JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 6, Number 3, Summer 1994

FAST NUMERICAL SOLUTION OF SINGULAR INTEGRAL EQUATIONS

P.P.B. EGGERMONT AND CH. LUBICH

ABSTRACT. We study a fast method for the numerical solution of Cauchy singular integral equations. The method is based on fast inversion of the principal part, and on multi-grid methods for the resulting Fredholm integral equation of the second kind. The inversion of the principal part is done via an approximate Wiener-Hopf factorization using FFTs. Under suitable smoothness assumptions, the algorithm requires $O(n \log n)$ operations to achieve an accuracy comparable to that of the trigonometric collocation method with n collocation points, which is known to give quasi-optimal approximations.

1. Introduction. We consider fast algorithms for the approximate solution of Cauchy integral equations over closed curves in the plane. We may write such equations as

$$c(\zeta)u(\zeta) + \frac{d(\zeta)}{\pi i}\int_\Gamma \frac{u(z)}{z-\zeta}\,dz + \frac{1}{2\pi i}\int_\Gamma k(\zeta,z)u(z)\frac{dz}{z} = f(\zeta) \tag{1.1}$$

over the unit circle in the complex plane $\Gamma = \{\zeta \in \mathbf{C} : |\zeta| = 1\}$, with integration direction counterclockwise. We will assume that c, d, k and f are fairly smooth functions. We write (1.1) in operator notation as

$$(A+K)u = f, \tag{1.2}$$

where $A = cI + dS$, with S denoting the singular integral operator, and where K is the above integral operator with kernel k. The operator $A+K$ can be invertible in its natural setting only if A is invertible, which is why A is called the principal part. The algorithm proposed in the present paper is actually a discretization of the "preconditioned" equation, a Fredholm integral equation of the second kind:

$$(I + A^{-1}K)u = A^{-1}f. \tag{1.3}$$

Received by the editors on February 21, 1994.

The proposed fast solution method is based on a collocation method combined with trigonometric interpolation and achieves its speed by an approximate Wiener-Hopf factorization using FFTs to approximate A^{-1} and a multi-grid method to solve (1.3). The fast Wiener-Hopf factorization was proposed and analyzed in a different context in our paper [**3**].

For the special case of *constant* coefficients $c, d \in \mathbf{C}$ in (1.1), a fast solution algorithm was previously proposed by Hackbusch [**6**, p. 272]. He noted that the collocation discretization of (1.2) then leads to a linear system $(A_n + K_n)u_n = f_n$ where A_n is a circulant matrix and is therefore readily inverted using FFT. The preconditioned equation $(I + A_n^{-1}K_n)u_n = A_n^{-1}f_n$, which can be viewed as a discretization of (1.3), is then solved by Hackbusch's multigrid method of the second kind. The algorithm of the present article reduces essentially to Hackbusch's method in the case of constant coefficients. The main contribution of the present paper is that it shows how to obtain similar computational efficiency in the general case of variable coefficients in (1.1).

We describe the algorithm in Section 2 and work out its error analysis in Section 3. As it turns out, by using n function evaluations and $O(n \cdot \log n)$ arithmetical operations, we get a solution approximation with a pointwise error bound that is asymptotically equivalent to that of the trigonometric collocation method. In our numerical experiments we actually observed errors of the same magnitude for both methods. The (fully discrete) trigonometric collocation method was recently studied by McLean et al. [**7**]. Our error analysis takes up some of their techniques.

2. The algorithm.

2.1. An inversion formula for the principal part. As usual in the analysis of singular integral equations, we rewrite $A = cI + dS$ as

$$A = aP + bQ, \tag{2.1}$$

where $a = c + d$, $b = c - d$ and P, Q are the projectors defined by $P = (I+S)/2$, $Q = (I-S)/2 = I - P$. We recall that P is the operator which cuts the coefficients with negative indices of an arbitrary Laurent series:

$$P\left(\sum_{\nu=-\infty}^{\infty} f_\nu \zeta^\nu\right) = \sum_{\nu=0}^{\infty} f_\nu \zeta^\nu.$$

When A is invertible (e.g., as a linear operator on a Hölder space), then its inverse can be expressed by an explicit formula which involves the Wiener-Hopf factorization of a/b. This is known from Gohberg and Fel'dman [**4**]. We discuss this for the moment in an informal manner.

The operator A in its natural setting is invertible if and only if (see, e.g., [**4, 8**])

$$\begin{gathered} a(\zeta) \neq 0, \qquad b(\zeta) \neq 0 \qquad \text{for } |\zeta| = 1, \\ \arg(a/b)(e^{it})|_{t=-\pi}^{\pi} = 0. \end{gathered} \tag{2.2}$$

The second condition says that the closed curve $(a/b)(\zeta)$, $|\zeta| = 1$, does not encircle the origin. Then a/b can be factored as

$$a/b = c^+ c^-, \tag{2.3}$$

where the Fourier coefficients $c_\nu^\pm$ of $c^\pm(\zeta)$ vanish for negative, respectively nonnegative, subscripts ν. The factors $c_\pm(\zeta)$ are obtained by computing the Fourier coefficients of $\log(a/b)$

$$\sum_{\nu=-\infty}^{\infty} l_\nu \zeta^\nu = \log \frac{a(\zeta)}{b(\zeta)}, \tag{2.4}$$

and setting

$$c^\pm(\zeta) := \exp(l^\pm(\zeta)), \tag{2.5}$$

where

$$l^-(\zeta) = \sum_{\nu=-\infty}^{-1} l_\nu \zeta^\nu, \qquad l^+(\zeta) = \sum_{\nu=0}^{\infty} l_\nu \zeta^\nu. \tag{2.6}$$

The solution $v = A^{-1} f$ of the equation $Av = f$ is then given by the formula

$$v = \left(\frac{1}{c^+} P + c^- Q\right) \frac{f}{b \cdot c^-}, \tag{2.7}$$

see Gohberg and Fel'dman [**4**, Chapter V.2]. This is readily verified upon noting that $(1/c^{+})P + c^{-}Q$ is the inverse of $c^{+}P + (1/c^{-})Q$.

2.2. Approximate inversion of the principal part. The following algorithm is a finite analog of the above procedure, which works with sequences of length n (typically a power of 2), such as $x = (x_\nu)_{\nu=0}^{n-1}$, which are extended n-periodically to arbitrary integer subscripts. We let r_n denote the restriction operator which evaluates a function on the unit circle at n equidistant points and collects the values in an n-vector. Thus, $r_n : C(\Gamma) \to \mathbf{C}^n$ is defined by

$$[r_n f]_\nu = f(\omega^\nu), \qquad \nu = 0, 1, \dots, n-1,$$

where $\omega = e^{2\pi i/n}$. We let F_n denote the discrete Fourier transform of length n:

$$[F_n x]_\nu = \sum_{\mu=0}^{n-1} \omega^{\nu\mu} x_\mu, \qquad \nu = 0, 1, \dots, n-1,$$

with its inverse F_n^{-1} given by

$$[F_n^{-1} y]_\mu = \frac{1}{n} \sum_{\nu=0}^{n-1} \omega^{-\nu\mu} y_\nu, \qquad \mu = 0, 1, \dots, n-1.$$

Further, we introduce the cutting operator Π by setting

$$[\Pi x]_\nu = \begin{cases} x_\nu, & \nu = 0, 1, \dots, n/2 - 1, \\ 0, & \nu = -n/2, \dots, -1. \end{cases}$$

Finally, we let $P_n = F_n \Pi F_n^{-1}$ and $Q_n = I - P_n$.

In the following algorithm, all operations on vectors, such as multiplication, division, taking the logarithm, and exponentiation are componentwise.

Algorithm 1. *Approximate computation of* $v = A^{-1}f$

Approximate Wiener-Hopf factorization [**3**]:

$$\begin{aligned} a_n &= r_n a, \\ b_n &= r_n b \\ c_n &= a_n/b_n \\ l_n &= \log c_n \\ l_n^+ &= P_n l_n \\ c_n^+ &= \exp l_n^+, \\ c_n^- &= c_n/c_n^+. \end{aligned}$$

Approximate inversion:

$$\begin{aligned} f_n &= r_n f \\ g_n &= f_n/(b_n \cdot c_n^-) \\ g_n^+ &= P_n g_n, \\ g_n^- &= g_n - g_n^+, \\ v_n &= g_n^+/c_n^+ + g_n^- c_n^-. \end{aligned}$$

Then v_n is the approximation to $r_n v$. This algorithm requires n evaluations of the data a, b, f and $O(n \log n)$ arithmetical operations using FFT. We write Algorithm 1 in shorthand as

$$v_n = B_n r_n f, \tag{2.8}$$

thus defining the matrix $B_n \in \mathbf{C}^{n\times n}$.

2.3. Discretization of the full equation. By means of the trapezoidal rule we approximate $r_n Kg$ for smooth g by $K_n r_n g \approx r_n Kg$ where

$$[K_n]_{\nu,\mu} = \frac{1}{n} k(\omega^\nu, \omega^\mu), \qquad \nu, \mu = 0, 1, \dots, n-1. \tag{2.9}$$

We get an approximation $u_n \in \mathbf{C}^n$ to $r_n u$ by solving the system of linear equations

$$u_n + B_n K_n u_n = B_n r_n f. \tag{2.10}$$

(We use in this paper the convention that a variable name with a *superscript* n denotes a continuous function, and the same variable name with *subscript* n refers to a related n-vector.) A continuous approximation $u^n = p^n u_n \in C(\Gamma)$ to u is obtained via the trigonometric interpolation operator $p^n : \mathbf{C}^n \to C(\Gamma)$ defined for data $y \in \mathbf{C}^n$ as

$$[p^n y](\zeta) = \sum_{\mu=-n/2}^{n/2-1} x_\mu \zeta^\mu, \qquad \zeta \in \Gamma,$$

where $x = (x_\mu)_{\mu=0}^{n-1}$ is determined by trigonometric interpolation: $[p^n y](\omega^\nu) = y_\nu$ for all ν. This interpolation condition amounts to $F_n x = y$, i.e.,

$$x = F_n^{-1} y.$$

If the standard multi-grid method of the second kind is used to solve (2.10), see Hackbusch [**5, 6**], then each iteration still requires $O(n^2)$ operations because of multiplications with the matrix K_n. In the case where the kernel k is smoother than f or a or b, the operation count can be brought down to $O(n \log n)$ without significant loss of accuracy through the following scheme (cf. Brandt and Lubrecht [**1**] for a related approach). In this scheme the $n \times n$ matrix K_n is replaced by its $m \times m$ dimensional approximation K_m with $m = \sqrt{n}$ (assumed to be an integer, e.g., for n a power of 4; otherwise m proportional to $\sqrt{n}$ would also do it):

$$(I + B_n p_{nm} K_m q_{mn}) u_n = B_n r_n f, \qquad \text{with } m = \sqrt{n}. \tag{2.11}$$

Here, $p_{nm} : \mathbf{C}^m \to \mathbf{C}^n$ is the coarse-to-fine grid mapping defined by trigonometric interpolation as

$$p_{nm} = r_n p^m, \tag{2.12}$$

and $q_{mn} : \mathbf{C}^n \to \mathbf{C}^m$ is the fine-to-coarse grid mapping, defined by duality as

$$\langle p_{nm} x, y \rangle_{\mathbf{C}^n} = \langle x, q_{mn} y \rangle_{\mathbf{C}^m}, \tag{2.13}$$

for all $x \in \mathbf{C}^m$, $y \in \mathbf{C}^n$, where $\langle \cdot, \cdot \rangle_{\mathbf{C}^n}$ denotes the scaled Euclidean inner product on $\mathbf{C}^n$,

$$\langle x, y \rangle_{\mathbf{C}^n} = \frac{1}{n} \sum_{\nu=0}^{n-1} x_\nu \overline{y_\nu}, \qquad x, y \in \mathbf{C}^n. \tag{2.14}$$

The actions of p_{nm} and q_{mn} can be computed in $O(n \log n)$ operations by FFT: $p_{nm}x$ is obtained by n/m FFTs of length m, and $q_{mn}y$ is composed of every (n/m)th entry of the discrete convolution $c * y$, where $c = p_{nm}d$ with $d = (m, 0, \dots, 0)^T \in \mathbf{C}^m$. A multi-grid method applied to (2.11) should reduce n by a factor of 4, and m by a factor of 2 in every coarsening step. It is reasonable to use trigonometric interpolation and pointwise restriction as grid transfer operators. For concreteness we formulate the two-grid version, cf. [**6**, Chapter 5]. For a given iterate $u_n^{(k)}$, the improved iterate $u_n^{(k+1)}$ is obtained via

$$
\begin{aligned}
v_n &= B_n r_n f,\\
\tilde{u}_n &= -B_n p_{nm} K_m q_{mn} u_n^{(k)} + v_n,\\
d_n &= (I + B_n p_{nm} K_m q_{mn})\tilde{u} - v_n,\\
e_n &= p_{nN}(I + B_N p_{NM} K_M q_{MN})^{-1} r_{Nn} d_n,\\
u_n^{(k+1)} &= \tilde{u}_n - e_n,
\end{aligned}
\tag{2.15}
$$

with $N = n/4,\ M = m/2$. In the multigrid version, the solution of the coarse-grid equation in the computation of e_n is replaced recursively by two iterations of the multigrid method on the coarser level. It will be seen that two iterations of the algorithm are sufficient to get an approximate solution of (2.11) whose error is below the approximation error of u_n.

The overall computational work is then $O(n \log n)$ for the approximate solution of problem (1.1) using n values of the right hand side f. The following section shows that the approximation obtained in this way is quasi-optimal.

3. Error analysis.

3.1. Analysis in Hölder-Zygmund spaces. Following the work of McLean et al. [**7**], Prössdorf and Silbermann [**8**], and reference therein, we find it appropriate to study equation (1.1) and its discretization in the framework of Hölder-Zygmund spaces $\mathcal{H}^s$, $s > 0$. We recall that for noninteger $s = m + \alpha$ ($m = 0, 1, 2, \dots, 0 < \alpha < 1$), the space $\mathcal{H}^s$ is just the space $C^{m,\alpha}$ of (periodic) functions whose mth derivative satisfies a

Hölder condition of order α. A norm on $\mathcal{H}^s$ is given by

$$||f||_s = ||f|| + \sup_{t;h\neq 0} \frac{|f^{(m)}(e^{i(t+h)}) - f^{(m)}(e^{it})|}{|h|^\alpha}. \tag{3.1}$$

Here and in the following $||\cdot||$ denotes the sup norm. For integer $s = m+1$, the space $\mathcal{H}^s$ consists of functions in $C^m(\Gamma)$ whose mth derivative satisfies the Zygmund condition. A norm on $\mathcal{H}^{m+1}$ is defined by

$$||f||_{m+1} = ||f|| + \sup_{t;h\neq 0} \frac{|f^{(m)}(e^{i(t+h)}) - 2f^{(m)}(e^{it}) + f^{(m)}(e^{i(t-h)})|}{|h|}. \tag{3.2}$$

Hölder-Zygmund spaces have several properties which make them an attractive setting for the numerical analysis of equation (1.1). One of these is their characterization as approximation spaces (see DeVore and Lorentz [**2**, Theorem 7.3.3]):

$$\mathcal{H}^s = \{f \in C : E_n(f) = O(n^{-s})\}, \tag{3.3}$$

where $E_n(f) = \min ||f - \varphi||$, with the minimum taken over all trigonometric polynomials φ of degree $\leq n$. Moreover,

$$||f||_{(s)} = ||f|| + \sup_n n^s E_n(f) \tag{3.4}$$

defines an equivalent norm on $\mathcal{H}^s$. This implies that $\mathcal{H}^s$ is actually a Banach algebra with identity. In particular, multiplication is continuous:

$$||f \cdot g||_s \leq M_s \cdot ||f||_s \cdot ||g||_s. \tag{3.5}$$

(This is obtained by taking φ, ψ as the best approximation polynomials of f, g in the identity $fg - \varphi\psi = (f-\varphi)g + f(g-\psi) + (f-\varphi)(g-\psi)$.) A consequence of (3.5) is that

$$\exp : \mathcal{H}^s \to \mathcal{H}^s \quad \text{is continuous.} \tag{3.6}$$

(This follows with the power series of the exponential function.) It is known from Prössdorf and Silbermann [**8**, assertion (iv), p . 239], that the spectrum of $f \in \mathcal{H}^s$ equals the image of the unit circle under f. Consequently, the usual symbolic calculus yields that

$$\text{(3.7)} \quad \log : \{f \in \mathcal{H}^s : f(\zeta) \neq 0 \text{ for } |\zeta| = 1, \\ \arg f(e^{it})|_{-\pi}^{\pi} = 0\} \to \mathcal{H}^s \quad \text{is continuous.}$$

Finally, from [**8**, p. 238], we have:

(3.8) The projectors $P, Q : \mathcal{H}^s \to \mathcal{H}^s$ are bounded linear operators.

These properties show that the procedure of Section 2.1 is well-defined on $\mathcal{H}^s$ and indeed provides the unique solution of the equation $Av = f$ posed in $\mathcal{H}^s$. We summarize this as follows.

Proposition 2. *Let $a, b \in \mathcal{H}^s$ satisfy condition* (2.2). *For $f \in \mathcal{H}^s$, the equation $Av = f$ has a unique solution $v \in \mathcal{H}^s$, which is given by* (2.3)–(2.6), *and*

$$\text{(3.9)} \qquad ||v||_s \leq C \cdot ||f||_s,$$

where the constant C depends only on the $\mathcal{H}^s$-norms of $a, b, 1/a, 1/b$.

In other words, the operator $A : \mathcal{H}^s \to \mathcal{H}^s$ is bounded and invertible, and

$$\text{(3.10)} \qquad ||A^{-1}||_{\mathcal{L}(\mathcal{H}^s)} \leq C.$$

Proposition 2 is actually a special case of Theorem 6.26 of Prössdorf and Silbermann [**8**], which also shows that condition (2.2) is necessary for the invertibility of $A = aP + bQ$.

For the error analysis that follows, we need bounds for the trigonometric interpolation operator $I^n = p^n r_n$ in various settings. If $f \in C(\Gamma)$, then $I^n f$ denotes the trigonometric polynomial $\sum_{\mu=-n/2}^{n/2-1} x_\mu \zeta^\mu$ that interpolates f at the points ω^ν, $\nu = 0, 1, \dots, n-1$. We have from McLean et al. [**7**, Section 5], that

$$\text{(3.11)} \qquad \begin{aligned} &\text{(i)} \quad ||I^n f|| + ||PI^n f|| \leq C \log n ||f||, \\ &\text{(ii)} \quad ||f - I^n f|| + ||P(f - I^n f)|| \leq C n^{-s} \log n ||f||_s, \end{aligned}$$

$$||I^n f||_s + ||PI^n f||_s \le C \log n ||f||_s, \tag{3.12}$$

and from Prössdorf and Silbermann [**8**, pp. 76–77], for $r < s$, that

$$\begin{aligned} &\text{(i)} \quad ||I^n f||_r \le C||f||_s, \\ &\text{(ii)} \quad ||f - I^n f||_r \le Cn^{r-s} \log n ||f||_s. \end{aligned} \tag{3.13}$$

3.2. Error bounds for the principal part. Let $v = A^{-1}f$, and let $v_n = B_n r_n f$ be the result of Algorithm 1. We will prove the following error bounds in the discrete and continuous maximum norms, both denoted by $|| \cdot ||$.

Theorem 3. *Let $a, b \in \mathcal{H}^t$ satisfy condition* (2.2), *and let $f \in \mathcal{H}^s$. If $t > s > 0$, then*

$$\begin{aligned} ||B_n r_n f - r_n A^{-1} f|| &\le C \cdot n^{-s} \log n \cdot ||f||_s, \\ ||p^n B_n r_n f - A^{-1} f|| &\le C \cdot n^{-s} (\log n)^2 \cdot ||f||_s. \end{aligned}$$

For $t = s$, such estimates still hold with one more factor of $\log n$. *The constant C depends only on s, t and on the $\mathcal{H}^t$-norms of $a, b, 1/a, 1/b$.*

Proof. We begin with a reformulation of Algorithm 1 that involves functions on the unit circle rather than n-vectors. From the properties of the trigonometric interpolation operator p^n, we see that

$$P_n \equiv F_n \Pi F_n^{-1} = r_n P p^n.$$

We can now reformulate Algorithm 1 in the following way: $v_n = B_n r_n f$ is obtained from

$$\begin{aligned} c_+^n &= \exp(PI^n \log(a/b)), \\ c_-^n &= \exp(QI^n \log(a/b)), \\ g_+^n &= PI^n \frac{f}{b \cdot c_-^n}, \\ g_-^n &= QI^n \frac{f}{b \cdot c_-^n}, \\ v_+^n &= g_+^n / c_+^n, \\ v_-^n &= g_-^n \cdot c_-^n, \\ v^n &= v_+^n + v_-^n, \\ v_n &= r_n v^n. \end{aligned} \tag{3.14}$$

We note that $c_\pm^n \in C(\Gamma)$ are such that $r_n c_\pm^n = c_n^\pm$ of Algorithm 1. We define the functions $c_\pm$, $g_\pm$, $v_\pm$, $v \in C(\Gamma)$ by analogous formulas in which the interpolation operator I^n is replaced by the identity operator. Then $v = A^{-1}f$ by Proposition 2. We now use the estimates in (3.11) to obtain that

$$||c_+^n - c_+|| \le Cn^{-t}\log n,$$

and the same bounds for the errors in c_-^n, $1/c_+^n$, $1/c_-^n$. Together with (3.11) and (3.5), this implies

$$\begin{aligned} ||g_+^n - g_+|| &\le \left\| PI^n\left\{\frac{f}{b}\left(\frac{1}{c_-^n} - \frac{1}{c_-}\right)\right\}\right\| + \left\|(PI^n - P)\frac{f}{bc_-}\right\| \\ &\le Cn^{-t}(\log n)^2||f|| + Cn^{-s}\log n||f||_s, \end{aligned}$$

and the same for g_-^n. We then get that the errors of v_+^n and v_-^n are bounded in the same way, and consequently we get the desired bound (3.7) for $v^n = v_+^n + v_-^n$ and hence also for $v_n = r_n v^n$. The claimed error bound for the interpolating polynomial $p^n v_n$ then follows upon writing

$$p^n v_n - v = p^n(v_n - r_n v) + (I^n v - v),$$

and using the above estimates for $v_n - r_n v$, the bound $||p^n|| \le C \cdot \log n$ (see (3.11.i)) as well as the bound from (3.11.ii)

$$||I^n v - v|| \le Cn^{-s}\log n||v||_s \le C'n^{-s}\log n||f||_s,$$

the last estimate coming from (3.9). $\square$

Corollary 4. *For the matrix B_n, we have the bound*

$$||B_n x|| \le C \cdot \log n \cdot ||x||, \qquad x \in \mathbf{C}^n.$$

Remarks. (a) Under the assumptions of Theorem 3, we have the additional error bound in Hölder-Zygmund spaces for $0 < r < s$:

$$(3.15) \qquad ||p^n B_n r_n f - A^{-1}f||_r \le C \cdot n^{-(s-r)}(\log n)^2 \cdot ||f||_s,$$

and also the bound

$$(3.16) \qquad ||p^n B_n r_n f||_s \le C \cdot (\log n)^2 \cdot ||f||_s.$$

This is obtained similarly to Theorem 3, by using properties like (3.5)–(3.8), and instead of (3.11) the bounds in (3.12) and (3.13).

(b) If a, b, k and f are analytic in an annulus around the unit circle (k analytic in both variables) and condition (2.2) is satisfied, then Algorithm 1 converges exponentially:

$$||p^n v_n - v|| \le C \cdot e^{-cn}, \tag{3.17}$$

with a $c > 0$. Compare [**3**, Theorem 8.3].

3.3. Error bounds for the discretization of the full equation. Combining Theorem 3 with techniques from Fredholm integral equations of the second kind, we will obtain the following error bounds. We assume throughout that (1.3) is uniquely solvable in $C(\Gamma)$ (and therefore also in $\mathcal{H}^s$ under our smoothness assumptions on the data).

Theorem 5. *Let $f \in \mathcal{H}^s$ and $a, b \in \mathcal{H}^t$ with $t > s > 0$. If $k \in \mathcal{H}^{2s}(\Gamma \times \Gamma)$, then the scheme* (2.11) *has a unique solution u_n for sufficiently large n which satisfies*

$$\begin{aligned} ||u_n - r_n u|| &\le C \cdot n^{-s} \log n \cdot ||f||_s, \\ ||p^n u_n - u|| &\le C \cdot n^{-s} (\log n)^2 \cdot ||f||_s. \end{aligned} \tag{3.18}$$

If k is only in $\mathcal{H}^s$, then this holds for the solution of (2.10).

Proof. The proof uses approximation and stability lemmas given below. We first prove the error estimate for the solution u_n of (2.10), assuming that $k \in \mathcal{H}^s(\Gamma \times \Gamma)$. We set $T = A^{-1}K$, $T_n = B_n K_n$. From (1.3) and (2.10) we get that the error vector $e_n = u_n - r_n u$ satisfies

$$e_n + T_n e_n = d_n$$

with

$$d_n = -(T_n r_n - r_n T)u + (B_n r_n - r_n A^{-1})f.$$

By Theorem 3, the last term is bounded in the maximum norm by $C \cdot n^{-s} \log n \cdot ||f||_s$. By Lemma 6 and because $||u||_s \le C' \cdot ||f||_s$, the maximum norm of $(T_n r_n - r_n T)u$ is bounded in the same way. The stability result, Lemma 7, then shows that $||e_n||$ is again bounded in this way, thus proving the first of the bounds (3.18) for the scheme (2.10). The second bound then follows as at the end of the proof of Theorem 3.

The proof for the solution of (2.11) is analogous, using the corresponding properties of $\widetilde{T}^n = B_n p_{nm} K_m q_{mn}$ with $m = \sqrt{n}$ instead of T^n. □

Lemma 6. *If* $k \in \mathcal{H}^s(\Gamma \times \Gamma)$*, then* $T_n = B_n K_n$ *satisfies*

$$||T_n x|| \leq C \cdot ||x||, \qquad x \in \mathbf{C}^n, \tag{3.19}$$

$$||p^n T_n r_n f||_s \leq C \cdot (\log n)^3 \cdot ||f||, \qquad f \in C(\Gamma), \tag{3.20}$$

$$||(T_n r_n - r_n T)g|| \leq C \cdot n^{-s} \log n \cdot ||g||_s, \qquad g \in \mathcal{H}^s, \tag{3.21}$$

where $T = A^{-1}K$*. If* $k \in \mathcal{H}^{2s}(\Gamma \times \Gamma)$*, then* $\widetilde{T}_n = B_n p_{nm} K_m q_{mn}$ *with* $m = \sqrt{n}$ *satisfies the same bounds.*

Proof. (a) Let $\pi^n : \mathbf{C}^n \to C(\Gamma)$ denote the piecewise linear interpolation operator. We have

$$T_n x = (B_n r_n)(p^n K_n r_n)\pi^n x.$$

Obviously we have $||\pi^n x|| \leq ||x||$, and Lemma 9 below shows that for $r < s$,

$$||(p^n K_n r_n)\pi^n x||_r \leq C \cdot ||\pi^n x||.$$

Theorem 3 gives us for $f \in \mathcal{H}^r$

$$||B_n r_n f|| \leq ||B_n r_n f - r_n A^{-1} f|| + ||r_n A^{-1} f|| \leq C \cdot ||f||_r,$$

and hence (3.19) follows by combining the above inequalities.

(b) We have

$$||p^n T_n r_n f||_s \leq ||p^n B_n r_n||_{\mathcal{L}(\mathcal{H}^s)} \cdot ||p^n K_n r_n f||_s,$$

and (3.20) follows from (3.16) and Lemma 9.

(c) We write

$$T_n r_n g - r_n T g = B_n (K_n r_n - r_n K) g + (B_n r_n - r_n A^{-1}) K g.$$

Using subsequently Theorem 3 and Lemma 9, we get (with $0 < r < s$)

$$\begin{aligned} ||B_n(K_n r_n - r_n K)g|| &\le C \cdot ||(p^n K_n r_n - K)g||_r \\ &\le C' \cdot n^{-s} \log n \cdot ||g||_s. \end{aligned}$$

Since K is a bounded operator on $\mathcal{H}^s$, Theorem 3 gives us the same bound for $||(B_n r_n - r_n A^{-1})Kg||$, and (3.21) follows.

(d) The results for $\widetilde{T}_n$ are obtained in the same way, using the properties of $\widetilde{K}_n = p_{nm} K_m q_{mn}$ stated in Lemma 9. □

Lemma 7. *Under the assumptions of Theorem* 5*, the matrices* $I+T_n$ *and* $I+\widetilde{T}_n$ *are invertible for sufficiently large* n*, with uniformly bounded inverses when* $\mathbf{C}^n$ *is equipped with the maximum norm.*

Proof. Let again $\pi^n : \mathbf{C}^n \to C(\Gamma)$ denote the piecewise linear interpolation operator. Then $I + r_n T \pi^n$ is the matrix of the piecewise linear collocation scheme for (1.3), which is known to have a uniformly bounded inverse for sufficiently large n, see, e.g., [**6**, Chapter 4]. We have

$$(T_n - r_n T \pi^n) T_n = (T_n r_n - r_n T) p^n T_n - r_n T (\pi^n r_n - p^n r_n) p^n T_n,$$

and Lemma 6 now shows that

$$||(T_n - r_n T \pi^n) T_n|| \to 0, \qquad n \to \infty.$$

A well-known argument from the numerical analysis of Fredholm integral equations, see, e.g., [**6**, p. 148], then shows that $I + T_n$ is invertible for sufficiently large n, and

$$(I+T_n)^{-1} = (I - (I + r_n T \pi^n)^{-1} (T_n - r_n T \pi^n) T_n)^{-1} (I - (I + r_n T \pi^n)^{-1} T_n).$$

As $||T_n||$ is uniformly bounded by (3.19), the result follows. The result for $\widetilde{T}_n$ is proved in the same way. □

3.4. Convergence of the multigrid method. Our previous estimates also give us the tools for showing convergence of the multigrid method.

Theorem 8. *Under the assumptions of Theorem* 5*, the multigrid method of the second kind for* (2.10) *or* (2.11)*, with trigonometric interpolation and pointwise restriction as grid transfer operators, converges in the maximum norm with a rate of* $O(n^{-s}(\log n)^5)$.

Proof. The result follows by using the estimates of Lemmas 6 and 7, and the bound (3.11) for trigonometric interpolation (and the triangle inequality and little else) in the multigrid convergence analysis of Hackbusch [**6**, Chapter 5]. In particular, the two-grid iteration matrix corresponding to (2.15) is (with $N = n/4$)

$$\begin{aligned}
&\{I - p_{nN}(I+\widetilde{T}_N)^{-1} r_{Nn}(I+\widetilde{T}_n)\}\widetilde{T}_n \\
&\quad = \{(I - p_{nN} r_{Nn}) + p_{nN}(I+\widetilde{T}_N)^{-1}[\widetilde{T}_N r_{Nn} - r_{Nn}\widetilde{T}_n]\}\widetilde{T}_n \\
&\quad = r_n\{(I - I^N) + p^N(I+\widetilde{T}_N)^{-1}[(\widetilde{T}_N r_N - r_N T) + r_{Nn}(r_n T - \widetilde{T}_n r_n)]\} \\
&\qquad \cdot (p^n \widetilde{T}_n).
\end{aligned}$$

Using subsequently the bounds (3.11)(ii) and (i) for trigonometric interpolation, the stability estimate of Lemma 7, the approximate estimate (3.21) twice and the smoothing property (3.20), we obtain that the norm of this matrix is bounded by $O(n^{-s}(\log n)^5)$. The generalization to the multigrid method is then obtained as in [**6**, Section 5.5.3]. □

3.5. Estimates for the trapezoidal rule. It remains to give the lemma for the (modified) trapezoidal rule approximation of the integral operator K to which we referred in the proof of Lemma 6.

Lemma 9. *If* $k \in \mathcal{H}^s(\Gamma \times \Gamma)$*, then* $K^n = p^n K_n r_n$ *satisfies*

$$||K^n f||_r \le \begin{cases} C \cdot \log n \cdot ||f||, & \text{for } r = s, \\ C \cdot ||f||, & \text{for } r < s, \end{cases} \tag{3.22}$$

$$||(K^n - K)g||_r \le C \cdot n^{-s} \log n \cdot ||g||_s, \quad \text{for } r < s, \tag{3.23}$$

for all $f \in C(\Gamma), g \in yl^s$.

If $k \in \mathcal{H}^{2s}(\Gamma \times \Gamma)$, *then the operator* $\widetilde{K}^n = p^n \widetilde{K}_n r_n$, *where* $\widetilde{K}_n = p_{nm} K_m q_{mn}$ *with* $m = \sqrt{n}$, *satisfies the same bounds.*

Proof. We give an outline of the proof for $\widetilde{K}^n$, which concerns the more interesting part of the lemma. The basic identity to observe is

$$\widetilde{K}^n f = I^m \kappa^n$$

with

$$\kappa^n(\zeta) = \frac{1}{2\pi i} \int_\Gamma I^n(f(*)[I^m k(\zeta, \cdot)](*))(z) \frac{dz}{z}, \tag{3.24}$$

where again $I^n = p^n r_n$ and $I^m = p^m r_m$ are the trigonometric interpolation operators on $C(\Gamma)$. This identity is seen from the definition (2.13) of q_{mn} by duality, which gives for $\mu = 0, 1, \dots, m-1$

$$[\widetilde{K}^n f](\omega_m^\mu) = \frac{1}{n} \sum_{\nu=0}^{n-1} f(\omega_n^\nu)[I^m k(\omega_m^\mu, \cdot)](\omega_n^\nu), \tag{3.25}$$

where $\omega_m = e^{2\pi i/m}$. This trapezoidal sum equals

$$[\widetilde{K}^n f](\omega_m^\mu) = \frac{1}{2\pi i} \int_\Gamma I^n(f(*)[I^m k(\omega_m^\mu, \cdot)](*))(z) \frac{dz}{z},$$

and hence (3.24) follows by noting $I^m = p^n p_{nm} r_m$. From (3.24), (3.25) and from the bounds (3.11) of the interpolation operators one deduces immediately bounds for $\widetilde{K}^n f$ and $(\widetilde{K}^n - K)g$ in the maximum norm, which are of the type (3.22) and (3.23). Taking finite differences of κ^n, which act only on the first argument of k in (3.24) whereas the interpolation operators act on the second argument, and then estimating as before, gives bounds of κ^n in the $\mathcal{H}^{2s}$-norm. The bounds (3.12) and (3.13) for I^m finally yield the result (even without the factor $\log n$ in (3.22) for $\widetilde{K}^n$). □

REFERENCES

1. A. Brandt and A. Lubrecht, *Multilevel matrix multiplication and fast solution of integral equations*, J. Comp. Phys. **90** (1991), 348–370.

2. R.A. DeVore and G.G. Lorentz, *Constructive approximation*, Springer-Verlag, Berlin, 1993.

3. P.P.B. Eggermont and Ch. Lubich, *Operational quadrature methods for Wiener-Hopf integral equations*, Math. Comp. **60** (1993), 699–718.

4. I.C. Gohberg and I.A. Fel'dman, *Convolution equations and projection methods for their solution*, Transl. Math. Monographs **41** (1974).

5. W. Hackbusch, *Multi-grid methods and applications*, Springer-Verlag, Berlin, 1985.

6. ———, *Integralgleichungen, Theorie und Numerik*, Teubner, Stuttgart, 1989.

7. W. McLean, S. Prössdorf and W.L. Wendland, *A fully-discrete trigonometric collocation method*, J. Integral Equations Appl. **5** (1993), 103–129.

8. S. Prössdorf and B. Silbermann, *Numerical analysis for integral and related operator equations*, Birkhäuser, Basel, 1991.

Department of Mathematical Sciences, University of Delaware, Newark, DE 19716

Mathematisch Institut, Universität Tübingen, Auf der Morgenstelle, D-72076 Tübingen