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VOLTERRA EQUATIONS WHICH MODEL EXPLOSION IN A DIFFUSIVE MEDIUM

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ABSTRACT. An investigation is made of certain nonlinear Volterra integral equations which model explosive behavior in a diffusive medium. The basic results provide criteria involving the kernel and the nonlinearity for the solution to experience blow-up. Supporting examples from solid combustion and shear band formation are provided. Also, the connection is made with one-dimensional parabolic partial differential equations.

1. Introduction. We examine a class of nonlinear Volterra equations motivated by certain models of a diffusive medium which can experience explosive behavior. We consider

$$u(t) = Tu(t) \equiv \int_{t_0}^{t} k(t-s)G[u(s),s]\,ds, \quad t \geq t_0, \tag{1.1}$$

where the nonlinearity has the form

$$G[u(t),t] = r(t)g[u(t)+h(t)]. \tag{1.2}$$

It is typical of explosive models that the nonlinear dependence on the solution is positive and increasing so that

$$g(u) > 0, \qquad g'(u) > 0, \qquad g''(u) > 0, \quad u > 0. \tag{1.3}$$

The given nontrivial functions $r(t)$ and $h(t)$ are allowed to enhance the explosive behavior by being nondecreasing. We will require that

$$r(t) > 0, \qquad r'(t) \geq 0, \quad t \geq t_0, \tag{1.4}$$

and

$$0 < h_0 \leq h(t) \leq h_\infty < \infty, \qquad h'(t) \geq 0, \quad t \geq t_0. \tag{1.5}$$

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The difference kernel in (1.1) reflects the diffusive nature of the problem and, hence, is taken to be nonnegative and decreasing,

$$k(t-s) \geq 0, \qquad k'(t-s) < 0, \quad t > s \geq t_0. \tag{1.6}$$

Two choices of the starting point in (1.1) will be allowed, namely, $t_0 = 0$ and $t_0 = -\infty$. The former corresponds to any finite value, after an appropriate shift of the independent variable. The latter arises in certain model problems where the time variable has been scaled so that its beginning is in the distant past.

Several examples of (1.1) which arise in the investigation of thermal ignition in solid combustible materials are special cases of

$$u(t) = \gamma \int_{-\infty}^{t} \left\{ \frac{1}{\sqrt{\pi(t-s)}} - \lambda e^{t-s} erfc\sqrt{t-s} \right\} e^{u(s)+\alpha s-\beta s^2}\, ds. \tag{1.7}$$

Here $u(t)$ is the perturbation of temperature above some noncombustible level, while $\alpha, \beta, \gamma, \lambda$ are various nonnegative parameters. The case of $\alpha = 1$, $\beta = \lambda = 0$, which corresponds to ignition driven by an imposed heat flux, has been examined in [**7,3,8**]. The case of $\alpha = \lambda = 1$, $\beta = 0$, which corresponds to ignition driven by a constant heat flux with allowance for convection cooling, was derived in [**4**]. The case of $\alpha = \lambda = 0$, $\beta = 1$, which corresponds to an imposed heat flux only marginally adequate for ignition, was shown in [**5**] to yield blow-up whenever γ is sufficiently large.

The formation of shear bands in steel, when subjected to very high strain rates, is accompanied by a dramatic rise in temperature. A theoretical model of that phenomenon has been developed recently in [**10**]. One particular variant of that model has the temperature $u(t)$ in the shear band governed by

$$u(t) = \gamma \int_{0}^{t} \frac{(1+s)^q [u(s)+1]^p}{\sqrt{\pi(t-s)}}\, ds. \tag{1.8}$$

Here γ, p, q are nonnegative material parameters related to the constitutive law for plastic straining.

The class of integral equations (1.1) with nonlinearity in the form of (1.2) also has an interesting connection to the theory of nonlinear parabolic partial differential equations with explosion solutions. A one-dimensional version of such parabolic problems has the form

$$\begin{gathered} v_t = v_{xx} + F[v(x,t),x], \quad -\infty < x < \infty,\ t > 0 \\ v(\pm\infty,t) = 0, \quad v(x,0) = v_0(x). \end{gathered} \tag{1.9}$$

Essentially all the results pertaining to such partial differential equations treat the nonlinearity as either having no explicit spatial dependence, i.e., $F = F[v]$, or at least as having very smooth spatial dependence. Certain situations in which the nonlinearity in (1.9) has a strongly singular spatial behavior can be related to (1.1). When

$$F[v(x,t),x] = 2\delta(x)g[v(x,t)], \tag{1.10}$$

then the initial-boundary value problem (1.9) can be related to the consideration of

$$u(t) = \int_0^t \frac{g[u(s)+h(s)]}{\sqrt{\pi(t-s)}}\,ds, \tag{1.11}$$

where

$$v(0,t) - u(t) = h(t) \equiv (\pi t)^{-1/2}\int_{-\infty}^{\infty} e^{-x^2/4t}v_0(x)\,dx. \tag{1.12}$$

Thus, we will be able to make some comparison of our results to those for certain nonlinear parabolic problems.

In the integral equation (1.1), there is a competition between the growing nonlinearity and the decaying kernel which makes the problem particularly relevant to a physical process in which explosive behavior might occur. Our objective is to characterize those properties of the nonlinearity and the kernel in such a manner as to yield sufficient conditions for the solution to exhibit blow-up.

With regard to the kernel in (1.1), it is evident from the examples shown that integrable singularities should be allowed. Thus, many of our results will be expressed in terms of the function

$$I(t) = \int_{t_0}^t k(t-s)r(s)\,ds, \quad t \geq t_0, \tag{1.13}$$

which is assumed to exist.

In Section 2 our analysis will show that an appropriate characterization of the nonlinearity is given by the value of the definite integral

$$\kappa = \int_{h_0}^{\infty} \frac{dz}{g(z)}. \tag{1.14}$$

It is found that $\kappa < \infty$ is essential for explosion. Clearly, the finiteness of this integral suggests that the nonlinearity $g(u)$ must grow sufficiently fast, as u increases, in order for blow-up to occur.

In Section 2 our analysis will establish how the properties of the kernel, as expressed by $I(t)$, and the properties of the nonlinearity, as expressed by κ, are related to blow-up. Among our results, we will show that whenever there exists a $t^{**} < \infty$ such that $I(t^{**}) = \kappa$, then (1.1) cannot have a continuous solution for $t \geq t^{**}$. This provides an upper bound for the occurrence of blow-up. In Section 3 we will apply our results to some examples suggested by (1.7) and (1.8). In Section 4 we discuss the connection of nonlinear parabolic partial differential equations to (1.11) for $g[v] = v^p$. In Section 5 an asymptotic analysis is carried out to determine the growth near blow-up for certain special cases of (1.1).

2. Principal results on blow-up. The results of this section assume that the properties of the nonlinearity and the kernel, as given in (1.2)–(1.6), hold throughout. We will begin by showing that any continuous and differentiable solution of (1.1) must be nonnegative and increasing. Then conditions will be given under which (1.1) has a unique continuous solution for all $t < t^*$. Finally, conditions under which (1.1) cannot have a continuous solution for $t \geq t^{**}$ will be presented. Altogether, these results suggest criteria under which (1.1) has a blow-up solution; that is, $u(t) \to \infty$ as $t \to \hat{t}$, $t^* \leq \hat{t} \leq t^{**}$.

The monotonic growth of the solution is established by the following:

Theorem 2.1. *Any continuous and differentiable solution of* (1.1) *must be positive and increasing for* $t > t_0$.

Proof. Since $u(t_0) = 0$ and $h(t_0) = h_0 > 0$, it follows that $u(t) + h(t) > 0$ on some sufficiently small interval $t_0 \leq t < t_1$. Then the properties of the nonlinearity and kernel together with (1.1) insure that $u(t) \geq 0$ for $t_0 \leq t \leq t_1$. This argument can be extended indefinitely to give that $u(t) \geq 0$ on any interval where it exists. Moreover, $u(t) \equiv 0$ cannot satisfy (1.1).

To see that $u(t)$ is increasing, differentiate (1.1) to obtain

$$\begin{aligned} u'(t) = k(t-t_0)r(t_0)g(h_0) + \int_{t_0}^{t} k(t-s)r'(s)g[u(s)+h(s)]\,ds \\ + \int_{t_0}^{t} k(t-s)r(s)g'[u(s)+h(s)][u'(s)+h'(s)]\,ds, \quad t > t_0. \end{aligned} \tag{2.1}$$

Since $u(t_0) = 0$ and $u(t) \geq 0$, then $u'(t) > 0$ on at least some small interval $t_0 < t < \bar{t}$. Assume that $u'(\bar{t}) = 0$, whereupon (2.1) becomes

$$\begin{aligned} 0 = k(\bar{t} - t_0)r(t_0)g(h_0) + \int_{t_0}^{\bar{t}} k(\bar{t} - s)r'(s)g[u(s) + h(s)]\,ds \\ + \int_{t_0}^{\bar{t}} k(\bar{t} - s)r(s)g'[u(s) + h(s)][u'(s) + h'(s)]\,ds. \end{aligned} \tag{2.2}$$

Since the right-hand side of (2.2) is positive for $t_0 < \bar{t} < \infty$, then no such $\bar{t}$ exists and, hence, $u'(t) > 0$ on any finite interval where (1.1) has a solution. This establishes Theorem 2.1. □

Both the lower and upper bounds on the blow-up time $\hat{t}$ can be expressed in terms of $I(t)$, as defined by (1.13). It is important to note that, under the given conditions, $I(t)$ is increasing. Differentiation of (1.13) yields

$$I'(t) = k(t-t_0)r(t_0) + \int_{t_0}^{t} k(t-s)r'(s)\,ds > 0, \quad t_0 < t < \infty. \tag{2.3}$$

Our lower bound on the occurrence of blow-up will rely upon the existence of a unique continuous solution of (1.1) which satisfies

$$0 \leq u(t) \leq M < \infty, \quad t_0 \leq t < t^*, \tag{2.4}$$

for some appropriate choice of M. This is established in the following:

Theorem 2.2. *There exists a unique continuous solution of* (1.1) *which satisfies* (2.4) *for any* $M < M^*$*, where* M^* *is the smallest solution of*

$$M^*/g[M^* + h_\infty] = 1/g'[M^* + h_\infty]. \tag{2.5}$$

Proof. It follows from (1.1) that, for any continuous function satisfying (2.4),

$$Tu(t) \le g[M + h_\infty]I(t). \tag{2.6}$$

Thus, T maps the space of continuous functions which satisfy (2.4) into itself provided that

$$I(t) \le M/g[M + h_\infty]. \tag{2.7}$$

The contraction property of T is established by considering two continuous functions u_1 and u_2, each satisfying (2.4). It follows that

$$\sup_{t \ge t_0} |Tu_1 - Tu_2| \le g'[M + h_\infty]I(t) \sup_{t \ge t_0} |u_1 - u_2|. \tag{2.8}$$

Thus, there is a contraction when

$$I(t) < 1/g'[M + h_\infty]. \tag{2.9}$$

From the given properties of $g(u)$ and $I(t)$, it is clear that there is always an $\overline{M}$ sufficiently small such that $I(t) < I(t^*)$ where

$$I(t^*) = \overline{M}/g[\overline{M} + h_\infty] \le 1/g'[\overline{M} + h_\infty]. \tag{2.10}$$

By increasing $\overline{M}$ until equality first holds in (2.10), i.e., $\overline{M} = M^*$, then the largest possible value for $I(t^*)$ is obtained. If, indeed, that value is in the range of $I(t)$, then (2.5) follows. This establishes Theorem 2.2. □

Our upper bound on the occurrence of blow-up will rely upon the nonexistence of a continuous solution of (1.1) for all $t \ge t^{**}$. The possibility of this is established by the following:

Theorem 2.3. *Whenever there exists a* $t^{**} < \infty$ *such that*

$$I(t^{**}) = \kappa = \int_{h_0}^{\infty} \frac{dz}{g(z)} < \infty, \tag{2.11}$$

then (1.1) *cannot have a continuous solution for* $t \geq t^{**}$.

Proof. Assume that (1.1) has a continuous solution for $t_0 \leq t \leq t_1$. Then

$$u(t) = Tu(t) \geq J(t), \quad t_0 \leq t \leq t_1, \tag{2.12}$$

where

$$J(t) = \int_{t_0}^{t} k(t_1 - s) r(s) g[u(s) + h(s)]\, ds. \tag{2.13}$$

It then follows that

$$\begin{aligned} J'(t) &= k(t_1 - t) r(t) g[u(t) + h(t)] \\ &\geq k(t_1 - t) r(t) g[J(t) + h_0]. \end{aligned} \tag{2.14}$$

Integration of the differential inequality (2.14) yields

$$\int_{h_0}^{J(t_1)+h_0} \frac{dz}{g(z)} \geq \int_{t_0}^{t_1} k(t_1 - s) r(s)\, ds = I(t_1). \tag{2.15}$$

The assumption of a continuous solution of (1.1) for $t_0 \leq t \leq t_1$ insures that $J(t_1) < \infty$; hence, we can replace $J(t_1)$ by that upper bound to obtain

$$\kappa > I(t_1). \tag{2.16}$$

If $I(t)$ is such that t_1 can be increased to some t^{**} where (2.11) holds, then (2.16) is contradicted. This establishes Theorem 2.3. □

The implication of the preceding results suggest criteria for a blow-up solution of (1.1). This is given in the following:

Theorem 2.4. *If the* M^* *defined by* (2.5) *and* κ *defined by* (1.14) *are such that the values of* $M^*/g[M^* + h_\infty]$ *and* $\kappa < \infty$ *are both in the range of* $I(t)$, *then the unique positive, continuous and increasing solution of* (1.1) *must cease to exist at some* $t = \hat{t}$, *where*

$$t_0 < I^{-1}(M^*/g[M^* + h_\infty]) = t^* \le \hat{t} \le t^{**} = I^{-1}(\kappa) < \infty. \tag{2.17}$$

Proof. By Theorem (2.2), there exists a unique continuous solution of (1.1) for all $t < t^*$. By Theorem (2.3), the continuous solution might exist beyond t^* but cannot exist for $t \ge t^{**}$. The monotonicity of $I(t)$ insures that the inverses indicated in (2.17) are properly defined. This establishes Theorem 2.4. □

3. Application of principal results. The results of Section 2 will now be applied to some of the examples of Section 1. We begin by considering (1.7) for $\alpha > 0$, $\gamma > 0$, $\beta = \lambda = 0$, which we express in the form

$$u(t) = \gamma \int_{-\infty}^{t} \frac{e^{\alpha s - c}}{\sqrt{\pi(t-s)}} [e^{u(s)+c}]\, ds. \tag{3.1}$$

Here we have chosen to fulfill the conditions (1.4) and (1.5) by taking $h(t) \equiv c = h_0 = h_\infty$ and $r(t) = \exp(\alpha t - c)$ for an arbitrary constant $c > 0$. The results will be independent of c, as should be expected.

It follows that (2.5) becomes

$$M^* e^{-(M^*+c)} = e^{-(M^*+c)} \tag{3.2}$$

which is satisfied by $M^* = 1$. It is further determined from (2.11) that

$$\kappa = \int_{c}^{\infty} e^{-z}\, dz = e^{-c}. \tag{3.3}$$

From (1.13) it is found that

$$I(t) = \gamma \int_{-\infty}^{t} \frac{e^{\alpha s - c}}{\sqrt{\pi(t-s)}}\, ds = \frac{\gamma}{\sqrt{\alpha}} e^{\alpha t - c}. \tag{3.4}$$

Thus, Theorem (2.4) gives that blow-up always occurs for (3.1) at $t = \hat{t}$, with

$$\frac{1}{\alpha}\log\frac{\sqrt{\alpha}}{\gamma} - \frac{1}{\alpha} = t^* \le \hat{t} \le t^{**} = \frac{1}{\alpha}\log\frac{\sqrt{\alpha}}{\gamma}. \tag{3.5}$$

Next we consider (1.7) for $\gamma > 0$, $\lambda = \alpha = 1$, $\beta = 0$, which we express in the form

$$u(t) = \int_{-\infty}^{t} k(t-s)e^{s-c}[e^{u(s)+c}]\,ds, \tag{3.6}$$

where again we have taken $h(t) \equiv c > 0$. The kernel is given by

$$k(t) = \gamma\left[\frac{1}{\sqrt{\pi t}} - e^t erfc\sqrt{t}\right] = \frac{\gamma e^t}{\sqrt{\pi t}}\int_1^{\infty}\frac{e^{-ty^2}}{y^2}\,dy, \tag{3.7}$$

with the latter representation making it clear that (1.6) is satisfied.

Since the nonlinearity is the same here as in (3.1), then (3.2) and (3.3) again apply. It further follows that

$$I(t) = \gamma\int_{-\infty}^{t}\left[\frac{1}{\sqrt{\pi(t-s)}} - e^{t-s}erfc\sqrt{t-s}\right]e^{s-c}\,ds = \frac{\gamma}{2}e^{t-c}. \tag{3.8}$$

Thus, Theorem (2.4) gives that blow-up always occurs for (3.6) at $t = \hat{t}$, with

$$\log\frac{2}{\gamma} - 1 = t^* \le \hat{t} \le t^{**} = \log\frac{2}{\gamma}. \tag{3.9}$$

For the shear band model (1.8), we consider the special case of $q = 0$, $p > 1$ with $\gamma > 0$, which gives

$$u(t) = \gamma\int_0^t \frac{1}{\sqrt{\pi(t-s)}}[u(s)+1]^p ds, \tag{3.10}$$

where $h(t) \equiv 1$. It follows that (2.5) becomes

$$M^*[M^*+1]^p = p[M^*+1]^{-p+1} \tag{3.11}$$

which implies that $M^* = 1/(p-1)$. It is further determined from (2.11) that

$$\kappa = \int_1^\infty z^{-p}dz = \frac{1}{p-1}. \tag{3.12}$$

From (1.13) it is found that

$$I(t) = \gamma \int_0^t \frac{ds}{\sqrt{\pi(t-s)}} = 2\gamma\sqrt{\frac{t}{\pi}}. \tag{3.13}$$

Thus, Theorem (2.4) gives that blow-up always occurs for (3.10) at $t = \hat{t}$, with

$$(1-p^{-1})^{2p}\frac{\pi}{4\gamma^2}(p-1)^{-2} = t^* \leq \hat{t} \leq t^{**} = \frac{\pi}{4\gamma^2}(p-1)^2. \tag{3.14}$$

To illustrate an example in which blow-up may or may not occur, we consider (1.7) with $\lambda = 0$, $\beta = 1$, $\gamma = 0$. We express this in the form

$$u(t) = \gamma e^{-c}\int_{-\infty}^t \frac{e^{\alpha s - s^2}}{\sqrt{\pi(t-s)}}[e^{u(s)+c}]\,ds. \tag{3.15}$$

As with (3.1), we have let $h(t) \equiv c > 0$ to fulfill (1.4) and (1.5). The results will be independent of c.

Since the nonlinearity is the same as (3.1), we have that

$$M^* e^{-(M^*+c)} = e^{-(1+c)}, \quad \kappa = e^{-c}. \tag{3.16}$$

In this example, some care must be taken for the fact that $r'(t) \geq 0$ only for $-\infty < t \leq \alpha/2$. Thus $u(t)$ cannot be guaranteed to increase for all $t > \alpha/2$. In the existence Theorem 2.2, the condition $r'(t) \geq 0$ is not used; and, in this problem if there is no blow-up, $u(t)$ will ultimately decay to zero. Moreover, $I(t)$ only increases to some maximum value and then decays to zero. That maximum value becomes part of a sufficient condition for blow-up to occur. It follows from (1.13) that

$$I(t) = \gamma e^{-c}\int_{-\infty}^t \frac{e^{\alpha s - s^2}}{\sqrt{\pi(t-s)}}\,ds = \gamma e^{-(\alpha^2/4)-c}Q\left(t - \frac{\alpha}{2}\right), \tag{3.17}$$

where

$$Q(\tau) = \int_{-\infty}^{\tau} \frac{e^{-\sigma^2}}{\sqrt{t-\sigma}}\, d\sigma \tag{3.18}$$
$$= \frac{\sqrt{\pi}}{2}|\tau|^{1/2} e^{-\tau^2/2}\left[I_{1/4}\left(\frac{\tau^2}{2}\right)\operatorname{sgn}\tau + I_{-1/4}\left(\frac{\tau^2}{2}\right)\right].$$

In [**5**], it was established that $Q(\tau)$ attains its maximum value only at $\tau = \bar{\tau}$, so that

$$0 \le Q(\tau) \le Q(\bar{\tau}) = 1.214\ldots, \quad \bar{\tau} = 0.541\ldots. \tag{3.19}$$

It follows that if

$$\gamma e^{\alpha^2/4} < 1/eQ(\bar{\tau}) = 0.303\ldots, \tag{3.20}$$

then the contraction mapping conditions hold for all $t > -\infty$, and hence $u(t)$ *does not experience blow-up*. In contrast, if

$$\gamma e^{\alpha^2/4} > 1/Q(\bar{\tau}) = 0.823\ldots, \tag{3.21}$$

then $u(t)$ *does experience blow-up at* $t = \hat{t}$, where

$$Q^{-1}(1/\gamma e^{1+\alpha^2/4}) + \frac{\alpha}{2} = t^* \le \hat{t} \le t^{**} \tag{3.22}$$
$$= Q^{-1}(1/\gamma e^{\alpha^2/4}) + \frac{\alpha}{2}.$$

4. Connection to parabolic differential equations. As indicated in Section 1, the nonlinear parabolic problem (1.9) can be converted to the integral equation (1.11) under the circumstances in which the nonlinearity has the strongly localized spatial behavior defined by (1.10). For this type of problem there is considerable interest in the power law nonlinearity,

$$g(v) = v^p, \quad p > 1, \tag{4.1}$$

whereupon (1.11) takes the form

$$u(t) = \int_0^t \frac{[u(s)+h(s)]^p}{\sqrt{\pi(t-s)}}\, ds \tag{4.2}$$

with $h(t)$ defined by (1.12).

Of particular importance here are the conditions for blow-up. As seen from Theorem 2.3, this essentially relies upon

$$\kappa = \int_{h_0}^{\infty} \frac{dz}{z^p} = \frac{1}{p-1}\left(\frac{1}{h_0}\right)^{p-1}, \qquad h_0 = h(0) = 2v_0(0). \tag{4.3}$$

Since $I(t) = 2\sqrt{t/\pi}$ for the given kernel, it follows that

$$t^{**} = \frac{\pi}{4(p-1)^2}\left(\frac{1}{h_0}\right)^{2p-2}. \tag{4.4}$$

It is difficult to make a direct comparison of this result with those found in the literature (see [**6**]). Those results mostly correspond to $F[v] = v^p$ in (1.9), as opposed to our case, $F[v,x] = 2\delta(x)v^p$, which has the spatial localization. The results of the literature generally indicate blow-up for $1 < p < p_c$, where p_c is a critical index; whereas, there is no blow-up for $p > p_c$ with sufficiently small initial data.

The implication of (4.4) is that there is blow-up for all $p > 1$. There is no p_c as found in the nonlocalized cases of the literature. However, for sufficiently small initial data, i.e., $h_0 \ll 1$, it is seen from (4.4) that $t^{**} \gg 1$ and hence blow-up can be very considerably delayed.

Perhaps this contrast of behavior between the localized and nonlocalized source terms is not surprising. For $p > p_c$ in the nonlocalized case, it is often explained that, with small initial data, the source remains small and can be offset by diffusion so as to prevent blow-up. In the case here, diffusion is never quite sufficient to offset the highly localized and concentrated source; consequently, blow-up will eventually occur.

5. Asymptotic behavior near blow-up. Here we examine the asymptotic growth of the solution to (1.1) near blow-up. In order to obtain some specific results, we confine our attention to situations in which (1.1) involves the diffusion kernel. Thus, we will consider

$$u(t) = \int_{t_0}^{t} [\pi(t-s)]^{-1/2} r(s) g[u(s) + h(s)]\, ds. \tag{5.1}$$

Our approach will be to obtain a formal asymptotic balance of (5.1) to determine the leading order behavior of $u(t)$ near blow-up, under the assumption that

$$u(t) \to \infty \quad \text{as} \quad t \to \hat{t}. \tag{5.2}$$

This analysis will not determine $\hat{t}$, but rather will demonstrate an asymptotic behavior which is self-consistent with (5.1) and (5.2).

We introduce the change of variables

$$\eta = (\hat{t} - t)^{-1} - \eta_0, \qquad \eta_0 = (\hat{t} - t_0)^{-1}, \qquad w(\eta) = u(t), \tag{5.3}$$

whereupon (5.1) takes the form

$$w(\eta) = (\eta + \eta_0)^{1/2} I^{1/2}\{\varphi(\eta)\}, \tag{5.4}$$

where

$$\varphi(\eta) = \frac{r(\hat{t} - (\eta + \eta_0)^{-1})}{(\eta + \eta_0)^{3/2}} g[w(\eta) + h(\hat{t} - (\eta + \eta_0)^{-1})], \tag{5.5}$$

and $I^{1/2}$ is the Riemann fractional integral operator of order 1/2 defined as

$$I^{1/2}\{\varphi(\eta)\} = \int_0^\eta [\pi(\eta - \xi)]^{-1/2} \varphi(\xi)\, d\xi. \tag{5.6}$$

In terms of the new independent variable η, the blow-up (5.2) becomes

$$w(\eta) \to \infty \quad \text{as } \eta \to \infty. \tag{5.7}$$

This is significant because the asymptotic behavior of integral equations like (5.6) as $\eta \to \infty$ has been the subject of several investigations (e.g., [**2, 9**]). Contained in the results of [**9**] is that, if

$$\varphi(\eta) \sim c\eta^{-m} + \dots, \quad m \geq 0, \quad \text{as } \eta \to \infty, \tag{5.8}$$

then

$$I^{1/2}\{\varphi(\eta)\} \sim \begin{cases} \frac{c\Gamma(1-m)}{\Gamma(3/2-m)} \eta^{-m+1/2}, & 0 \leq m < 1, \\ \frac{c}{\sqrt{\pi}} (\log \eta) \eta^{-1/2}, & m = 1, \end{cases} \tag{5.9}$$

as $\eta \to \infty$. This is the fundamental relationship required to determine the leading order behavior of $w(\eta)$ as $\eta \to \infty$ in (5.4), and hence the behavior of $u(t)$ as $t \to \hat{t}$ in (5.1).

We first consider the case in which

$$g[v] = v^p, \quad p > 3/2. \tag{5.10}$$

In anticipation of algebraic growth near blow-up, we consider

$$w(\eta) \sim A\eta^l, \quad l > 0, \quad \text{as } \eta \to \infty. \tag{5.11}$$

It then follows from (5.5) that

$$\varphi(\eta) \sim r(\hat{t})A^p\eta^{pl-3/2}, \quad \text{as } \eta \to \infty. \tag{5.12}$$

Utilizing (5.9) with $m = 3/2 - pl$, an asymptotic balance to leading order in (5.4) requires that

$$A\eta^l \sim \frac{A^p r(\hat{t})\Gamma(pl-1/2)}{\Gamma(pl)}\eta^{pl-1/2}, \quad \text{as } \eta \to \infty. \tag{5.13}$$

Satisfaction of (5.13) yields

$$l = \frac{1}{2(p-1)}, \qquad A = \left\{ \frac{\Gamma[\frac{p}{2(p-1)}]}{r(\hat{t})\Gamma[\frac{1}{2(p-1)}]} \right\}^{1/p-1}. \tag{5.14}$$

Thus, for the case of (5.10), if there is a blow-up solution of (5.1), its asymptotic growth is given by

$$u(t) \sim A(\hat{t} - t)^{-1/(2(p-1))}, \quad \text{as } t \to \hat{t}. \tag{5.15}$$

We next consider the case in which

$$g[v] = e^v, \tag{5.16}$$

the nonlinearity typically associated with combustion problems. In anticipation of logarithmic growth near blow-up, we consider

$$\begin{aligned} w(\eta) &\sim \log(A\eta^l) \\ &\sim l\log\eta + \log A, \quad \text{as } \eta \to \infty. \end{aligned} \tag{5.17}$$

It then follows from (5.5) that

$$\varphi(\eta) \sim Ar(\hat{t})e^{h(\hat{t})}\eta^{l-3/2}, \quad \text{as } \eta \to \infty. \tag{5.18}$$

Utilizing (5.9) with $m = 3/2 - l$, an asymptotic balance to leading order in (5.4) requires that

$$l \log \eta \sim \frac{Ar(\hat{t})e^{h(\hat{t})}}{\sqrt{\pi}} \log \eta, \qquad m = \frac{3}{2} - l = 1. \tag{5.19}$$

Satisfaction of (5.19) yields

$$l = \frac{1}{2}, \qquad A = \frac{\sqrt{\pi}e^{-h(\hat{t})}}{2r(\hat{t})}. \tag{5.20}$$

Thus, for the case of (5.16), if there is a blow-up solution of (5.1), its asymptotic growth is given by

$$u(t) \sim \frac{1}{2} \log \frac{1}{\hat{t} - t}, \quad \text{as } t \to \hat{t}. \tag{5.21}$$

This result has been previously determined (e.g., [**3, 8**]).

The asymptotic behavior of (1.1) near blow-up for more general kernels and other nonlinearities requires that (5.4) be replaced by

$$w(\eta) = L\{\psi(\eta)\} \equiv \int_0^\eta k\left[\frac{\eta - \xi}{(\eta + \eta_0)(\xi + \eta_0)}\right]\psi(\xi)\,d\xi \tag{5.22}$$

where

$$\begin{aligned}\psi(\eta) = r(\hat{t} - (\eta + \eta_0)^{-1})g[w(\eta) \\ + h(\hat{t} - (\eta + \eta_0)^{-1})](\eta + \eta_0)^{-2}.\end{aligned} \tag{5.23}$$

To proceed requires more explicit information about the nature of the kernel and the nonlinearity. The key to the asymptotic analysis of (5.22) is to have a relationship between $\psi(\eta)$ and $L\{\psi(\eta)\}$, as $\eta \to \infty$, analogous to (5.8)–(5.9). The development of such relationships may require specialized techniques for the asymptotic expansion of integrals, as are found in [**1**]. For example, when $L\{\psi(\eta)\}$ can be expressed in terms of the Riemann fractional integral operator of order μ, $0 < \mu < 1$, there is an analog of (5.8)–(5.9) provided by [**1**, Chapter 4]. Then, for specific nonlinearities, an asymptotic analysis similar to that shown above could be done on (5.22).

REFERENCES

1. N. Bleistein and R.A. Handelsman, *Asymptotic expansion of integrals*, Holt, Rinehardt and Winston, New York, 1975.

2. R.A. Handelsman and W.E. Olmstead, *Asymptotic solution to a class of nonlinear Volterra integral equations*, SIAM J. Appl. Math. **22** (1972), 373–384.

3. A.K. Kapila, *Evolution of deflagration in a cold combustible subjected to a uniform energy flux*, Int. J. Engrg. Sci. **43** (1981), 495–509.

4. D. Glenn Lasseigne and W.E. Olmstead, *Ignition of a combustible solid by convection heating*, J. Appl. Math. and Phys. (ZAMP) **34** (1983), 886–898.

5. ———, *Ignition or nonignition of a combustible solid with marginal heating*, Quart. Appl. Math. **49** (1991), 309–312.

6. H.A. Levine, *The role of critical exponents in blowup theorems*, SIAM Rev. **32** (1990), 262–288.

7. A. Linan and F.A. Williams, *Theory of ignition of a reactive solid by a constant energy flux*, Comb. Sci. Tech. **3** (1971), 91–98.

8. W.E. Olmstead, *Ignition of a combustible half space*, SIAM J. Appl. Math. **43** (1983), 1–15.

9. W.E. Olmstead and R.A. Handelsman, *Asymptotic solution to a class of nonlinear Volterra integral equations* II, SIAM J. Appl. Math. **30** (1976), 180–189.

10. W.E. Olmstead, S. Nemat-Nasser and L. Ni, *Shear bands as discontinuities*, Pittsburgh, Center for Nonlinear Analysis, Carnegie Mellon University, 92-NA-013, 1992.

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