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COUPLED VOLTERRA EQUATIONS WITH BLOW-UP SOLUTIONS

W.E. OLMSTEAD, CATHERINE A. ROBERTS AND KENG DENG

ABSTRACT. A pair of coupled nonlinear Volterra equations are examined for possible blow-up solutions. The system is motivated by certain models of explosion phenomena in a diffusive medium. Criteria for a blow-up to occur as well as bounds on the time of its occurrence are derived for a general class of nonlinearities. Specific results are obtained for two special cases involving power law and exponential nonlinearities. Also, the asymptotic growth rate near blow-up is determined for these two special cases when the kernel behaves like that of the one-dimension heat equation.

1. Introduction. We examine a pair of coupled nonlinear Volterra equations, which are motivated by certain models of a diffusive medium that can experience explosive behavior. The particular models of interest are described by the vector integral equation,

$$u(t) = Tu(t) \equiv \int_0^t k(t-s)F[u(s)+h(s)]\,ds, \qquad t \geq 0, \tag{1}$$

where the components of the solution $u = [u_1, u_2]$ and the given data $h = [h_1, h_2]$ are real functions. The nonlinear operator $T = [T_1, T_2]$ is defined by the real components of the function $F = [F_1, F_2]$ and the real scalar kernel $k(t-s)$. The kernel is taken to be continuously differentiable and have the properties

$$k(t-s) \geq 0, \qquad k'(t-s) < 0, \qquad t > s \geq 0, \tag{2}$$

which are characteristic of problems involving diffusion.

The arguments of the component functions F_j, $j = 1, 2$, are restricted so that

$$F_1 = F_1[u_2 + h_2], \qquad F_2 = F_2[u_1 + h_1]. \tag{3}$$

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Furthermore, each component is taken to be twice continuously differentiable with the properties that

$$F_j(v) > 0, \qquad F_j'(v) > 0, \qquad F_j''(v) > 0, \qquad v > 0, \quad j = 1, 2, \tag{4}$$

which are typical of problems modelling explosions.

The given functions $h_j(t)$, $j = 1, 2$, typically are determined by the initial data of an explosion problem. We will assume that

$$0 < a_j \le h_j(t) \le b_j < \infty, \qquad h_j'(t) \ge 0, \qquad t \ge 0, \quad j = 1, 2. \tag{5}$$

Our goal is to explore general conditions for blow-up solutions of (1) and to derive explicit criteria for the special cases of power law and exponential nonlinearities. We will also establish the asymptotic growth near blow-up for each of these special cases when the kernel is that associated with the one-dimensional diffusion equation. The techniques of our analysis will be analogous to those employed in [**9**] and [**10**] for a scalar equation like (1).

The connection of (1) with certain systems of nonlinear initial-boundary value problems for parabolic partial differential equations should be noted. The first example of such related problems is

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) w_j(x,t) = \delta(x-l) F_j[w_{3-j}(x,t)], \quad 0 < x < L, \quad t > 0 \tag{6}$$

$$\frac{\partial w_j}{\partial x}(0,t) = 0, \qquad \frac{\partial w_j}{\partial x}(L,t) = 0, \qquad t \ge 0, \tag{7}$$

$$w_j(x,0) = w_j^0(x), \qquad 0 \le x \le L, \quad j = 1, 2. \tag{8}$$

Here the delta function acts to localize the nonlinear effect at a specified position $x = l$, $0 < l < L$. This type of problem can serve as a model for combustion phenomena in which ignition leading to thermal run-away is confined to a very thin zone (e.g., [**1**]). A single component version of (6)–(8) has recently been investigated in [**8**]. Considerable attention has been given to the single component version of (6)–(8) without the localizing effect of the delta function; see [**7**] for a review of that field.

The conversion of (6)–(8) to a vector integral equation in the form of (1) is accomplished by utilizing the Green's function $G(x,t|\xi,s)$ derived from the linear heat operator with Neumann boundary conditions. It takes the form in spectral and image representations, respectively, as

$$\begin{aligned} G(x,t\mid\xi,s) &= H(t-s)\biggl\{\frac{1}{L}+\frac{2}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi\xi}{L}\cos\frac{n\pi x}{L} \\ &\qquad \exp\biggl[-\frac{n^2\pi^2}{L^2}(t-s)\biggr]\biggr\} \\ &= \frac{H(t-s)}{2[(t-s)]^{1/2}}\sum_{n=-\infty}^{\infty}\biggl\{\exp\biggl[-\frac{(x-\xi-2nL)^2}{4(t-s)}\biggr] \\ &\qquad +\exp\biggl[-\frac{(x+\xi-2nL)^2}{4(t-s)}\biggr]\biggr\}. \end{aligned} \tag{9}$$

It follows that

$$\begin{aligned} w_j(x,t) &= \int_0^t\int_0^L G(x,t\mid\xi,s)\delta(\xi-l)F_j[w_{3-j}(\xi,s)]\,d\xi\,ds \\ &\quad + \tilde{h}_j(x,t), \qquad j=1,2, \end{aligned} \tag{10}$$

where

$$\tilde{h}_j(x,t) = \int_0^L G(x,t\mid\xi,0)w_j^0(\xi)\,d\xi. \tag{11}$$

Then set $x=l$ in (10) and use the sifting property of the delta function to obtain

$$\begin{aligned} u_j(t) &= \int_0^t G(l,t\mid l,s)F_j[u_{3-j}(s)+h_{3-j}(s)]\,ds, \\ &\qquad t\ge 0, \quad j=1,2, \end{aligned} \tag{12}$$

where, in this example,

$$u_j(t) = w_j(l,t)-h_j(t), \qquad h_j(t)=\tilde{h}_j(l,t). \tag{13}$$

Clearly (12) is an example of (1) in component form, where the kernel $G(l, t \mid l, s)$ is of difference type and satisfies the properties of (2). It is assumed that the properties of the initial data $w_j^0(x)$, $j = 1, 2$ are such that (5) is satisfied.

Another example of a parabolic system that can be reduced to (1) is given by

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) w_j(x,t) = 0, \qquad 0 < x < L, \quad t > 0, \tag{14}$$

$$\frac{\partial w_j}{\partial x}(0,t) = 0, \qquad \frac{\partial w_j}{\partial x}(L,t) = F_j[w_{3-j}(L,t)], \quad t \geq 0, \tag{15}$$

$$w_j(x,0) = w_j^0(x), \qquad 0 \leq x \leq L, \quad j = 1, 2. \tag{16}$$

Single component versions of (14)–(16) have been considered in [**3**] and [**6**], while the system form has been examined in [**4**] and [**5**].

The conversion of (14)–(16) to (1) utilizes the same Green's function (9) to obtain

$$w_j(x,t) = \int_0^t G(x,t \mid L,s) F_j[w_{3-j}(L,s)]\, ds + \tilde{h}_j(x,t), \quad j = 1, 2, \tag{17}$$

where $\tilde{h}_j$ is again given by (11). Evaluation of (17) at $x = L$ yields

$$u_j(t) = \int_0^t G(L,t \mid L,s) F_j[u_{3-j}(s) + h_{3-j}(s)]\, ds, \quad t \geq 0, \quad j = 1, 2, \tag{18}$$

where, in this example,

$$u_j(t) = w_j(L,t) - h_j(t), \qquad h_j(t) = \tilde{h}_j(L,t). \tag{19}$$

Here (18) is another example of (1) in component form, where the kernel $G(L, t \mid L, s)$ is again of difference type and satisfies the properties of (2).

2. Properties and existence of the solution. Here we will examine some basic properties of the solution to (1) and establish both the existence and uniqueness of a bounded solution for $0 \le t < t^*$. This will provide a lower bound t^* on any blow-up that might occur. The general approach is similar to that used in [**9**] for a scalar equation.

The nonnegative property of the solution to (1) is established through examination of its components. By noting that $u_j(0) = 0$ and $h_j(0) > 0$, it follows that $u_j(t) + h_j(t) > 0$, $j = 1, 2$, at least on some common small interval $0 \le t \le \tilde{t}$. Then the properties of the nonlinearity and the kernel insure that $u_j(t) \ge 0$ for $0 \le t \le \tilde{t}$. This argument can be extended indefinitely to demonstrate that $u = [u_1, u_2]$ is such that

$$u_j(t) \ge 0, \qquad t \ge 0, \quad j = 1, 2, \tag{20}$$

as long as (1) has a solution.

To demonstrate the monotone growth of the solution, we consider the derivative of (1) in component form,

$$\begin{aligned} u_j'(t) &= k(t)F_j(a_{3-j}) \\ &\quad + \int_0^t k(t-s)F_j'[u_{3-j}(s) + h_{3-j}(s)][u_{3-j}'(s) + h_{3-j}'(s)]\, ds, \\ &\qquad t \ge 0, \quad j = 1, 2. \end{aligned} \tag{21}$$

Since $u_j(0) = 0$ and $u_j(t) \ge 0$, then $u_j'(t) > 0$, $j = 1, 2$, at least on some common small interval $0 < t < \bar{t}$. Assume, for example, that $u_1'(\bar{t}) = 0$, $u_2'(\bar{t}) \ge 0$. Then, from (21) with $j = 1$,

$$\begin{aligned} 0 &= k(\bar{t})F_1(a_2) \\ &\quad + \int_0^{\bar{t}} k(\bar{t} - s)F_1'[u_2(s) + h_2(s)][u_2'(s) + h_2'(s)]\, ds. \end{aligned} \tag{22}$$

Clearly this is a contradiction since the right side of (22) must be positive for $\bar{t} < \infty$. Thus, $u_1'(\bar{t}) \ne 0$, and analogously it follows that $u_2'(\bar{t}) \ne 0$. Hence, no such $\bar{t}$ exists, and consequently, $u' = [u_1', u_2']$ is such that

$$u_j'(t) > 0, \qquad 0 < t < \infty, \quad j = 1, 2, \tag{23}$$

as long as (1) has a solution.

To establish the existence of a unique solution to the system (1), we consider the vector space C^0 of pairs of functions continuous on $[0,\infty)$. The properties of T are such that we can restrict our attention to the closed subset $X \subset C^0$ where

$$X = \{u = [u_1, u_2] \mid 0 \le u_j(t) \le M < \infty, 0 \le t < \hat{t},\ j = 1,2\}, \tag{24}$$

with each $u_j(t)$ continuous on the given interval. For $u \in X \subset C^0$, we use the norm

$$||u|| = \sup_{0 \le t < \hat{t}} \{|u_1(t)| + |u_2(t)|\}. \tag{25}$$

The goal is to find the limitation on $\hat{t}$ under which the operator T is a contraction mapping of the closed subset X into itself.

Given the properties of the kernel and the nonlinearities, it follows from (1) that for each component

$$0 \le u_j(t) \le F_j(M + b_{3-j}) I(t), \qquad j = 1,2, \tag{26}$$

where

$$I(t) = \int_0^t k(t-s)\, ds. \tag{27}$$

In order to insure that T maps X into itself, then $\hat{t}$ must be such that

$$I(\hat{t}) \le M/m, \qquad m = \max_{j=1,2} \{F_j(M + b_{3-j})\}. \tag{28}$$

To establish the contraction property of the operator T, defined by the right side of (1), we consider $u, v \in X$, whereupon it follows that

$$\begin{aligned} Tu - Tv &= \int_0^t k(t-s)\{F[u(s)+h(s)] - F[v(s)+h(s)]\}\, ds \\ &= \int_0^t k(t-s) F'[\theta(s)+h(s)][u(s)-v(s)]\, ds. \end{aligned} \tag{29}$$

Here the mean value theorem implies that $\theta = [\theta_1, \theta_2]$ is such that θ_j lies between u_j and v_j, $j = 1,2$. Given the properties of (4), it follows from (29) that

$$||Tu - Tv|| \le \max_{j=1,2} \{F_j'(M + b_{3-j})\} I(\hat{t}) ||u - v||. \tag{30}$$

Thus, T is a contracting mapping of the closed subset X into itself provided that

$$I(\hat{t}) < 1/\tilde{m}, \qquad \tilde{m} = \max_{j=1,2}\{F_j'(M + b_{3-j})\}. \tag{31}$$

To determine a bounding value on $\hat{t}$ for which existence holds, it is convenient to define

$$I(t^*) = \sup_{M \geq 0} I_M, \tag{32}$$

where, for each $M \geq 0$,

$$I_M = \min\{M/m, 1/\tilde{m}\}. \tag{33}$$

This insures that both (28) and (31) are fulfilled for $0 \leq \hat{t} < t^*$.

Thus, we are able to conclude that there exists a unique, continuous solution $u(t)$ of (1) which is nonnegative and increasing for $0 \leq t < t^*$, where t^* is given by (32). This t^* represents a lower bound on any possible blow-up of the solution to (1).

3. Nonexistence of the solution. Here we determine conditions under which the solution of (1) can experience a blow-up. In particular, we will derive criteria that establish a $t^{**} < \infty$, such that (1) cannot possess a continuous solution for $t \geq t^{**}$.

We begin by assuming that (1) has a continuous solution for $0 \leq t \leq \hat{t}$. It then follows from (1) and (2) that the individual components of u satisfy the inequalities,

$$u_j(t) \geq J_j(t) \equiv \int_0^t k(\hat{t} - s) F_j[u_{3-j}(s) + h_{3-j}(s)]\, ds, \qquad 0 \leq t \leq \hat{t}, \quad j = 1, 2. \tag{34}$$

Differentiation of $J_j(t)$ gives

$$J_j'(t) = k(\hat{t} - t) F_j[u_{3-j}(t) + h_{3-j}(t)], \qquad 0 \leq t \leq \hat{t}, \quad j = 1, 2. \tag{35}$$

Then, combining (34) and (35) yields the coupled pair of differential inequalities

$$J_j'(t) \geq k(\hat{t}-t)F_j[J_{3-j}(t)+a_{3-j}], \qquad J_j(0)=0, \quad 0 \leq t \leq \hat{t}, \quad j=1,2. \tag{36}$$

If we demonstrate nonexistence by reason of blow-up for either of the $J_j(t)$, then (34) implies a nonexistence for the corresponding $u_j(t)$. To investigate the possibility of such behavior of $J_j(t)$, we introduce a comparison problem for the coupled differential equations,

$$V_j'(t) = k(\hat{t}-t)F_j[V_{3-j}(t)], \qquad V_j(0)=a_j-\delta>0, \quad 0 \leq t \leq \hat{t}, \quad j=1,2, \tag{37}$$

where $\delta > 0$ is taken to be sufficiently small to insure the positivity of the initial conditions. To establish that $J_j(t)+a_j > V_j(t)$, $j=1,2$, we define

$$U_j(t) = J_j(t)+a_j-V_j(t), \qquad j=1,2. \tag{38}$$

It then follows from (36) and (37) that

$$\begin{aligned} U_j'(t) &\geq k(\hat{t}-t)\{F_j[J_{3-j}(t)+a_{3-j}]-F_j[V_{3-j}(t)]\} \\ &\geq k(\hat{t}-t)F_j'[\tilde{\theta}_{3-j}(t)]U_{3-j}(t), \qquad U_j(0)=\delta, \quad j=1,2, \end{aligned} \tag{39}$$

where, by the mean value theorem, $\tilde{\theta}_j$ lies between J_j+a_j and V_j, both of which are positive. Integration of (39) gives

$$U_j(t) \geq \delta + \int_0^t k(\hat{t}-s)F_j'[\tilde{\theta}_{3-j}(s)]U_{3-j}(s)\,ds, \quad 0 \leq t \leq \hat{t}, \quad j=1,2. \tag{40}$$

By recombination of the two inequalities of (40), it follows that

$$\begin{aligned} U_j(t) \geq \delta &+ \int_0^t k(\hat{t}-s)[F_j'(\tilde{\theta}_{3-j}(s)]\left\{\delta + \int_0^s k(\hat{t}-\zeta)F_{3-j}'[\tilde{\theta}_j(\zeta)]U_j(\zeta)\,d\zeta\right\}ds, \\ &0 \leq t \leq \hat{t}, \quad j=1,2. \end{aligned} \tag{41}$$

From (41) it is clear that $U_j(t) > 0$ for all $\delta > 0$, and hence

$$J_j(t) + a_j \geq [V_j(t)]_{\delta=0}, \qquad 0 \leq t \leq \hat{t}, \quad j = 1, 2. \tag{42}$$

Thus, we are able to conclude that

$$u_j(t) \geq J_j(t) \geq [V_j(t)]_{\delta=0} - a_j, \qquad 0 \leq t \leq \hat{t}, \quad j = 1, 2, \tag{43}$$

so that a nonexistence by blow-up of either $V_j(t)$ implies a nonexistence for the corresponding $u_j(t)$.

To investigate the differential system (37), it is convenient to define the anti-derivatives $G_j(V)$ as

$$F_j(V) \equiv \frac{dG_j}{dV}(V), \qquad j = 1, 2. \tag{44}$$

We can then put (37), with $\delta = 0$, into the equivalent form

$$\begin{gathered} \frac{d}{dt}\{G_{3-j}[V_j(t)]\} = k(\hat{t} - t)F_j[V_{3-j}(t)]F_{3-j}[V_j(t)], \\ V_j(0) = a_j, \quad j = 1, 2. \end{gathered} \tag{45}$$

In view of the invariance of the right side of (45) with $j = 1, 2$, it follows that

$$G_2[V_1(t)] - G_1[V_2(t)] = G_2(a_1) - G_1(a_2), \qquad 0 \leq t \leq \hat{t}. \tag{46}$$

Moreover, the positivity of the $F_j(V)$ with $V > 0$ insures that the $G_j(V)$ are strictly increasing and hence invertible so that

$$V_j(t) = G_{3-j}^{-1}\{G_j[V_{3-j}(t)] - G_j(a_{3-j}) + G_{3-j}(a_j)\}, \qquad j = 1, 2. \tag{47}$$

The system (37), with $\delta = 0$, can now be decoupled into

$$\begin{gathered} V_j'(t) = k(\hat{t} - t)F_j[G_j^{-1}\{G_{3-j}[V_j(t)] - G_{3-j}(a_j) + G_j(a_{3-j})\}], \\ V_j(0) = a_j, \qquad 0 \leq t \leq \hat{t}, \quad j = 1, 2. \end{gathered} \tag{48}$$

Each of the equations (48) can be solved independently in an implicit form for each of the V_j. This yields

$$\begin{gathered} \int_{a_j}^{V_j(t)} \frac{dz}{F_j\{G_j^{-1}[G_{3-j}(z) - G_{3-j}(a_j) + G_j(a_{3-j})]\}} = I(t), \\ 0 \leq t \leq \hat{t}, \quad j = 1, 2. \end{gathered} \tag{49}$$

From (49) we can infer criteria for the blow-up of V_j and hence for u_j. It is convenient to define

$$\kappa_j = \int_{a_j}^{\infty} \frac{dz}{F_j\{G_j^{-1}[G_{3-j}(z) - G_{3-j}(a_j) + G_j(a_{3-j})]\}}, \quad j = 1, 2. \tag{50}$$

Thus, whenever the smallest of the κ_j is in the range of $I(t)$, then (49) implies that the corresponding V_j must blow-up. Consequently, it is clear from (37) that, if either V_j experiences a blow-up, then so does the other. That is, if there exists a $t^{**} < \infty$ such that

$$I(t^{**}) = \min_{j=1,2} \kappa_j, \tag{51}$$

then

$$V_j(t) \longrightarrow \infty \quad \text{as } t \longrightarrow t^{**}, \qquad j = 1, 2. \tag{52}$$

In view of (43), the implication of (52) is that each of the u_j will experience blow-up. Moreover, when one component of $u = [u_1, u_2]$ ceases to exist as some t_c, the continuity of (1) implies that the other component must also cease to exist at the same t_c. Thus, if (51) is satisfied, then

$$u(t) \longrightarrow \infty \quad \text{as} \quad t \longrightarrow t_c, \tag{53}$$

where

$$t^* \le t_c \le t^{**}. \tag{54}$$

4. Applications. Here we will apply the results of Sections 2 and 3 to two specific classes of nonlinearities $F(v)$ that are frequently encountered in problems with blow-up solutions. In particular, we examine a case of power law nonlinearities and a case of exponential nonlinearities.

For the power law nonlinearities, we consider $F = [F_1, F_2]$ to have the form

$$F_j(v) = [v]^{p_j}, \qquad p_j > 1, \quad j = 1, 2, \tag{55}$$

where $p_1 \geq p_2$ without loss of generality. For simplicity, we set $a_1 = a_2 = a$, $b_1 = b_2 = b$.

For the lower bound t^* on blow-up developed in Section 2, it follows from (32) and (33) that

$$I(t^*) = [(p_1 - 1)/b]^{p_1 - 1}(p_1)^{-p_1}. \tag{56}$$

Assuming that the right side of (56) lies in the range of $I(t)$, it follows that (1) has a unique, continuous solution for $0 \leq t < t^*$.

To apply the nonexistence results of Section 3, it is first noted from (44) that

$$G_j(v) = \frac{1}{p_j + 1}[v]^{p_j + 1}, \qquad j = 1, 2. \tag{57}$$

It then follows from (50) that

(58)
$$\kappa_j = \frac{1}{\alpha_j}\int_a^\infty [z^{p_{3-j}+1} - a^{p_{3-j}+1} + \beta_j]^{-p_j/(p_j+1)}\,dz,$$
$$\alpha_j = \left(\frac{p_j + 1}{p_{3-j} + 1}\right)^{p_j/(p_j+1)}, \qquad \beta_j = \frac{p_{3-j} + 1}{p_j + 1}a^{p_j + 1}, \qquad j = 1, 2.$$

Convergence of the integrals κ_j necessitates that $p_1 p_2 > 1$, which is assured by (55). These integrals can be evaluated in terms of the hypergeometric function ${}_2F_1$ as

$$\begin{aligned}\kappa_j = \frac{(p_j + 1)a^{1-p_j}}{(p_1 p_2 - 1)}\,{}_2F_1\Bigg[\frac{p_j}{p_j + 1}, 1; \frac{p_1 + p_2 + 2p_1 p_2}{(p_1 + 1)(p_2 + 1)};\\ 1 - \frac{p_{j+1}}{p_{3-j} + 1}a^{p_{3-j} - p_j}\Bigg], \qquad j = 1, 2.\end{aligned} \tag{59}$$

It then follows from (51) that

$$I(t^{**}) = \min_{j=1,2} \kappa_j. \tag{60}$$

Assuming that the right side of (56) lies in the range of $I(t)$, it follows that (1) has no continuous solution for $t \geq t^{**}$.

For the exponential nonlinearity, we consider $F = [F_1, F_2]$ to have the form

$$F_j(v) = e^{\gamma_j v}, \qquad \gamma_j > 0, \quad j = 1, 2, \tag{61}$$

where $\gamma_1 \geq \gamma_2$ without loss of generality. Again for simplicity, we set $a_1 = a_2 = a$, $b_1 = b_2 = b$.

For the lower bound t^* on blow-up developed in Section 2, it follows that

$$I(t^*) = \frac{1}{\gamma_1} e^{-(1+\gamma_1 b)}. \tag{62}$$

Assuming that the right side of (56) lies in the range of $I(t)$, it follows that (1) has a unique, continuous solution for $0 \leq t < t^*$.

To apply the nonexistence results of Section 3, it is first noted from (44) that

$$G_j(v) = \frac{1}{\gamma_j} e^{\gamma_j v}, \qquad j = 1, 2. \tag{63}$$

It then follows from (50) that

$$\kappa_j = \frac{\gamma_{3-j}}{\gamma_j} \int_a^\infty \left(e^{\gamma_{3-j} z} - e^{\gamma_{3-j} a} + \frac{\gamma_{3-j}}{\gamma_j} e^{\gamma_j a} \right)^{-1} dz, \qquad j = 1, 2. \tag{64}$$

Evaluation of the integrals yields

$$\kappa_j = (\gamma_{3-j} e^{\gamma_j a} - \gamma_j e^{\gamma_{3-j} a})^{-1} \left[(\gamma_j - \gamma_{3-j}) a - \log \frac{\gamma_{3-j}}{\gamma_j} \right], \qquad j = 1, 2. \tag{65}$$

It then follows from (51) that

$$I(t^{**}) = \min_{j=1,2} \kappa_j. \tag{66}$$

Assuming that the right side of (65) lies in the range of $I(t)$, it follows that (1) has no continuous solution for $t \geq t^{**}$.

5. Growth rates near blow-up. The asymptotic behavior of the solution near blow-up is highly dependent on the explicit form of the nonlinearity, as well as certain specific properties of the kernel. We will present results here for a kernel which behaves like that associated with the parabolic problems (6)–(8) and (14)–(16). For those problems, the kernel takes the form

$$k(t-s) = G(\hat{l}, t \mid \hat{l}, s), \qquad 0 \le s \le t, \tag{67}$$

as given by (9). Using the image representation in (9), it was shown in [**8**], for a single equation, that the leading order contribution to the asymptotic analysis arises from the $n = 0$ term. This allows (67) to be replaced by a similar specification, namely that the kernel has the asymptotic behavior,

$$k(t-s) \sim \frac{1}{2}[\pi(t-s)]^{-1/2}, \quad \text{as} \quad t \longrightarrow s. \tag{68}$$

Kernels with this type of asymptotic behavior were treated in a slightly more general context in [**10**] for a scalar equation. For kernels that behave like (68), we will determine the growth rate near blow-up for the two types of nonlinearities considered in Section 4.

Our analysis parallels that of [**10**] for a scalar equation. We assume that blow-up occurs simultaneously for each component of $u = [u_1, u_2]$, so that

$$u_j(t) \longrightarrow \infty, \quad \text{as} \quad t \longrightarrow t_c < \infty, \quad j = 1, 2. \tag{69}$$

To carry out the asymptotic analysis of (1), it is convenient to introduce the transformation,

$$\eta = (t_c - t)^{-1} - \eta_0, \qquad \eta_0 = (t_c)^{-1}, \qquad w_j(\eta) = u_j(t), \qquad j = 1, 2, \tag{70}$$

whereupon (69) implies that

$$w_j(\eta) \longrightarrow \infty, \quad \text{as} \quad \eta \longrightarrow \infty, \quad j = 1, 2. \tag{71}$$

Our analysis does not determine t_c, but rather demonstrates an asymptotic balance as $\eta \to \infty$ for (1) whereby the leading order growth rate near blow-up is determined.

In terms of the transformation (70), the components of (1) take the form,

$$w_j(\eta) = \int_0^\eta k\{(\eta-\sigma')[(\sigma'+\eta_0)(\eta+\eta_0)]^{-1}\}\Phi_j(\sigma')\,d\sigma', \quad j=1,2, \tag{72}$$

where

$$\Phi_j(\eta) = (\eta+\eta_0)^{-2}F_j\{w_{3-j}(\eta)+h_{3-j}[t_c-(\eta+\eta_0)^{-1}]\}. \tag{73}$$

Our analysis of (72) as $\eta \to \infty$ is based upon certain techniques developed in [**2**] for the asymptotic expansion of integrals, as adapted in [**10**] for scalar integral equations like (1). By setting $\sigma' = \eta\tau$ in (72) and utilizing (68), it follows that

$$w_j(\eta) \sim \eta I_j(\eta) = \eta\int_0^1 [\pi(1-\tau)]^{-1/2}\varphi_j(\eta\tau)\,d\tau, \quad \text{as } \eta \longrightarrow \infty, \tag{74}$$

where

$$\begin{aligned}\varphi_j(\eta\tau) &= (\eta\tau+\eta_0)^{1/2}\Phi_j(\eta\tau)\\ &= (\eta\tau+\eta_0)^{-3/2}F_j\{w_{3-j}(\eta\tau)+h_{3-j}[t_c-(\eta\tau+\eta_0)^{-1}]\},\\ &\qquad j=1,2.\end{aligned} \tag{75}$$

To investigate the asymptotic behavior of $I_j(\eta)$ as $\eta \to \infty$, the technique of [**2**] is to employ the Parseval formula for Mellin transforms to convert that integral to a more suitable form. It follows from [**10**] that (74) becomes

$$w_j(\eta) \sim \frac{\eta}{4\pi i}\int_{c-i\infty}^{c+i\infty} \eta^{-z}\frac{\Gamma(1-z)}{\Gamma(3/2-z)}M[\varphi_j(\tau);z]\,dz, \quad \text{as} \quad \eta \longrightarrow \infty, \quad j=1,2, \tag{76}$$

where the Mellin transform is defined by

$$M[\varphi_j(\tau);z] = \int_0^\infty z^{z-1}\varphi_j(\tau)\,d\tau. \tag{77}$$

In (76) the vertical path of integration in the complex z-plane lies within the strip of analyticity of the integrand. The advantage of (76), as demonstrated in [**2**], is that the asymptotic behavior of the integral is determined by the asymptotic behavior of $\varphi_j(\eta)$ as $\eta \to \infty$. It follows from (75) that

$$\varphi_j(\eta) \sim \eta^{-3/2} F_j[w_{3-j}(\eta)], \quad \text{as} \quad \eta \longrightarrow \infty, \quad j = 1, 2. \tag{78}$$

To proceed further requires that the explicit form of the nonlinearity be given.

For the case of the power law nonlinearities (55), we have

$$\begin{aligned} \varphi_j(\eta) \sim \eta^{-3/2}[w_{3-j}(\eta)]^{p_j}, \qquad p_j > 1, \\ \text{as} \quad \eta \longrightarrow \infty, \quad j = 1, 2, \end{aligned} \tag{79}$$

where $p_1 \geq p_2$. Then, to achieve an asymptotic balance to leading order in (76), we assume that

$$w_j(\eta) \sim A_j(\eta)^{l_j}, \quad \text{as} \quad \eta \longrightarrow \infty, \quad j = 1, 2. \tag{80}$$

It then follows that

$$\begin{aligned} \varphi_j(\eta) \sim A_{3-j}^{p_j}(\eta)^{p_j l_{3-j} - 3/2}, \\ \text{as} \quad \eta \longrightarrow \infty, \quad j = 1, 2. \end{aligned} \tag{81}$$

From (81) it can be determined (see [**2**]) that $M[\varphi_j; z]$ has a simple pole at $z = 3/2 - p_j l_{3-j}$ and

$$\begin{aligned} M[\varphi_j; z] \sim -\frac{(A_{3-j})^{p_j}}{z - (3/2 - p_j l_{3-j})} \\ \text{as} \quad z \longrightarrow \frac{3}{2} - p_j l_{3-j}, \quad j = 1, 2. \end{aligned} \tag{82}$$

To compute the asymptotic behavior of the integral in (76), the vertical path is displaced to the right. In doing so, the pole implied by (82) is encountered before that of $\Gamma(1-z)$ at $z = 1$ whenever

$$3/2 - p_j l_{3-j} < 1, \qquad j = 1, 2, \tag{83}$$

and hence it provides the leading order contribution. Thus, for (80), the asymptotic equality (76) takes the form

$$A_j(\eta)^{l_j} \sim A_{3-j}^{p_j} \frac{\Gamma(p_j l_{3-j} - 1/2)}{2\Gamma(p_j l_{3-j})} (\eta)^{p_j l_{3-j} - 1/2}, \quad \text{as} \quad \eta \longrightarrow \infty, \quad j = 1, 2. \tag{84}$$

The asymptotic balance of (84) requires that

$$l_j = \frac{p_j + 1}{2(p_1 p_2 - 1)}, \quad A_j = \left\{ \frac{2^{p_j+1}\Gamma(p_j l_{3-j})}{\Gamma(p_j l_{3-j} - 1/2)} \left[\frac{\Gamma(p_{3-j} l_j)}{\Gamma(p_{3-j} l_j - 1/2)} \right]^{p_j} \right\}^{1/(p_1 p_2 - 1)}, \quad j = 1, 2. \tag{85}$$

These results verify that the restriction of (83) is appropriate.

Thus, for the case of the power law nonlinearities (55), it follows that the growth rates near blow-up for $u = [u_1, u_2]$ are given by

$$u_j(t) \sim A_j(t_c - t)^{-(p_j+1)/(2(p_1 p_2 - 1))}, \quad \text{as} \quad t \longrightarrow t_c, \quad j = 1, 2, \tag{86}$$

where A_j is given by (85).

For the case of the exponential nonlinearities (61), we have

$$\varphi_j(\eta) \sim \eta^{-3/2} \exp[\gamma_j w_{3-j}(\eta)], \qquad \gamma_j > 0, \quad \text{as} \quad \eta \longrightarrow \infty, \quad i = 1, 2, \tag{87}$$

where $\gamma_1 \geq \gamma_2$. Then to achieve an asymptotic balance to leading order in (76), we assume that

$$w_j(\eta) \sim \log(A_j \eta^{l_j}) \sim l_j \log \eta + \log A_j, \quad \text{as} \quad \eta \longrightarrow \infty, \quad j = 1, 2. \tag{88}$$

It then follows that

$$\varphi_j(\eta) \sim (A_{3-j})^{\gamma_j} (\eta)^{\gamma_j l_{3-j} - 3/2}, \quad \text{as} \quad \eta \longrightarrow \infty, \quad j = 1, 2, \tag{89}$$

and analogous to (82), we have the principal pole of $M[\varphi_j; z]$ given by

$$M[\varphi_j; z] \sim -\frac{(A_{3-j})^{\gamma_j}}{z - (3//2 - \gamma_j l_{3-j})}, \quad \text{as} \quad z \longrightarrow 3/2 - \gamma_j l_{3-j}, \quad j = 1, 2. \tag{90}$$

Again the asymptotic behavior of the integral in (76) is found by displacing the vertical path to the right. In order for this to yield a logarithmic term to balance (88), it is necessary for the pole implied by (90) to coalesce with that of $\Gamma(1-z)$ at $z = 1$. Thus we must impose the conditions

$$3/2 - \gamma_j l_{3-j} = 1, \quad j = 1, 2. \tag{91}$$

Thus the asymptotic equality (76) takes the form

$$l_j \log \eta \sim \frac{(A_{3-j})^{\gamma_j}}{2\pi^{1/2}} \log \eta, \quad \text{as} \quad \eta \longrightarrow \infty, \quad j = 1, 2. \tag{92}$$

It follows from (91) and (92) that an asymptotic balance is achieved with

$$l_j = \frac{1}{2\gamma_{3-j}}, \qquad A_j = \left(\frac{\pi^{1/2}}{\gamma_j}\right)^{1/\gamma_{3-j}}. \tag{93}$$

Thus for the case of the exponential nonlinearities (61), it follows that the growth rates for $u = u[u_1, u_2]$ near blow-up are given by

$$u_j(t) \sim \frac{1}{2\gamma_{3-j}} \log\left(\frac{1}{t_c - t}\right), \quad \text{as} \quad t \longrightarrow t_c, \quad j = 1, 2. \tag{94}$$

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Department of Engineering Sciences and Applied Mathematics, Northwestern University, Evanston, IL 60208

Department of Mathematics, Northern Arizona University, Flagstaff, AZ 86011

Department of Mathematics, University of Southwestern Louisiana, Lafayette, LA 70504