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A "NATURAL" STATE-SPACE FOR AN AEROELASTIC CONTROL SYSTEM

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ABSTRACT. In [**16**] a set of sufficient conditions was formulated to guarantee the "proper" invertibility of a finite Hilbert transform appearing in the derivation of a dynamic model for an aeroelastic system. Here we outline how these conditions, with some modification, can be used to construct a "natural" (i.e., motivated by the model derivation) state-space, appropriate for control design purposes. In the process we also provide a detailed discussion on a somewhat "controversial" statement in [**16**].

1. Introduction. Well posed state-space formulations for control systems governed by singular integro-differential equations have been studied in a sequence of papers, [**6, 15, 7, 13, 8, 14**], hoping to achieve the following objectives: (i) find an (infinite dimensional) state-space such that the control problem for the singular integro-differential equation can be equivalently formulated as the abstract Cauchy problem

$$\dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}u(t), \qquad z(0) = z_0, \tag{1.1}$$

where the linear operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup on the selected state-space; (ii) make the selection in (i) in such a way that $\mathcal{A}$ in (1.1) satisfies a dissipative estimate on the selected space.

To summarize the findings of the papers listed above, we note that for a large class of singular integro-differential equations of neutral type the well-posedness (i.e., (i)) has been established on state-spaces of the type $R^n \times L_{p,g}$, where $g(\cdot)$ denotes a weight function. To achieve objective (ii) one has to find the appropriate weight function which is not straightforward in the sense that there is no systematic procedure

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using the properties of the underlying equation for its construction. We believe that the results in [**16**] give some insights in this direction (see discussion below). In the remaining part of this section, we explain the interest in singular integro-differential equations of neutral-type in relation to the approximate solutions of certain control problems arising in aeroelasticity.

In [**4**] a complete dynamical model in terms of a set of singular integro-differential equations of neutral type was derived to study the elastic motions of a three degree-of-freedom structure (an airfoil with trailing edge flap) placed into two-dimensional unsteady flow. The main motivation there (i.e., in [**4**]) was to establish a theoretical and computational framework, more suitable than the original coupled ODE-PDE initial-boundary value problem, for the determination of a control law (using the flap as an active control surface) for the stabilization of the structure (flutter suppression) when subjected to arbitrary disturbances. In view of our beginning remarks, objectives (i) and (ii) have to be satisfied by the abstract equation (1.1) associated with the neutral equation describing the model. In addition, conditions required on initial data (i.e., state-space) need to be compatible with those which justify the validity of the model derivation. Concerning the latter, the key step is to establish the applicability of Söhngen's inversion formula [**18**] for finite Hilbert transforms [**12**] for all initial data in the state-space where (i) and (ii) hold.

We note that this point in [**16**] is completely overlooked and therefore the otherwise interesting results in [**16**] on the solutions of the airfoil equation (or on a correctly posed formulation for inverting the finite Hilbert transform) have no correspondence, along the lines of the study in [**5**], to aeroelastic control applications. On the positive side, in the next section we discuss how the observation in [**16**] can lead to the construction of a weighted L_p space as a potential state-space candidate for the modeling singular neutral equation.

2. The evolution equation for $\dot{\Gamma}$. We begin by recalling (see, e.g., [**4**]) that the derivation of the dynamic model for the fluid-structure interaction problem leads to the equation

$$\int_{-1}^{1} \frac{\gamma(t,\alpha)}{\alpha - x}\, d\alpha = 2\pi w_a(t,x) + \int_0^{\infty} \frac{\dot{\Gamma}(t-\sigma)}{1 - x + U\sigma}\, d\sigma \tag{2.1}$$

where w_a is the downwash function (describing the motions of the airfoil), $\gamma(\cdot,\cdot)$ is the circulation per unit distance (angular velocity of the fluid) on the airfoil, and $\dot{\Gamma}$ is the derivative of the extended total airfoil circulation function

$$\Gamma(t) = \begin{cases} \int_{-\infty}^{t} \psi(s)\,ds & t < 0, \\ \eta + \int_0^t (\partial/\partial s) \int_{-1}^{1} \gamma(s,\alpha)\,d\alpha\,ds & t \geq 0, \end{cases} \tag{2.2}$$

with initial data $(\eta, \psi(\cdot)) \in R \times L_{2,g}(-\infty, 0)$ ($g(\cdot)$ is a weight function).

Note that (2.2) indicates a relationship between $\dot{\Gamma}$ and γ and consequently both sides of (2.1) have $\dot{\Gamma}$ dependence. Note also that we are not interested in finding γ, rather we are interested in obtaining an evolution equation for $\dot{\Gamma}$.

Remark 2.1. In [**16**], for fixed $t = t_0$, essentially the equation

$$\int_{-1}^{1} \frac{\gamma_0(\alpha)}{\alpha - x}\,d\alpha = 2\pi w_{a,0}(x) + \int_0^{\infty} \frac{G_0(-\sigma)}{1 - x + U\sigma}\,d\sigma \tag{2.3}$$

is investigated, where $\gamma_0(\alpha) = \gamma(t_0, \alpha)$, $w_{a,0}(x) = w_a(t_0, x)$ and $G_0(\cdot) = \dot{\Gamma}_{t_0}$, i.e., $G_0(-\sigma) = \dot{\Gamma}(t_0 - \sigma)$, $\sigma \in (0, \infty)$. There are several problems with equation (2.3) if one wants to claim that it has use in aeroelastic applications, as found in [**1, 3**], for example,

(i) Both sides of (2.3) contain unknown quantities, i.e., $\gamma_0(\cdot)$ and $G_0(-\sigma)$ for $0 < \sigma \leq t_0$.

(ii) Any assumptions imposed on $G_0(-\sigma) = \dot{\Gamma}(t_0 - \sigma)$ for $0 < \sigma \leq t_0$ will have implications on $\gamma_0(\sigma)$ via the defining equation (2.2).

In sum, equation (2.3) alone has nothing to do with aeroelastic models, and consequently the results of the papers [**16**] and [**5**] are not comparable.

Assume for the moment that (2.1) can be inverted using Söhngen's formula (see, e.g., [**18, 12**] to give

$$\gamma(t,x) = \frac{2}{\pi}\sqrt{\frac{1-x}{1+x}}\left\{ \int_{-1}^{1} \sqrt{\frac{1+y}{1-y}}\, \frac{w_a(t,y)}{x-y}\,dy + \frac{1}{2\pi} \int_{-1}^{1} \sqrt{\frac{1+y}{1-y}}\, \frac{dy}{x-y} \int_0^{\infty} \frac{\dot{\Gamma}(t-\sigma)}{1-y+U\sigma}\,d\sigma \right\}. \tag{2.4}$$

Now integrating both sides of equation (2.4) with respect to x over $(-1,1)$, we obtain

$$\begin{aligned}\int_{-1}^{1}\gamma(t,x)\,dx &= 2\int_{-1}^{1}\sqrt{\frac{1+y}{1-y}}w_a(t,y)\,dy\\ &\quad+\int_0^\infty\Big(1-\sqrt{1+\frac{2}{U\sigma}}\Big)\dot\Gamma(t-\sigma)\,d\sigma.\end{aligned}\tag{2.5}$$

Considering the term $\int_0^\infty \dot\Gamma(t-\sigma)\,d\sigma$, on the righthand side of (2.5) we can write that

$$\begin{aligned}\int_0^\infty \dot\Gamma(t-\sigma)\,d\sigma &= \int_{-\infty}^0 \dot\Gamma(t+s)\,ds = \int_{-\infty}^t \dot\Gamma(u)\,du\\ &= \int_{-\infty}^0 \psi(u)\,du + \Gamma(t) - \Gamma(0^+)\\ &= \int_{-\infty}^0 \psi(u)\,du + \eta + \int_{-1}^1 \gamma(t,x)\,dx\\ &\quad - \int_{-1}^1 \gamma(0^+,x)\,dx - \eta,\end{aligned}\tag{2.6}$$

where we have used the defining equation (2.2). Substituting (2.6) into (2.5), we obtain

$$\begin{aligned}\int_{-1}^{1}\gamma(t,x)\,dx &= 2\int_{-1}^{1}\sqrt{\frac{1+y}{1-y}}w_a(t,y)\,dy\\ &\quad-\int_0^\infty\sqrt{1+\frac{2}{U\sigma}}\dot\Gamma(t-\sigma)\,d\sigma\\ &\quad+\int_{-\infty}^0\psi(u)\,du+\int_{-1}^1\gamma(t,x)\,dx\\ &\quad-\int_{-1}^1\gamma(0^+,x)\,dx.\end{aligned}$$

After obvious manipulations, and taking $\eta = \int_{-1}^1 \gamma(0^+,x)\,dx$, we obtain the following evolution equation for $\dot\Gamma(\cdot)$

$$\begin{aligned}\int_{-\infty}^0\sqrt{1-\frac{2}{Us}}\dot\Gamma(t+s)\,ds &= 2\int_{-1}^1\sqrt{\frac{1+y}{1-y}}w_a(t,y)\,dy\\ &\quad+\Big[\int_{-\infty}^0\psi(u)\,du-\eta\Big].\end{aligned}\tag{2.7}$$

Note that the righthand side of (2.7) is normally a smooth function of t (see, e.g., [**4, 5**]) and thus we can differentiate the basic circulation equation (2.7) to get the evolution equation of neutral type

(2.8)

$$\frac{d}{dt}\int_{-\infty}^{0}\sqrt{1-\frac{2}{Us}}\dot{\Gamma}(t+s)\,ds = 2\int_{-1}^{1}\sqrt{\frac{1+y}{1-y}}\frac{\partial}{\partial t}(w_a(t,y))\,dy, \quad t>0,$$
$$\dot{\Gamma}(s)=\psi(s), \quad s\in(-\infty,0).$$

Now we can attempt to justify the inversion formula we used to obtain (2.4) from (2.1) as follows:

(i) Conducting a study similar to the ones undertaken in [**16, 12**], "desired' properties of $\dot{\Gamma}(t-\sigma)$ can be deducted from equation (2.1) to validate the applicability of Söhngen's inversion formula.

(ii) "Desired" properties of $\dot{\Gamma}(t-\sigma)$, $t>0$, $\sigma\in(0,\infty)$, appearing on the righthand side of (2.1) can possibly be guaranteed by selecting an appropriate state-space X, $\psi(\cdot)\in X$, and showing well-posedness of the neutral-equation (2.8) on that state-space.

The difficult part of the process is the selection of the appropriate state-space X in (ii) (which is sort of an inverse problem) and clearly any information obtained in (i) in this direction is highly beneficial.

Remark 2.2. If, modifying a condition in [**16**, Proposition 1], we require $\dot{\Gamma}_t(\cdot)$ to satisfy

$$\text{(2.9)} \qquad \int_0^\infty \sigma^{-\beta}||\dot{\Gamma}(t-\sigma)||^p\,d\sigma<\infty; \quad \beta>0; \quad 1\le p<\infty,\ t\ge 0,$$

then applicability of the inversion formula is guaranteed as long as we can find X, such that for any $\psi\in X$ we have $\dot{\Gamma}_t\in X$ and for any function $\psi\in X$ condition (2.9) is satisfied. Of course, a natural candidate for X is $X=L_{p,k}(-\infty,0)$ with $k(\sigma)=\sigma^{-\beta}$, $\sigma>0$, i.e., X is a weighted L_p-space with weight-function $k(\cdot)$. Note that although condition (2.9) looks very reasonable it has not been used before to generate state-space candidates for the neutral equation (2.8).

Remark 2.3. Concerning the well-posedness of the neutral-equation (2.8), a variety of studies were carried out (see, e.g., [**6, 15, 7**]), mostly on the finite delay version of (2.8), and well-posedness results were produced on product spaces of the type $R\times L_p$, but no dissipative estimates were established on the operator A appearing in the associated abstract equation (1.1). Finally, the search for dissipative estimates led to the consideration of weighted L_2-spaces [**14, 8**], i.e., conditions similar to condition (2.9) were introduced to provide dissipativeness (needed for approximation) for operator A in (1.1). The results for neutral equations with finite-delay were extended in [**13**] (and later in [**9**]) for a class of infinite delay neutral equations including (2.8). In particular, Theorem 2.1 in [**9**] can be used to show well-posedness of (2.8) on the state-space $L_{2,k}(-\infty,0)$ with $k(\sigma)=e^{w\sigma}\sqrt{1+2/(U\sigma)}$, $\sigma\in(0,\infty)$.

In [**4, 5**], the evolution equation (2.8) describing the flow is coupled to the rigid-body dynamics of the airfoil in order to obtain a complete set of functional differential equations that describes the composite system. (The coupling is represented by the "downwash" function, w_a, in (2.8) and the presence of $\dot{\Gamma}_t$ in the expressions for the aerodynamic loads in the rigid body equations (see [**5, 4**] for details).) The resulting model for the aeroelastic system has the form

$$\frac{d}{dt}\left[Ax(t)+\int_{-\infty}^{0}A(s)x(t+s)\,ds\right] = Bx(t)+\int_{-\infty}^{0}B(s)x(t+s)\,ds+Gu(t) \tag{2.10}$$

for $t>0$, where $x(t)=\operatorname{col}(h(t),\theta(t),\beta(t),\dot{h}(t),\dot{\theta}(t),\dot{\beta}(t),\Gamma(t),\dot{\Gamma}_t)=\operatorname{col}(x_1(t),[x_2]_t)$. The functions h,θ,β denote the plunge, pitch angle and flap angle, respectively. The 8×8 matrix A is singular (each entry of the last row is zero) while the $A_{88}(\cdot)$ component of the 8×8 matrix function $A(s)$ is weakly singular ($A_{88}(s)=((Us-2)/Us)^{1/2}$). The control $u(\cdot)$ in (2.10) is a torque applied at the flap-hinge line.

3. A singular neutral system. In this section we study the neutral equation (2.10) in the context of functional analytic semi-group theory. In particular, we formulate an equivalent abstract evolution equation of the form

$$\dot{z}(t)=\mathcal{A}z(t)+F(t)$$

and show that linear operator $\mathcal{A}$ is the infinitesimal generator of a

C_0-semigroup on the weighted product space $Z = R^7 \times L_{2,k}$, where $k(s) = e^{-\omega s}\sqrt{1 - 2/Us}$, $s \in (-\infty, 0)$.

Consider the neutral system

$$\begin{cases} (d/dt)D_1(x_1(t), [x_2]_t) = L_1(x_1(t), [x_2]_t) + f(t), & t \geq 0, \\ (d/dt)D_2([x_2]_t) = L_2(x_1(t)), & t \geq 0, \end{cases} \tag{3.1}$$

with initial conditions

$$x_1(0) = \eta, \qquad [x_2]_0 = \varphi(\cdot). \tag{3.2}$$

The linear operators D_1, D_2, L_1 and L_2 appearing in (3.1) are assumed to have the following representations for $(\eta, \varphi) \in R^7 \times C([-\infty, 0]; R)$

$$D_1(\eta, \varphi) = I\eta + \int_{-\infty}^{0} A_{12}(s)\varphi(s)\, ds, \tag{3.3}$$

$$L_1(\eta, \varphi) = B_{11}\eta + B_{12}\varphi(0) + \int_{-\infty}^{0} B_{12}(s)\varphi(s)\, ds, \tag{3.4}$$

$$D_2(\varphi) = \int_{-\infty}^{0} g(s)\varphi(s)\, ds, \tag{3.5}$$

$$L_2(\eta) = B_{21}\eta, \tag{3.6}$$

where I, B_{11}, B_{12}, B_{21}, $A_{12}(\cdot)$, $g(\cdot)$ and $B_{12}(\cdot)$ denote nonzero blocks in the system matrices A, B, $A(\cdot)$ and $B(\cdot)$, i.e.,

$$A = \begin{bmatrix} I_{7\times 7} & 0 \\ 0 & 0_{1\times 1} \end{bmatrix}, \qquad A(s) = \begin{bmatrix} 0_{7\times 7} & A_{12}(s) \\ 0 & g(s)_{1\times 1} \end{bmatrix}, \tag{3.7}$$

$$B = \begin{bmatrix} B_{11_{7\times 7}} & B_{12} \\ B_{21} & 0_{1\times 1} \end{bmatrix}, \qquad B(s) = \begin{bmatrix} 0_{7\times 7} & B_{12}(s) \\ 0 & 0_{1\times 1} \end{bmatrix}. \tag{3.8}$$

$A_{12}(\cdot)$ and $B_{12}(\cdot)$ are smooth functions and

$$g(s) = \sqrt{1 - \frac{2}{Us}}. \tag{3.9}$$

Define the linear operator $\mathcal{A}$ by

$$\mathcal{A}(D_1(\eta, \varphi), \varphi) = (L_1(\eta, \varphi), \dot{\varphi}) \tag{3.10}$$

on domain

$$D(\mathcal{A}) = \{(\eta, \varphi) \in R^7 \times L_{2,k} \mid \dot{\varphi} \in L_{2,k}, D_2\dot{\varphi} = L_2\eta\}. \tag{3.11}$$

Remark 3.1. Introducing the function $z(t) = (D_1(x_1(t), [x_2]_t)$, it can be shown that the neutral system (3.1)–(3.2) with conditions (3.3)–(3.9) is equivalent to the abstract Cauchy-problem

$$\dot{z}(t) = \mathcal{A}z(t) + F(t) \tag{3.12}$$

with initial data

$$z(0) = (D_1(\eta, \varphi), \varphi), \tag{3.13}$$

where $(\eta, \varphi) \in Z$, $\mathcal{A}$ is defined by (3.10)–(3.11) and

$$F(t) = (f(t), \theta), \tag{3.14}$$

where θ denotes the zero function in $L_{2,k}$.

Next we show the well-posedness of (3.12)–(3.13), i.e., that $\mathcal{A}$ generates a C_0-semigroup on $R^7 \times L_{2,k}$.

Theorem 3.2. *Let* $g : (-\infty, 0] \to R^+$ *be given by* (3.9) *and for* $(\eta, \varphi) \in Z = R^7 \times L_{2,k}$ *define* $||\cdot||_Z$ *by*

$$||(\eta, \varphi)||_Z = \left(|\eta|^2 + \int_{-\infty}^0 |\varphi(s)|^2 k(s)\, ds\right)^{1/2}. \tag{3.15}$$

Then the linear operator $\mathcal{A}$ *defined by* (3.10)–(3.11) *is the infinitesimal generator of a* C_0*-semigroup* $\{S(t)\}_{t\geq 0}$ *on* Z*.*

Proof. The claim of the theorem follows (see [**17**]) by showing that $\mathcal{D}(\mathcal{A})$ is a dense subspace of Z, and for sufficiently large $\gamma > 0$, the operator $\mathcal{A} - \gamma$ is dissipative and its range is the whole space Z.

We note first that density of $\mathcal{D}(\mathcal{A})$ in Z is an easy consequence of the density of the subspace $\{\varphi \in L_{2,k} : \int_{-\infty}^0 g(s)\dot{\varphi}(s)\, ds = 0\}$ in $L_{2,k}$ (see [**9**, Lemma 2.3]).

Next we show that for large enough $\gamma > 0$ we have

$$\langle (L_1(\eta,\varphi),\dot\varphi) - \gamma(D_1(\eta,\varphi),\varphi),(D_1(\eta,\varphi),\varphi)\rangle_Z \le 0, \qquad (D_1(\eta,\varphi),\varphi)\in\mathcal{D}(\mathcal{A}), \tag{3.16}$$

to guarantee dissipativeness of $\mathcal{A}-\gamma$. In particular, we have

$$\begin{aligned}\langle \mathcal{A}(D_1(\eta,\varphi),\varphi) - \gamma(D_1(\eta,\varphi),\varphi),(D_1(\eta,\varphi),\varphi)\rangle_Z &= [L_1(\eta,\varphi)-\gamma D_1(\eta,\varphi)]^T D_1(\eta,\varphi)\\ &\quad + \int_{-\infty}^0 (\dot\varphi(s)-\gamma\varphi(s))\varphi(s)e^{-\omega s}g(s)\,ds\\ &= \mathrm{I}+\mathrm{II}.\end{aligned} \tag{3.17}$$

Expression (3.4) and obvious manipulations yield the estimate

$$\begin{aligned}\mathrm{I} &= \bigg(B_{11}D_1(\eta,\varphi) - B_{11}\int_{-\infty}^0 A_{12}(s)\varphi(s)\,ds + B_{12}\varphi(0)\\ &\qquad + \int_{-\infty}^0 B_{12}(s)\varphi(s)\,ds\bigg)^T D_1(\eta,\varphi) - \gamma|D_1(\eta,\varphi)|^2\\ &\le (c_1-\gamma)|D_1(\eta,\varphi)|^2 + c_2||\varphi||_k^2 + c_3|\varphi(0)|^2,\end{aligned} \tag{3.18}$$

where c_1, c_2 and c_3 are appropriate constants.

Straightforward calculations, the domain condition (3.11), an integration by parts, and the positivity of $\dot g(\cdot)$ yield for the second term (II) in (3.17) the following:

(3.19)

$$\begin{aligned}\mathrm{II} &= \int_{-\infty}^0 (\dot\varphi(s)-\gamma\varphi(s))\varphi(s)e^{-\omega s}g(s)\,ds\\ &= \int_{-\infty}^0 \dot\varphi(s)(\varphi(s)-e^{\omega s}(\varphi(0))e^{-\omega s}g(s)\,ds\\ &\quad + \varphi(0)\int_{-\infty}^0 \dot\varphi(s)g(s)\,ds - \gamma||\varphi||_k^2\\ &= \varphi(0)L_2\eta + \int_{-\infty}^0 (\dot\varphi(s)-\omega e^{\omega s}\varphi(0))(\varphi(s)-e^{\omega s}\varphi(0))e^{-\omega s}gs\,ds\\ &\quad + \omega\varphi(0)\int_{-\infty}^0 (\varphi(s)-e^{\omega s}\varphi(0))g(s)\,ds - \gamma||\varphi||_k^2\end{aligned}$$

$$
\begin{aligned}
&= \varphi(0)L_2\eta \\
&\quad - \frac{1}{2}\int_{-\infty}^{0}(\varphi(s) - e^{\omega s}\varphi(0))^2(e^{-\omega s}\dot{g}(s) - \omega e^{-\omega s}g(s))\,ds \\
&\quad + \omega\varphi(0)\int_{-\infty}^{0}(\varphi(s) - e^{\omega s}\varphi(0))g(s)\,ds - \gamma||\varphi||_k^2 \\
&\le c_4\varphi^2(0) + c_5|D_1(\eta,\varphi)|^2 + \left(c_6 + \frac{\omega}{2} - \gamma\right)||\varphi||_k^2 \\
&\quad - \frac{\omega}{2}\int_{-\infty}^{0}\varphi^2(0)e^{\omega s}g(s)\,ds.
\end{aligned}
$$

Combining (3.18) and (3.19), we have

$$
\begin{aligned}
\langle \mathcal{A}(D_1(\eta,\varphi),\varphi) - \gamma(D_1(\eta,\varphi),\varphi), (D_1(\eta,\varphi),\varphi)\rangle_Z
&\le (c_1 + c_5 - \gamma)|D_1(\eta,\varphi)|^2 \\
&\quad + \left(c_2 + c_6 + \frac{\omega}{2} - \gamma\right)||\varphi||_k^2 \\
&\quad + \left(c_3 + c_4 - \frac{\omega}{2}\right)\int_{-\infty}^{0}\varphi^2(0)e^{\omega s}g(s)\,ds.
\end{aligned} \tag{3.20}
$$

Note that since $\omega\int_{-\infty}^{0} e^{\omega s}g(s)\,ds \to \infty$, as $\omega \to \infty$, it is possible to select $\omega > 0$ such that the righthand side of (3.20) is nonpositive and that guarantees the dissipativeness of $\mathcal{A} - \gamma$.

Finally, the range condition is established by noting the nonsingularity of the matrix

$$
\begin{bmatrix}
\gamma I - B_{11} & \gamma\int_{-\infty}^{0} A_{12}(s)e^{\gamma s}\,ds - B_{12} - \int_{-\infty}^{0} B_{12}(s)e^{\gamma s}\,ds \\
-B_{21} & \gamma\int_{-\infty}^{0} g(s)e^{\gamma s}\,ds
\end{bmatrix}
$$

for sufficiently large values of γ. □

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