^{\infty}(-\overline{K})^{n+1} = \sum_{n=0}^{\infty}(\overline{K} - \overline{K}^2)\overline{K}^n. \tag{14}$$

It is clear that (10) and (14) imply that $(I + \overline{K})^{-1}\overline{K}$ is generated by some nonnegative continuous kernel and so is $(I + mK)^{-1}K$. Notice that $(1/m)F\varphi + \varphi = (1/m)F_m\varphi$ so we see that (13) is the same as (11). The proof is therefore complete. □

Remark 2. If $0 < \alpha \leq k(x,y) \leq c$ on $\Omega \times \Omega$, $k(x,y) \geq (m+\varepsilon)k_2(x,y)$ is then satisfied with any $m < \alpha/(c^2\mu(\Omega))$ and some $\varepsilon > 0$.

3. Some applications. Because of the equivalence of (3) and (4), the well-known results of fixed point equations for increasing operators can be applied to (3), provided $f_m(x,u)$ is increasing in u. To illustrate this, we prove some simple results which only need the following lemma.

Lemma 2 [**3**]. *Let (X,K) be an ordered Banach space with a* normal cone K. *Assume that for $u < v$, $A : [u,v] \to [u,v]$ is compact and increasing. Then A has maximal and minimal fixed points in $[u,v]$.*

Theorem 3. *Assume that $-1 \notin \sigma(K)$, $(I+K)^{-m}K$ is order-preserving and $f_m(x,u)$ is increasing in u, for some integer $m > 0$. Assume further that there exist $v, \bar{v} \in X$ with $v \le \bar{v}$ such that*

$$(I+K)^{-m}(KFv - v) \ge 0, \qquad (I+K)^{-m}(\bar{v} - KF\bar{v}) \ge 0. \tag{15}$$

Then the equation (1) *has minimal and maximal solutions in $[v,\bar{v}]$.*

Proof. By Theorem 1, Equation (1) is equivalent to

$$\varphi = A\varphi \quad \text{with} \quad A\varphi = K^0 F_m \varphi + K^{00}\varphi.$$

It is clear that A is compact and increasing since F_m is. Therefore, in order to apply Lemma 2, it remains to prove

$$v \le Av, \qquad A\bar{v} \le \bar{v}. \tag{16}$$

We only prove $v \le Av$ since $A\bar{v} \le \bar{v}$ can be similarly proved. From (15), one has $(I+K)^{-m}(KFv - v) \ge 0$, i.e.,

$$(I+K)^{-m}(KF_m v - mKv - v) \ge 0 \tag{17}$$

and we claim that

$$(I+K)^m v - mKv - v = \sum_{n=1}^{m-1} [(I+K)^n - I]Kv. \tag{18}$$

This is clearly true when $m = 1$. Assume that it is true for $m = p$. Then if $m = p + 1$, we have

$$\begin{aligned}(I+K)^{p+1}v - (p+1)Kv - v \\ &= (I+K)^p - pKv - v + [(I+K)^p - I]Kv \\ &= \sum_{n=1}^{p-1}[(I+K)^n - I]Kv + [(I+K)^p - I]Kv \\ &= \sum_{n=1}^{p}[(I+K)^n - I]Kv.\end{aligned}$$

Thus, by mathematical induction we see that (18) is true. Now (17) and (18) yield

$$(I+K)^{-m}\left(KF_m v + \sum_{n=1}^{m-1}[(I+K)^n - I]Kv - (I+K)^m v\right) \geq 0$$

which is the same as $v \leq Av$. Therefore, Lemma 2 is immediately applicable and the proof is thus complete. □

In Theorem 3 the condition that $(I+K)^{-m}K$ is order preserving may appear to be difficult to verify. However, from Remark 1, we may instead verify that $(I-K)^m K$ is increasing which may be easier to determine in some instances. We note also that the restriction in Remark 1 that $r(K) < 1$ is harmless since one may always replace K by αK for some $\alpha > 0$. It is interesting to determine conditions under which $(I-K)^m K$ is increasing for some (or for all) $m \in \mathbf{N}$. Along this line we consider the following integral equation

$$\varphi(x) = \int_a^b e^{-(x-y)} f(y, \varphi(y))\, dy. \tag{19}$$

Example 1. Assume that $a < b$, $f : [a,b] \times \mathbf{R} \to \mathbf{R}$ is graph measurable, and $f_{\hat{m}}(x,u)$ is increasing in u for some integer $\hat{m} > 0$. Assume further that

$$\lim_{u \to 0+} \frac{\inf_{a \leq y \leq b} f(y,u)}{u} > e^{b-a} \tag{20}$$

and

$$\lim_{u\to\infty} \frac{\sup_{a\le y\le b} f(y,u)}{u} < e^{-(b-a)}. \tag{21}$$

Then (19) has a solution φ such that $\varphi(y) \ge 0$ for all $y \in [a,b]$.

Proof. It is clear that (19) is equivalent to $\varphi = KF\varphi$ with $(K\psi)(x) = \int_a^b e^{-(x-y)}\psi(y)\,dy$ and $F\varphi = f(y,\varphi(y))$. Now it is easy to see that $KK = (b-a)K$. Thus we can find $n \in \mathbf{N}$ such that, with $\hat{K} = (1/n)K$, $r(\hat{K}) < 1$ and $\hat{K}\hat{K} = \alpha\hat{K}$ with $\alpha = (b-a)/n < 1$. By condition (20) we can choose n so large that for all $y \in [a,b]$,

$$nf(y,v(y)) \ge 1 \tag{22}$$

where $v(y) = (1/n)\int_a^b e^{-(y-x)}\,dx$. Now $n \in \mathbf{N}$ is fixed. A direct calculation shows that for any $m \in \mathbf{N}$ we have $(I-\hat{K})^m\hat{K} = (1-\alpha)^m\hat{K}$.

Hence, by Remark 1, $(I+\hat{K})^{-m}\hat{K}$ is increasing for all $m \in \mathbf{N}$. If we let $\hat{f} = nf$ and $\hat{F} = nF$, then (19) is equivalent to

$$\varphi = \hat{K}\hat{F}\varphi \tag{23}$$

and $\hat{F}_m(x,y)$ is increasing in u for $m = n\hat{m}$. Thus, if we fix $m = n\hat{m}$ then by (22) it is easy to check that

$$(I+\hat{K})^{-m}(\hat{K}\hat{F}(v) - v)) \ge 0 \tag{24}$$

since $(I+\hat{K})^{-m}(\hat{K}\hat{F}(v) - v) = (I+\hat{K})^{-m}(nf(y,v(y)) - 1)$ and $(I+\hat{K})^{-m}\hat{K}$ is increasing.

By (21) we can choose a sufficiently large $\beta > 1$ such that, with $\bar{v}(y) = (\beta/n)\int_a^b e^{-(y-x)}\,dx$ we will have

$$\beta \ge nf(y,\bar{v}(y) \quad \text{for all} \quad y \in [a,b]. \tag{25}$$

As in the proof of (24), we may now obtain

$$(I+\hat{K})^{-m}(\hat{v} - \hat{K}\hat{F}(\bar{v})) \ge 0. \tag{26}$$

Thus the condition (15) is satisfied by $\hat{K}$ and $\hat{F}$. Therefore, we may apply Theorem 3 to get minimal and maximal solutions of (23) in $[v, \bar{v}]$, which are also solutions of (19). □

Remark 3. In order to achieve the same result for (19) without using Theorem 3, we would have to assume that $f(x,)$ is jointly continuous in x and u, and increasing in u.

The next theorem can be analogously proved by applying Theorem 2 and Lemma 2.

Theorem 4. *Assume that for some $m > 0$ and $\varepsilon > 0$,*

$$k(x,y) \geq (m+\varepsilon) k_2(x,y) \quad \text{on } \Omega \times \Omega$$

and $f_m(x,u)$ is increasing in u. Assume further that there are $\varphi, \bar{\varphi} \in X$ with $\varphi \leq \bar{\varphi}$ such that

$$\varphi \leq FK\varphi, \qquad FK\bar{\varphi} \leq \bar{\varphi}. \tag{27}$$

Then Equation (1) *has maximal and minimal solutions in $[K\varphi, K\bar{\varphi}]$.*

Proof. By Theorem 2, equation (1) is equivalent to

$$\varphi = A\varphi := (I + mK)^{-1} K F_m \varphi$$

and $(I+mK)^{-1}K$ is increasing. Let $v = K\varphi$ and $\bar{v} = K\bar{\varphi}$. Equation (27) implies

$$(I+mK)^{-1}(KFv - v) \geq 0 \quad \text{and} \quad (I+mK)^{-1}(\bar{v} - KF\bar{v}) \geq 0.$$

Hence,

$$v \leq Av \quad \text{and} \quad A\bar{v} = \bar{v}$$

and Lemma 2 applies to $[v, \bar{v}]$. The proof is complete. □

4. Final remarks. In general, it is difficult to check if $(I+K)^{-m}K$ is indeed increasing. Therefore we also proved the more concrete Theorem

2 and used the easy-to-check condition (27) instead of (15) in Theorem 4. Hence, it is very interesting to know when $(I+K)^{-m}K$ is an increasing operator.

When $m=1$, (15) reduces to

$$(I+K)^{-1}(KFv-v)\geq 0 \quad \text{and} \quad (I+K)^{-1}(\bar{v}-KF\bar{v})\geq 0. \tag{28}$$

This may seem to be an artificial condition. However, we show next that it is automatically satisfied by the Dirichlet problem for differential equations

Example 2. Consider

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{29}$$

where we assume that $\partial\Omega$ is sufficiently smooth. Let v be a lower solution of the BVP (29), namely,

$$\begin{cases} -\Delta v \leq f(v) & \text{in } \Omega \\ v \leq 0 & \text{on } \partial\Omega. \end{cases} \tag{30}$$

We claim that v must automatically satisfy

$$(I+K)^{-1}(KFv-v)\geq 0$$

where K is the Green's function for $-\Delta u = 0$ subject to $u=0$ on $\partial\Omega$. To this end, let $\xi(x)$ be a solution of

$$\begin{cases} -\Delta \xi = f(v) + \Delta v & \text{in } \Omega \\ \xi = -x & \text{on } \partial\Omega. \end{cases}$$

Then $\eta = v + \xi$ satisfies

$$\begin{cases} -\Delta \eta = f(v) & \text{in } \Omega \\ \eta = 0 & \text{on } \partial\Omega. \end{cases}$$

That is,

$$\eta = KFv = \int_\Omega K(x,y) f(v(y))\, dy.$$

Let $(I_K)^{-1}\xi = p$, i.e., $p + Kp = \xi$. Hence, by (30),

$$\begin{cases} -\Delta p + p = -\Delta\xi = f(v) + \Delta v \geq 0 & \text{in } \Omega \\ p = \xi \geq 0 & \text{on } \partial\Omega. \end{cases}$$

The maximum principle thus implies $p(x) \geq 0$ on Ω, i.e., $(I+K)^{-1}\xi \geq 0$. Consequently,

$$\begin{aligned} (I+K)^{-1}v &\leq (I+K)^{-1}v + (I+K)^{-1}\xi \\ &= (I+K)^{-1}\eta = (I+k)^{-1}KFv. \end{aligned}$$

Similarly, any upper solution of $\bar{v}$ of the BVP (29) also automatically satisfies (28).

If K is a Green's function as in Example 1, the maximum principle implies that $(I + mK)^{-1}K$ is increasing for all $m \geq 0$. Of course, $K(x, y)$ is nonnegative and symmetric in that situation. One might conjecture that perhaps all nonnegative symmetric $K(x, y)$ enjoy this property. Unfortunately, this is false, even for matrices as the next example reveals.

Example 3. Take three orthogonal unit vectors

$$\vec{a} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}, \qquad \vec{c} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\sqrt{2}2 \end{pmatrix}$$

and assume that they are eigenvectors of a symmetric matrix K corresponding to the eigenvalues $\lambda_1 = 3$, $\lambda_2 = 2$ and $\lambda_3 = 1$, respectively. Then

$$\begin{aligned} K &= \lambda_1 \vec{a}\vec{a}^{\perp} + \lambda_2 \vec{b}\vec{b}^{\perp} + \lambda_3 \vec{c}\vec{c}^{\perp} \\ &= 3 \begin{pmatrix} \frac{1}{4} & \frac{1}{4}\,\frac{\sqrt{2}}{4} & \\ \frac{1}{4} & \frac{1}{4} & \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{2}{4} \end{pmatrix} + 2 \begin{pmatrix} \frac{2}{4} & -\frac{2}{4} & 0 \\ -\frac{2}{4} & \frac{2}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{2}}{4} & \\ \frac{1}{4} & \frac{1}{4} & -\frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{2}{4} \end{pmatrix} \geq 0. \end{aligned}$$

We claim that $(I+mK)^{-1}K$ is not an increasing operator for large m. More specifically, we shall show that

$$\frac{\lambda_1}{1+m\lambda_1}a_1a_2+\frac{\lambda_2}{1_m\lambda_2}b_1b_2+\frac{\lambda_3}{1+m\lambda_3}c_1c_2>0$$

for large m. To this end, we have

$$\begin{aligned}\bigg(\prod_{i=1}^{3}&\frac{\lambda_i}{1+m_i\lambda_i)}\bigg)a_1a_2+\frac{\lambda_2}{1+m\lambda_2}b_1b_2+\frac{\lambda_3}{1+m\lambda_3}c_1c_2\bigg)\\ &=\lambda_1(1+m\lambda_2)(1+m\lambda_3)\frac{1}{4}+\lambda_2(1+m\lambda_1)+(1+m\lambda_3)\bigg(-\frac{2}{4}\bigg)\\ &\quad+\lambda_3(1+m\lambda_1)(1+m_2)\frac{1}{4}\\ &=\bigg(\frac{\lambda_1}{4}-\frac{2\lambda_2}{4}+\frac{\lambda_3}{4}\bigg)+m\bigg(\frac{9}{5}-4+\frac{5}{4}\bigg)<0\end{aligned}$$

for large m.

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