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ASYMPTOTIC PROPERTIES VIA AN INTEGRODIFFERENTIAL INEQUALITY

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ABSTRACT. Recent results about asymptotic properties for integrodifferential equations in $\Re^n$ are studied in Hara, et al. [**6**] by analyzing a Liapunov function $v(\cdot)$ satisfying

$$v'(t) \leq -\alpha v(t) + \int_0^t \omega(t,s)v(s)\,ds, \quad t \geq t_0 \geq 0.$$

We will extend the techniques in [**6**] to the study of integrodifferential equations

$$x'(t) = A(t)\left[x(t) + \int_{\#}^t F(t,s)x(s)\,ds\right], \quad t \geq t_0 \geq 0, \ (\# = 0 \text{ or } -\infty),$$

in real Hilbert spaces with unbounded linear operators $A(\cdot)$, when a Liapunov function $v(\cdot)$ satisfies

$$v'(t) \leq -\alpha v(t) + \sqrt{v(t)}\int_{\#}^t \omega(t,s)\sqrt{v(s)}\,ds, \quad t \geq t_0 \geq 0 \ (\# = 0 \text{ or } -\infty).$$

The results include uniform stability and asymptotic stability, as well as uniform boundedness and ultimate boundedness, which are not studied in [**6**]. The above integrodifferential equations occur in viscoelasticity and in heat conduction for materials with memory.

1. Introduction. In qualitative studies of differential or integrodifferential equations, Liapunov or Liapunov-Razumikhin methods are

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very effective in analyzing the asymptotic properties, which include stabilities and boundedness.

It is commonly required that the derivative of a Liapunov function (or functional) along a solution be negative all the time, or Razumikhin conditions are imposed to only require that the derivative be negative when the Liapunov function reaches its maximum at t on $[0,t]$ or on $(-\infty,t]$. See, e.g., [**3, 4**].

Recently, Hara, Yoneyama and Miyazaki [**6**] presented some new general results about the asymptotic properties of integrodifferential equations in $\Re^n$, in which the condition on a Liapunov function $v(\cdot)$ is such that

$$v'(t) \leq -\alpha v(t) + \int_0^t \omega(t,s)v(s)\,ds, \qquad t \geq t_0 \geq 0, \tag{1.1}$$

with

$$\limsup_{t\to\infty} \int_0^t \omega(t,s)\,ds < \alpha$$

and

$$\lim_{u\to\infty} \int_0^t \omega(u,s)\,ds = 0 \quad \text{for } t > 0. \tag{1.2}$$

It can be seen that, for large t, $v'(t) < 0$ if $v(t)$ reaches its maximum on $[0,t]$. So it is in the right spirit of a Razumikhin condition. However, due to the special forms of inequalities (1.1)–(1.2), the proofs of the results are very simple and elementary.

We are interested in the asymptotic properties of integrodifferential equations in real Hilbert spaces, especially equations of the form

$$\begin{aligned} &x'(t) = A(t)\left[x(t) + \int_0^t F(t,s)x(s)\,ds\right], \qquad t \geq t_0 \geq 0,\\ &x(s) = \phi(s), \qquad 0 \leq s \leq t_0, \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} &x'(t) = A(t)\left[x(t) + \int_{-\infty}^t F(t,s)x(s)\,ds\right], \qquad t \geq t_0 \geq 0,\\ &x(s) = \phi(s), \qquad s \leq t_0, \end{aligned} \tag{1.4}$$

with unbounded linear operators $A(\cdot)$, which can be used to model heat conduction or viscoelasticity for materials with memory. For example, as stated in Grimmer and Liu [**3**], equations from heat conduction

$$\begin{aligned} q(t,x) &= -Eu_x(t,x) \\ &\qquad - \int_{\#}^{t} b(t-s)u_x(s,x)\,ds, \quad (\# = 0 \text{ or } -\infty) \\ u_t(t,x) &= -\partial q(t,x)/\partial x + f(t,x), \end{aligned}$$

can be rewritten as (assuming $E=1$)

$$u_t(t,x) = \frac{\partial^2}{\partial x^2}\left[u(t,x) + \int_{\#}^{t} b(t-s)u(s,x)\,ds\right] + f(t,x).$$

Similar to [**3**], we can construct a Liapunov function $v(\cdot)$ for equations (1.3) and (1.4) in a natural way. Then, deriving an inequality in the spirit of (1.1), we end up with

$$\begin{gathered} v'(t) \le -\alpha v(t) + \sqrt{v(t)} \int_{\#}^{t} \omega(t,s)\sqrt{v(s)}\,ds, \\ t \ge t_0 \ge 0, \ (\# = 0 \text{ or } -\infty) \end{gathered} \tag{1.5}$$

where ω is determined by F.

It will be seen from Lemmas 3.7 and 3.8 that transformations can be made so that we may assume $v(t) > 0$ in (1.5). And then, if we divide $2\sqrt{v(t)}$ in (1.5) and set $y(t) = \sqrt{v(t)}$, the inequality for $y(\cdot)$ will have the same form as (1.1). This indicates that techniques used in [**6**] can be extended to the study of (1.3) and (1.4).

In Section 2 we will summarize and generalize those results in [**6**] concerning inequality (1.1). They are then applied in Section 3 to establish results concerning uniform stability and asymptotic stability, as well as uniform boundedness and ultimate boundedness, which are not studied in [**6**]. Transformations from inequality (1.5) to (1.1) are also given in Section 3. Finally, in Section 4 the results in Section 3 are applied to equations (1.3) and (1.4) to obtain the uniform stability, asymptotic stability, uniform boundedness and ultimate boundedness. An example in [**6**] indicates that, in general, uniform asymptotic stability is not expected under conditions (1.1) and (1.2). See [**5, 7**] for studies of uniform asymptotic stability with other conditions.

We remark that similar results about stability and asymptotic stability for (1.3) and (1.4), when $A(t) = A$, $t \geq 0$ and $F(t,s) = F(t-s)$ are given in Grimmer and Liu [**3**] by Razumikhin techniques. We will see that the treatment here is much simpler; only an elementary integrodifferential inequality is used. And results about uniform boundedness and ultimate boundedness can also be obtained in a unified way.

Finally, note that for the equation

$$\begin{gathered} x'(t) = Ax(t) + \int_{\#}^{t} F(t,s)x(s)\,ds, \\ t \geq t_0 \geq 0, \; (\# = 0 \text{ or } -\infty) \end{gathered} \tag{1.6}$$

in $\Re^n$ with inner product $\langle , \rangle$, we can define $v = \langle x, x\rangle = ||x||^2$; then

$$\begin{aligned} v'(t) &= 2\langle x(t), x'(t)\rangle \\ &= 2\Big\langle x(t), Ax(t) + \int_{\#}^{t} F(t,s)x(s)\,ds \Big\rangle \\ &\leq 2\langle x(t), Ax(t)\rangle + 2\sqrt{v(t)} \int_{\#}^{t} ||F(t,s)|| \sqrt{v(s)}\,ds. \end{aligned} \tag{1.7}$$

If A is a negative definite or a stable matrix, then inequality (1.5) can also occur naturally in this situation. Therefore the study of the asymptotic properties of (1.6) an also be carried out using inequality (1.5).

2. A lemma. In this section we prove a lemma which will be used in the next section to study asymptotic properties. The lemma summarizes and generalizes those results in [**6**] concerning inequality (1.1). The proofs are similar to those in [**6**]; we give them here for completeness.

Lemma 2.1. *Let $\alpha > 0$ be a constant. Assume that $\omega(t,s) \geq 0$ is continuous for $0 \leq s \leq t$, with*

$$\limsup_{t\to\infty} \int_{\#}^{t} \omega(t,s)\,ds < \alpha \tag{2.1}$$

and

$$\lim_{u\to\infty}\int_{\#}^{t}\omega(u,s)\,ds=0 \quad \textit{for } t>0\ (\#=0 \textit{ or } -\infty).$$

Consider all functions $v(\cdot)=v(\cdot,t_0):[0,\infty)\to[0,\infty)$, *or* $(-\infty,\infty)\to[0,\infty)$, *such that*

$$v'(t)\le-\alpha v(t)+\int_{\#}^{t}\omega(t,s)v(s)\,ds, \qquad t\ge t_0\ge 0\ (\#=0 \textit{ or } -\infty). \tag{2.2}$$

(a) *For any constant* $M>0$ *fixed, if* $v(s)=v(s,t_0)\le M$ *for* $0\le s\le t_0$ *when* $\#=0$, *or for* $s\le t_0$ *when* $\#=-\infty$, *then* $v(t)\le M$, $t\ge t_0$.

(b) *For any constants* $B>0$, $B_0>0$ *and* $t_0\ge 0$, *there is a constant* $T=T(B,B_0,t_0)>0$ *such that if* $v(\cdot)=v(\cdot,t_0)\ge 0$ *is a function satisfying (2.2) with* $v(s)\le B_0$, *(here* $0\le s\le t_0$ *when* $\#=0$ *or* $s\le t_0$ *when* $\#=-\infty$*), then* $v(t)<B$, $t\ge T+t_0$.

Proof. We consider only $\#=0$ since the proof for $\#=-\infty$ is the same. From (2.1), there exist $\gamma>0$ and $T_0>0$, such that

$$0<\gamma<\alpha \quad\text{and}\quad \int_0^t\omega(t,s)\,ds<\gamma, \qquad t\ge T_0. \tag{2.3}$$

(a) Assume that $v(s)\le M$, $0\le s\le t_0$. If $\{v(t)\le M, t\ge t_0\}$ is not true, then there exist $M_1>M$ and $t_1>t_0$ such that $v(t_1)=M_1$ and $v(t)\le M_1$, $t\in[0,t_1]$. Here we only consider the case $T_0\le t_0$. The proof for $T_0>t_0$ is the same as in [**6**] and is omitted for simplicity. Now, from (2.2), we have for $T_0\le t_0\le t\le t_1$,

$$v(t)\le v(t_0)e^{-\alpha(t-t_0)}+\int_{t_0}^{t}e^{-\alpha(t-r)}\left(\int_0^r\omega(r,s)v(s)\,ds\right)dr. \tag{2.4}$$

Hence, from (2.3),

$$M_1=v(t_1)\le Me^{-\alpha(t_1-t_0)}+M_1\alpha\int_{t_0}^{t_1}e^{-\alpha(t_1-r)}\,dr \tag{2.5}$$

$$< M_1\left[e^{-\alpha(t_1-t_0)}+\alpha\int_{t_0}^{t_1}e^{-\alpha(t_1-r)}\,dr\right]=M_1, \tag{2.6}$$

which is a contradiction. Therefore, (a) is proven.

(b) If the result is not true, then there exist constants $B > 0$, $B_0 > 0$ and $t_0 \geq 0$, and a sequence $\{v_k(\cdot) = v_k(\cdot, t_0)\}$ satisfying (2.2) and $t_k \to \infty$, as $k \to \infty$, such that $v_k(s) \leq B_0$, $0 \leq s \leq t_0$ and $v_k(t_k) \geq B$.

Accordingly, we can denote P the nonempty set of all such sequences $\{v_k\}$, and

$$P^* = \{\limsup_{k\to\infty} v_k(t_k) \mid \{v_k\} \in P\}. \tag{2.7}$$

Now, from result (a), we see that for any $\{v_k\} \in P$, $\{v_k(s) \leq B_0, 0 \leq s \leq t_0\}$ implies $v_k(t_k) \leq B_0$ for $t_k \in [0, \infty)$. Thus, $P^* \subseteq [B, B_0]$. Therefore,

$$\infty > L \equiv \max\{p \mid p \in P^*\} \geq B > 0.$$

By (2.3) there is a θ with $\gamma/\alpha < \theta < 1$. As $(\alpha\theta + \gamma)L/(2\theta\alpha) < L$, there is a $\{v_k\} \in P$ such that

$$\limsup_{k\to\infty} v_k(t_k) > (\alpha\theta + \gamma)L/(2\theta\alpha). \tag{2.8}$$

From the definition of L, it is easily seen that for this $\{v_k\} \in P$, there is an $H > 0$ such that

$$v_k(t) \leq L/\theta, \qquad k \geq H, \qquad t \geq t_H. \tag{2.9}$$

Now from (2.1) we can find $t^* > \max\{H, t_H, t_0, T_0\}$ such that

$$\int_0^{t_H} \omega(u, s)\, ds < \frac{(\alpha\theta - \gamma)L}{2\theta B_0}, \qquad u \geq t^*. \tag{2.10}$$

Thus, from (2.2), (2.3), (2.9) and (2.10), one has for $k \geq H$ and $t \geq t^*$ (note that from result (a), $\{v_k(s) \leq B_0, 0 \leq s \leq t_0\}$ implies $v_k(t) \leq B_0$,

$t \geq 0$)

(2.11)
$$v_k(t) \leq v_k(t^*)e^{-\alpha(t-t^*)} + \int_{t^*}^{t} e^{-\alpha(t-r)}\left(\int_0^{t_H} \omega(r,s)v_k(s)\,ds\right)dr$$

(2.12)
$$+ \int_{t^*}^{t} e^{-\alpha(t-r)}\left(\int_{t_H}^{r} \omega(r,s)v_k(s)\,ds\right)dr$$

(2.13)
$$\leq B_0 e^{-\alpha(t-t^*)} + B_0\int_{t^*}^{t} e^{-\alpha(t-r)}\left(\int_0^{t_H} \omega(r,s)\,ds\right)dr$$

(2.14)
$$+ \frac{L}{\theta}\int_{t^*}^{t} e^{-\alpha(t-r)}\left(\int_0^{r} \omega(r,s)\,ds\right)dr$$

(2.15)
$$\leq B_0 e^{-\alpha(t-t^*)} + \frac{(\alpha\theta - \gamma)L}{2\theta\alpha} + \frac{L\gamma}{\theta\alpha}$$

(2.16)
$$\leq B_0 e^{-\alpha(t-t^*)} + \frac{(\alpha\theta + \gamma)L}{2\theta\alpha}.$$

Let k be large so that $t_k > t^*$. Hence, from (2.8) and (2.16),

$$\frac{(\alpha\theta + \gamma)L}{2\theta\alpha} < \limsup_{k\to\infty} v_k(t_k) \leq \frac{(\alpha\theta + \gamma)L}{2\theta\alpha},$$

which is a contradiction. This proves (b). $\square$

Remark 2.2. If $\omega(t,s) = \omega(t-s)$, then condition (2.1) is equivalent to $\int_0^\infty \omega(s)\,ds < \alpha$.

3. The asymptotic properties. In this section we will study the asymptotic properties under conditions (2.1) and (2.2), or conditions (2.1) and (1.5). The results include uniform stability, asymptotic stability, uniform boundedness and ultimate boundedness. They can be applied to any differential or integrodifferential equations, if for which a Liapunov function satisfying (2.1) and (2.2), or (2.1) and (1.5) can be constructed.

First, for convenient reference, we give the following standard definitions for the case $\# = 0$. The definitions for $\# = -\infty$ can be stated accordingly. Note that, in the following, we use "system" to denote any differential or integrodifferential equation in a space with norm $||\cdot||$.

Definition 3.1. For any $t_0 \geq 0$ and any continuous function ϕ on $[0, t_0]$, a *solution* $u(\cdot, t_0, \phi)$ of a "system" is a function on $[0, \infty)$ satisfying the "system" for $t \geq t_0$, and $u(s) = \phi(s)$ for $s \in [0, t_0]$.

Definition 3.2. Solutions $u(\cdot) = u(\cdot, t_0, \phi)$ of a "system" are *uniformly bounded* if, for each $B_1 > 0$ there is a $B_2 = B_2(B_1) > 0$, such that $\{||\phi(s)|| \leq B_1, 0 \leq s \leq t_0\}$ implies $||u(t)|| < B_2$, $t \geq t_0$.

Definition 3.3. Solutions $u(\cdot) = u(\cdot, t_0, \phi)$ of a "system" are *ultimate bounded* if there is a bound $B > 0$ such that, for each $B_3 > 0$ and $t_0 \geq 0$, there is a $T = T(B, B_3, t_0) > 0$ such that $\{||\phi(s)|| \leq B_3,\ 0 \leq s \leq t_0\}$ implies $||u(t)|| < B$, $t \geq T + t_0$.

Definition 3.4. Assume that $u \equiv 0$ is a solution of a "system." Then solution $u \equiv 0$ is *stable* if, given $\varepsilon > 0$ and $t_0 \geq 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that $||\phi(s)|| < \delta$ on $[0, t_0]$ implies $||u(t, t_0, \phi)|| < \varepsilon$ for $t \geq t_0$. It is uniformly stable if it is stable and the δ is independent of t_0.

Definition 3.5. Assume $u \equiv 0$ is a solution of a "system." Then solution $u \equiv 0$ is asymptotically stable if it is stable and, for any $t_0 \geq 0$, there exists a constant $r = r(t_0) > 0$ such that $||\phi(s)|| < r$ on $[0, t_0]$ implies $u(t, t_0, \phi) \to 0$ as $t \to \infty$.

Applying Lemma 2.1, we now have the following result.

Theorem 3.6. *Assume that there exist functions* ("wedges") W_i, $i = 1, 2$, *with* $W_i : [0, \infty) \to [0, \infty)$ *and* W_1 *strictly increasing. Further, assume that there exists a* (*Liapunov*) *function* V *such that, for solutions* $u(\cdot)$ *of a "system,"*

(c1) $W_1(||u(t)||) \leq V(u(t)) \leq W_2(||u(t)||)$, (c2) $v(t) \equiv V(u(t))$ *satisfies* (2.1) *and* (2.2).

Then solutions of the "system" are uniformly bounded and ultimate bounded. Also, if $u \equiv 0$ *is a solution of the "system", then the zero solution is uniformly stable and asymptotically stable.*

Proof. Again, we consider only $\# = 0$ since the proof for $\# = -\infty$ is the same. From the definitions of W_i and condition (c1), we only need to prove the corresponding statements for $v(t) \equiv V(u(t))$. (Note that even $v(\cdot)$ may not be a solution of any "system.")

Uniform boundedness: For $B_1 > 0$, choose $B_2 = B_1$. Then, from Lemma 2.1(a), $v(s) \leq B_1$, $0 \leq s \leq t_0$ implies $v(t) \leq B_1 = B_2$, $t \geq t_0$.

Ultimate boundedness: Choose $B = 1$. Then from Lemma 2.1(b), for any B_3 (treat it as B_0 in Lemma 2.1(b)) > 0 and $t_0 \geq 0$, there is a $T = T(B, B_3, t_0) > 0$ such that $v(s) \leq B_3$, $0 \leq s \leq t_0$ implies $v(t) < B$, $t \geq T + t_0$.

Uniform stability: Given $\varepsilon > 0$, choose $\delta(\varepsilon) = \varepsilon$. Then, from Lemma 2.1(a), $||v(s)|| \leq \delta(\varepsilon)$ $(= \varepsilon)$, $0 \leq s \leq t_0$ implies that $||v(t)|| \leq \varepsilon$, $t \geq t_0$.

Asymptotic stability: Stability is proven already. Next, for any $t_0 \geq 0$, choose $r = r(t_0) = 1$. Then, from Lemma 2.1(b), for any ε (treat it as B in Lemma 2.1(b)) > 0, $B_0 \equiv r$ $(= 1)$, there is a $T = T(\varepsilon, 1, t_0) = T(\varepsilon, t_0) > 0$ such that $v(s) \leq B_0 = r$, $0 \leq s \leq t_0$ implies $v(t) \leq \varepsilon$, $t \geq T + t_0$. (Thus $v(t) \to 0$, $t \to \infty$.) □

Now we demonstrate that if $v(\cdot)$ satisfies (1.5), then transformations can be made so that inequality (1.1) is satisfied. Hence, the asymptotic properties stated in Theorem 3.6 can also be obtained using inequality (1.5). Once again, the results are shown for $\# = 0$.

Lemma 3.7 [**8**]. *Assume that* $v(t) \geq 0$, $t \geq 0$ *and satisfies* (1.5), *where* $\omega(t, s) \geq 0$. *Then for* $\bar{\varepsilon} > 0$, $g(t) \equiv v(t) + \bar{\varepsilon} e^{-\alpha t} > 0$ *and satisfies*

(1.5), *and* $y(t) \equiv \sqrt{g(t)}$ *satisfies*

$$y'(t) \leq -\frac{\alpha}{2} y(t) + \int_0^t \frac{\omega(t,s)}{2} y(s)\, ds, \qquad t \geq t_0 \geq 0. \tag{3.1}$$

Proof. The results are established by taking a derivative in t. For example, we have

$$\begin{aligned} g'(t) &\leq -\alpha v(t) + \sqrt{v(t)} \int_{\#}^t \omega(t,s)\sqrt{v(s)}\, ds - \alpha\bar{\varepsilon}e^{-\alpha t} \\ &\leq -\alpha[v(t) + \bar{\varepsilon}e^{-\alpha t}] + \sqrt{v(t)} \int_{\#}^t \omega(t,s)\sqrt{v(s)}\, ds \\ &\leq -\alpha g(t) + \sqrt{g(t)} \int_{\#}^t \omega(t,s)\sqrt{g(s)}\, ds, \end{aligned}$$

and

$$y'(t) = \frac{g'(t)}{2\sqrt{g(t)}} \leq -\frac{\alpha}{2}\sqrt{g(t)} + \int_{\#}^t \frac{\omega(t,s)}{2}\sqrt{g(s)}\, ds. \qquad \square$$

Lemma 3.8 [**8**]. *If* $g(t) \equiv v(t) + \bar{\varepsilon}e^{-\alpha t}$ *satisfies the conditions stated in Definition* 3.2 *through Definition* 3.5, *then so does* $v(\cdot)$. (*Note that even* $g(\cdot)$ *may not be a solution of any* "*system*").

Proof. The results can be verified by checking the conditions. For example, for Definition 3.4: Given $\varepsilon > 0$ and $t_0 \geq 0$. Since $g(\cdot)$ satisfies conditions in Definition 3.4, there exists a $\delta_1 = \delta_1(\varepsilon, t_0) > 0$ such that $||g(s)|| < \delta_1$ on $[0, t_0]$ implying $||g(t)|| < \varepsilon$ for $t \geq t_0$. Now let $\bar{\varepsilon} = \delta_1(\varepsilon, t_0)/2$ and choose $\delta = \delta_1(\varepsilon, t_0)/2$. Then $||v(s)|| < \delta$ on $[0, t_0]$ implying $||g(s)|| \leq ||v(s)|| + \bar{\varepsilon} \leq \delta_1$. Thus, $||v(t)|| \leq ||g(t)|| < \varepsilon$ for $t \geq t_0$. Similarly, other conditions in other definitions can be checked. (Note that $\alpha > 0$ and hence $e^{-\alpha t} \to 0$ as $t \to \infty$.) $\square$

The following result verifies that Theorem 3.6 is still valid if inequality (1.1) is replaced by inequality (1.5).

Theorem 3.9 [**8**]. *Assume that there exist functions* W_i, $i = 1, 2$, *with* $W_i : [0, \infty) \to [0, \infty)$, *and* W_1 *strictly increasing. Further, assume that there exists a function* V *such that, for solutions* $u(\cdot)$ *of a "system,"*

(c1) $W_1(||u(t)||) \le V(u(t)) \le W_2(||u(t)||)$,

(c2) $v(t) \equiv V(u(t))$ *satisfies* (2.1) *and* (1.5).

Then solutions of the "system" are uniformly bounded and ultimate bounded. Also, if $u \equiv 0$ *is a solution of the "system," then the zero solution is uniformly stable and asymptotically stable.*

Proof. Consider the functions g and y defined in Lemma 3.7. From (3.1) and (2.1), we see that the proof of Theorem 3.6 can be carried over to show that $y(\cdot)$ satisfies conditions in Definition 3.2 through Definition 3.5. Then it is clear that $g(\cdot) = y^2(\cdot)$ also satisfies conditions in Definition 3.2 through Definition 3.5. Now the results are established by applying Lemma 3.8. □

4. Integrodifferential equations. In this section we will apply the results in Theorem 3.9 to

$$
\begin{aligned}
x'(t) &= A(t)\left[x(t) + \int_0^t F(t,s)x(s)\,ds\right], \qquad t \ge t_0 \ge 0, \\
x(s) &= \phi(s), \qquad 0 \le s \le t_0,
\end{aligned} \tag{4.1}
$$

and

$$
\begin{aligned}
x'(t) &= A(t)\left[x(t) + \int_{-\infty}^t F(t,s)x(s)\,ds\right], \qquad t \ge t_0 \ge 0, \\
x(s) &= \phi(s), \qquad s \le t_0,
\end{aligned} \tag{4.2}
$$

with unbounded operators $A(\cdot)$ in real Hilbert space X with inner product $\langle , \rangle$. Since we only study asymptotic properties here, we will assume the existence and uniqueness of solutions, which can be found in, e.g., [**1, 2**]. We define a Liapunov function V for $z = (x, w) \in X \times X$ by

$$
V(z) = \langle x, x\rangle - 2\langle x, w\rangle + \frac{3}{2}\langle w, w\rangle. \tag{4.3}
$$

Also, we define $||z||^2 \equiv ||x||^2 + ||w||^2$ and let $z(t) = (x(t), w(t))$ with $x(\cdot)$ a solution of equation (4.1) or (4.2), and

$$w(t) = x(t) + \int_{\#}^{t} F(t,s)x(s)\,ds,$$
$$t \geq 0 \ (\# = 0 \text{ or } -\infty).$$

Then it is clear that, in order to prove the asymptotic properties of solutions $x(\cdot)$ of equation (4.1) or (4.2), we only need to prove the corresponding statements for $z(\cdot)$. We also note that $x \equiv 0$ is a solution of (4.1) and of (4.2).

Theorem 4.1. *Suppose that solutions $x(\cdot)$ of* (4.1) *and* (4.2) *exist and are unique on* $[0,\infty)$, *and suppose that for some constants $\lambda > 0$ and $\beta > 0$,*

$$\langle A(t)x, x\rangle \leq -\lambda\langle x, x\rangle, \qquad x \in D(A(t)), \quad t \geq 0, \tag{4.4}$$

and

$$\langle F(t,t)x, x\rangle \geq \beta\langle x, x\rangle, \qquad x \in X, \quad t \geq 0,$$

where D means domain. Then

(a) $||z||^2/5 \leq V(z) \leq 3||z||^2$,

(b) $v(t) \equiv V(z(t))$ *satisfies inequality* (1.5) *with $\# = 0$ for* (4.1) *and with $\# = -\infty$ for* (4.2), *where*

$$\alpha \equiv \min_{t\geq 0} \frac{1}{3}\left\{\lambda - \frac{3}{2}||F(t,t)||, 2\beta - \frac{3}{2}||F(t,t)||\right\},$$

and

$$\omega(t,s) \equiv (6 + 3\sqrt{6})\left\|\frac{\partial}{\partial t}F(t,s)\right\|.$$

So, if

$$\limsup_{t\to\infty} \int_0^t \omega(t,s)\,ds < \alpha$$

and

$$\lim_{u\to\infty} \int_0^t \omega(u,s)\,ds = 0 \quad \textit{for } t \geq 0,$$

then solutions of (4.1) *are uniformly bounded and ultimate bounded, and the zero solution of* (4.1) *is uniformly stable and asymptotically stable.*

Similarly, if

$$\limsup_{t\to\infty}\int_{-\infty}^{t}\omega(t,s)\,ds<\alpha$$

and

$$\lim_{u\to\infty}\int_{-\infty}^{t}\omega(u,s)\,ds=0\quad \textit{for } t\geq 0,$$

then solutions of (4.2) *are uniformly bounded and ultimate bounded, and the zero solution of* (4.2) *is uniformly stable and asymptotically stable.*

Proof. First, we have

$$\begin{aligned} V(z) &\geq ||x||^2-2||x||\,||w||+\frac{3}{2}||w||^2\\ &=(||x||-||w||)^2+\frac{1}{2}||w||^2\\ &=\frac{1}{6}(3||w||-2||x||)^2+\frac{1}{3}||x||^2. \end{aligned}\tag{4.5}$$

Thus, we obtain $||z||^2/5 \leq V(z) \leq 3||z||^2$. Next, differentiating $v(t)\equiv V(z(t))$ with respect to t yields (note that $*$ denotes convolution)

(4.6)

$$\begin{aligned} v'(t)&=\frac{d}{dt}V(z(t))\\ &=2\langle x'(t),x(t)\rangle-2\langle x'(t),w(t)\rangle-2\langle w'(t),x(t)\rangle+3\langle w'(t),w(t)\rangle\\ &=\langle A(t)w(t),w(t)\rangle-2\langle F(t,t)x(t),x(t)\rangle-2\left\langle x(t),\frac{\partial}{\partial t}F*x(t)\right\rangle\\ &\quad+3\langle F(t,t)x(t),w(t)\rangle+3\left\langle\frac{\partial}{\partial t}F*x(t),w(t)\right\rangle\\ &\leq-\lambda||w(t)||^2-2\beta||x(t)||^2+3||F(t,t)||\,||x(t)||\,||w(t)||\\ &\quad+(2||x(t)||+3||w(t)||)\left\|\frac{\partial}{\partial t}F*x(t)\right\| \end{aligned}$$

$$
\begin{aligned}
&\leq -\lambda ||w(t)||^2 - 2\beta ||x(t)|| + \frac{3}{2} ||F(t,t)|| (||x(t)||^2 + ||w(t)||^2) \\
&\quad + (2\sqrt{3} + 3\sqrt{2}) \sqrt{v(t)} \left\| \frac{\partial}{\partial t} F * x(t) \right\| \quad \text{(from (4.5))} \\
&\leq \left(-\lambda + \frac{3}{2} ||F(t,t)|| \right) ||w(t)||^2 + \left(-2\beta + \frac{3}{2} ||F(t,t)|| \right) ||x(t)||^2 \\
&\quad + (2\sqrt{3} + 3\sqrt{2}) \sqrt{v(t)} \left\| \frac{\partial}{\partial t} F * x(t) \right\| \\
&\leq -3\alpha (||x(t)||^2 + ||w(t)||^2) + (2\sqrt{3} + 3\sqrt{2}) \sqrt{v(t)} \left\| \frac{\partial}{\partial t} F * x(t) \right\| \\
&\leq -\alpha v(t) + (2\sqrt{3} + 3\sqrt{2}) \sqrt{v(t)} \left(\sqrt{3} \left\| \frac{\partial}{\partial t} F \right\| * \sqrt{v}(t) \right) \quad \text{(from (a))} \\
&\leq -\alpha v(t) + \sqrt{v(t)}\, (6 + 3\sqrt{6}) \left(\left\| \frac{\partial}{\partial t} F \right\| * \sqrt{v}(t) \right),
\end{aligned}
$$

where

$$\left\| \frac{\partial}{\partial t} F \right\| * \sqrt{v}(t) = \int_0^t \left\| \frac{\partial}{\partial t} F(t,s) \right\| \sqrt{v(s)}\, ds$$

and

$$\int_{-\infty}^t \left\| \frac{\partial}{\partial t} F(t,s) \right\| \sqrt{v(s)}\, ds$$

for (4.1) and (4.2), respectively. Thus, the results are established by applying Theorem 3.9. □

Remark 4.2. It is known that $A(t) \equiv \partial^2/\partial x^2$, $t \geq 0$ with domain $H_0^1(0,1) \cap H^2(0,1)$ satisfying (4.4) on $X = L^2(0,1)$ with $\lambda = 1$. Thus applications can be carried out. We omit them here.

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