JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 5, Number 2, Spring 1993

THE QUASI-ELASTIC METHOD OF SOLUTION FOR A CLASS OF INTEGRODIFFERENTIAL EQUATIONS

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1. Introduction. In this paper we consider initial value problems of the form

$$(1)\quad y''+A(x)y+B(x)L(y,x)=f(x), \qquad y(0)=\alpha, y'(0)=\beta, \quad 0\le x\le b$$

or boundary value problems of the form

$$(2)\quad y''+A(x)y+B(x)L(y,x)=f(x), \qquad y(0)=\alpha, y(b)=\gamma, \quad 0\le x\le b,$$

where $L(y,x)$ is a Volterra integral operator, viz. $L(y,x)=\int_0^x K(x,t)\cdot y(t)\,dt$. It is easy to reduce equations of the form (1) to a system of linear Volterra integral equations of the second kind. Thus, if $A(x), B(x), f(x)$ and $K(x,t)$ are continuous, (1) has a unique solution (cf. [**7**, p. 50, Exercise 3.19]).

If $A(x)$ and $B(x)$ are constant, and $L(y,x)$ is a convolution operator, then (1) or (2) can be solved using Laplace transforms. The quasi-elastic solution method was originally proposed by Schapery [**8**] as an approximate technique for evaluating the inverse Laplace transform. In [**1, 6**], the method has been applied directly to solve integral equations of Volterra type. Also in these publications, the convergence and accuracy of the quasi-elastic solution method (which is called by these authors the method of variable moduli) is considered.

Boundary value problems in viscoelasticity are usually defined by integrodifferential equations involving the same integrals as those appearing in the respective viscoelastic constitutive equations. Accordingly,

Received by the editors on November 25, 1992 and in revised form on January 26, 1993.
The examples in this paper were computed with the aid of MACSYMA, a large symbolic manipulation program developed at the MIT Laboratory for Computer Science and supported from 1975 to 1983 by the National Aeronautics and Space Administration under grant NSG 1323, by the Office of Naval Research under grant N00014-77-C-0641, by the U.S. Department of Energy under grant ET-78-C-02-4687, and by the U.S. Air Force under grant F49620-79-C-020, and since 1982 by Symbolics, Inc. of Burlington, Mass.

the quasi-elastic method can be used to solve these equations, which is very important for those problems that are difficult to solve by means of Laplace transforms. In [**4, 5, 10**], the quasi-elastic method is applied to solve some viscoelastic buckling problems represented by equations related to (2). Since the method is so widely used, it is appropriate to establish it on a firm mathematical foundation.

In this paper we study the convergence of the quasi-elastic solution method when applied to initial value problems of the form (1), or boundary value problems of the form (2). However, it will be clear that the ideas developed here can be used to study other types of integrodifferential equations that contain Volterra integral operators.

2. Description of the quasi-elastic solution method and statement of results. We first consider the initial value problem (1). Shapery's idea, when applied to this problem, consists in approximating $L(y,x)$ by means of $y(x)L(1,x)$. Thus, the solution to (1) is approximated by the solution to

$$y'' + H(x)y = f(x), \qquad y(0) = \alpha, y'(0) = \beta, \quad 0 \le x \le b, \tag{3}$$

where $H(x) = A(x) + B(x)L(1,x)$, which in turn can be solved either numerically or by an approximation method that we shall describe in the sequel. However, this first rudimentary quasi-elastic approximation may not always be sufficiently accurate and a more sophisticated approach may be required.

Vinogradov [**10**] noted that, if y is the solution to (1), v_0 is the solution to (3), and $G_0 = y - v_0$, then $G_0'' + A(x)G_0 + B(x)L(G_0,x) = B(x)[L(1,x)v_0(x) - L(v_0,x)]$, $G_0(0) = G_0'(0) = 0$. Since $v_0(x)$ is assumed known, this equation is of the form (1), and therefore G_0 can be approximated by the solution v_1 of $v_1'' + H(x)v_1 = B(x)[L(1,x)v_0(x) - L(v_0,x)]$.

Continuing in this fashion, we see that $y(x) = S_n(x) + G_n(x)$, with

$$S_n(x) = \sum_{k=0}^{n} v_k(x),$$

where

$$G_{k+1} = G_k - v_{k+1}, \quad k = 0, 1, 2, \dots ,$$

$$\begin{aligned} &v_k''+H(x)v_k = e_k(x); \qquad v_0(0)=\alpha, v_0'(0)=\beta; \\ &v_k(0)=v_k'(0)=0, \quad k=1,2,3,\dots \end{aligned} \tag{4}$$

$$\begin{aligned} &G_k''+A(x)G_k+B(x)L(G_k,x)=e_{k+1}(x), \qquad G_k(0)=G_k'(0)=0, \\ &k=0,1,2,\dots \end{aligned} \tag{5}$$

$$\begin{aligned} &e_0(x)=f(x); \qquad e_k(x)=B(x)[L(1,x)v_{k-1}(x)-L(v_{k-1},x)], \\ &k=1,2,3,\dots . \end{aligned} \tag{6}$$

We shall call $S_n(x)$ the n^{th} order viscoelastic approximation to the initial value problem (1). We shall also use the following notation:

$$A=\sup\{|A(t)|,\ 0\le t\le b\}, \qquad B=\sup\{|B(t)|,\ 0\le t\le b\},$$

$$K=\sup\{|K(x,t)|,\ 0\le x,t\le b\},$$

$$m_0(x)=\int_0^x\int_0^\xi |f(s)|\,ds\,d\xi+|\alpha|+|\beta|x,$$

$$m_k(x)=\int_0^x\int_0^\xi |e_k(s)|\,ds\,d\xi, \quad k=1,2,3,\dots$$

$$m_k=\sup\{|m_k(x)|:0\le x\le b\}, \quad k=0,1,2,\dots ,$$

$$H=\sup\{|H(x)|:0\le x\le b\},$$

$$P=\sup\left\{\int_0^x |K(x,t)|[\cosh(\sqrt{H}x)-\cosh(\sqrt{H}t)]\,dt : 0\le x\le b\right\},$$

$$S(x)=\exp[Ax^2/2+BKx^3/6], \qquad R(x)=\int_0^x S(t)\,dt,$$

$$Q(x)=\int_0^x R(t)\,dt.$$

Our first result is:

Theorem 1. *Let $A(x), B(x), f(x) \in C[0,b]$, $K(x,t) \in C([0,b] \times [0,b])$, and let $y(x)$ be the solution to the initial value problem* (1). *Let $v_k(x)$, $k = 0, \dots, n$ be defined by* (4), *and let $G_n(x)$ be defined by* (5). *Then*

$$|G_n(x)| \le Q(x)(BP)^{n+1} m_0 x^{2n}/(2n)!,$$
$$|G_n'(x)| \le R(x)(BP)^{n+1} m_0 x^{2n}/(2n)!,$$

and

$$|G_n''(x)| \le S(x)(BP)^{n+1} x^{2n}/(2n)!.$$

In particular, $y(x) = \sum_{k=0}^{\infty} v_k(x)$, uniformly on $[0,b]$.

The idea of approximating systems of integral Volterra equations by means of series of functions that are solutions of differential equations has been used before. For example, Bownds approximates $K(x,t)$ by a sum of the form $\sum_{j=1}^{n} \phi_j(x)\psi_j(t)$, where the functions $\psi_j(x)$ form an orthogonal sequence, and the functions $\phi_j(t)$ satisfy certain differential equations (cf., e.g., [**3**]). This is, of course, a completely different approach.

Assume now that the boundary value problem (2) has a solution v, and let $\beta = v'(0)$. Then $v = v_1 + \beta v_2$, where

$$\begin{gathered} v_1'' + A(x)v_1 + B(x)L(v_1, x) = f(x), \\ v_1(0) = \alpha, \qquad v_1'(0) = 0, \quad 0 \le x \le b, \end{gathered} \tag{7}$$

and

$$\begin{gathered} v_2'' + A(x)v_2 + B(x)L(v_2, x) = 0, \\ v_2(0) = 0, \qquad v_2'(0) = 1, \quad 0 \le x \le b. \end{gathered} \tag{8}$$

If $v_2(b) = 0$, then $v_1(b) = v(b) = \gamma$, and therefore $v_1(x)$ is a solution of (2). Assume therefore that $v_2(b) \neq 0$. We see readily that $\beta = (\gamma - v_1(b))/v_2(b)$. Thus, if $A(x), B(x), f(x)$ and $K(x,t)$ are continuous, the solution $v(x)$ is unique. If $S_n(x), S_{n,1}(x)$ and $S_{n,2}(x)$ are the n^{th} order viscoelastic approximations to (2), (7), and (8), respectively, it is readily seen that $S_n(x) = S_{n,1}(x) + \beta S_{n,2}(x)$.

Let $\beta_n = (\gamma - S_{n-1}(b))/S_{n,2}(b)$ if $S_{n,2}(b) \neq 0$, and let $\beta_n = 0$ otherwise. If

$$m_{0,1}(x) = \int_0^x |f(t)|\,dt + |\alpha|, \qquad m_{0,2}(x) = x,$$
$$m_{0,j} = \sup\{|m_{0,j}(x)|,\ 0 \le x \le b\}, \quad j = 1, 2,$$

and

$$V_n(x) = S_{n,1}(x) + \beta_n S_{n,2}(x), \tag{9}$$

applying Theorem 1 we obtain:

Theorem 2. *Let $A(x), B(x), f(x) \in C[0,b]$, $K(x,t) \in C([0,b] \times [0,b])$, let $v(x)$ be a solution of the boundary value problem* (2)*, and let $V_n(x)$ be defined by* (9)*. Assume that $v_2(b) \neq 0$ (where $v_2(x)$ is defined by* (8)*), and that*

$$m_{0,2} Q(b)(BP)^{n+1} b^{2n}/(2n)! < |S_{n,2}(b)|.$$

Then

$$|v(x) - V_n(x)| \le c_n (BP)^{n+1} b^{2n} Q(b)/(2n)!,$$

where

$$c_n = \frac{|S_{n,1}(b) - \gamma| m_{0,2} + |S_{n,2}(b)| m_{0,1}}{[|S_{n,2}(b)| - m_{0,2} Q(b)(BP)^{n+1} b^{2n}/(2n)!]|S_{n,2}(b)|} \cdot \cosh(\sqrt{H}b)\cosh(\sqrt{BP}b).$$

To decide whether to use Theorem 2, one could find an approximate solution $S_{n,2}(x)$ to (8) and, since $v_2(b) = S_{n,2}(b) + G_{n,2}(b)$, use the upper bound for $|G_n(x)|$ given by Theorem 1 to determine whether $|G_{n,2}(b)| < |S_{n,2}(b)|$. If $v_2(b) \neq 0$, this inequality will hold for sufficiently large n.

The proof of Theorem 1 depends on parts of the following proposition, which is of independent interest.

Lemma. *Let $H(x), f(x) \in C[0,b]$, and assume that $y(x)$ satisfies* (3)*. Then, for $0 \le t \le x \le b$:*

a) $y(x) = \sum_{n=0}^{\infty} y_n(x)$, *where*

$$y_0(x) = \int_0^x \int_0^\xi f(s)\, ds\, d\xi + \alpha + \beta x, \tag{10}$$

$$y_n(x) = -\int_0^x \int_0^\xi H(s) y_{n-1}(s)\, ds\, d\xi, \quad n = 1, 2, 3, \dots . \tag{11}$$

b) $|y(x)| \le m_0 \cosh(\sqrt{H} x)$

c) $|y(x) - \sum_{n=0}^p y_n(x)| \le m_0(x) H^{p+1} \cosh(\sqrt{H} x) x^{2p+2}/(2p+2)!$

d) $|y(x) - y(t)| \le m_0(x)[\cosh(\sqrt{H} x) - \cosh(\sqrt{H} t)]$.

Applying part a) of the Lemma to the functions $v_k(x)$ defined by (4), we have:

$$v_k(x) = \sum_{r=0}^\infty v_{kr}(x), \quad k = 0, 1, 2, \dots , \tag{12}$$

where

$$\begin{aligned} v_{00}(x) &= \int_0^x \int_0^\xi f(s)\, ds\, d\xi + \alpha + \beta x; \\ v_{k0}(x) &= \int_0^x \int_0^\xi e_k(s)\, ds\, d\xi, \quad k = 1, 2, 3, \dots ; \end{aligned} \tag{13}$$

(14)

$$v_{kr}(x) = -\int_0^x \int_0^\xi H(s) v_{kr-1}(s)\, ds\, d\xi, \quad k=0, 1, 2, \dots ,\ r=1, 2, 3, \dots .$$

In some cases, e.g., when $A(x), B(x), f(x)$ and $K(x,t)$ are polynomials, it is easy to compute the functions $v_{kr}(x)$ exactly using computer algebra software like, e.g., MACSYMA. One could then try to approximate $v_k(x)$ by truncating the right hand side of (12). Because we need to know $v_{k-1}(x)$ first in order to evaluate $e_k(x)$ for $k \ge 1$, ((6)), we replace the functions $v_{kr}(x)$ by approximations $y_{kr}(x)$, obtained by means of the recursive scheme described forthwith:

Let $\{r_k : k = 0, 1, 2, \dots\}$ be a given sequence of positive integers,

$$d_0(x) = f(x); \qquad y_{0r}(x) = v_{0r}(x), \quad r = 0, 1, 2, \dots ;$$

$$y_0(x) = \sum_{r=0}^{r_0} y_{0r}(x).$$

For $k = 1, 2, 3, \dots,$

$$d_k(x) = B(x)[L(1,x)y_{k-1}(x) - L(y_{k-1}, x)]; \tag{15}$$

$$y_{k0}(x) = \int_0^x \int_0^\xi d_k(s)\, ds\, d\xi; \tag{16}$$

$$y_{kr}(x) = -\int_0^x \int_0^\xi H(s) y_{kr-1}(s)\, ds\, d\xi, \quad r = 1, 2, 3, \dots ; \tag{17}$$

$$y_k(x) = \sum_{r=0}^{r_k} y_{kr}(x).$$

If $T_n(x) = \sum_{k=0}^n y_k(x)$, and $r = \min\{r_k,\ 0 \le k \le n\}$, we have:

Theorem 3. *Under the hypotheses of Theorem* 1, *let* $T = 2BKb \cdot \cosh(\sqrt{H}b)$, $T_1 = BK\cosh(\sqrt{H}b)$, $D = P/(2Kb\cosh(\sqrt{H}b))$, *and* $D_1 = P/(K\cosh(\sqrt{H}b))$. *Then* $S_n(x) = T_n(x) + U_n(x)$, *where*

$$|U_n(x)| \le m_0 \cosh^2(\sqrt{H}b)\cosh(\sqrt{D}b)T^n H^{r+1} b^{2r+2}/(2r+2)!$$

and

$$|U_n'(x)| \le m_0 \sinh^2(\sqrt{H}b)\cosh(\sqrt{D_1}b)T_1^n H^{r+1} b^{2r+1}/(2r+1)!.$$

Let

$$\gamma_n = [\gamma - T_{n,1}(b)]/T_{n,2}(b),$$

$$p_{n,1} = \cosh^2(\sqrt{H}b)\cosh(\sqrt{D}b)T^n H^{r+1} b^{2r+2}/(2r+2)!,$$

$$p_{n,2} = m_{0,2}(p_{n,1} + Q(b)(BP)^{n+1} b^{2n}/(2n)!)$$

$$q_n = \frac{[|T_{n,1}(b) - \gamma| + m_{0,2}p_{n,1}]m_{0,2} + [|T_{n,2}(b)| + m_{0,2}p_{n,1}]m_{0,1}}{[|T_{n,2}(b)| - p_{n,2}][|T_{n,2}(b)| - m_{0,2}p_{n,1}]}$$
$$\cdot \cosh(\sqrt{H}b)\cosh(\sqrt{BP}b),$$

$$\sigma_n = \frac{|T_{n,1}(b) - \gamma| m_{0,2} + |T_{n,2}(b)| m_{0,1}}{[|T_{n,2}(b)| - m_{0,2} p_{n,1}] |T_{n,2}(b)|} p_{n,1},$$

$$\delta_n = \frac{|T_{n,1}(b) - \gamma| + m_{0,1} p_{n,1}}{|T_{n,2}(b)| - m_{0,2} p_{n,1}}.$$

If $T_{n,1}(x)$ and $T_{n,2}(x)$ are the approximations to $S_{n,1}(x)$ and $S_{n,2}(x)$, respectively, defined in the manner described in the paragraph preceding the statement of Theorem 3, then, combining Theorems 2 and 3, we shall prove:

Theorem 4. *Let* $A(x), B(x), f(x) \in C[0,b]$, $K(x,t) \in C([0,b] \times [0,b])$, *let* $v(x)$ *be a solution of the boundary value problem* (2), *and assume that* $v_2(b) \neq 0$ *and* $p_{n,2} < |T_{n,2}(b)|$. *Then, if* $W_n(x) = T_{n,1}(x) + \gamma_n T_{n,2}(x)$,

$$|v(x) - W_n(x)| \leq q_n (BP)^{n+1} b^{2n} Q(b)/(2n)! \\ + (m_{0,1} + \delta_n m_{0,2}) p_{n,1} + \sigma_n |T_{n,2}(x)|.$$

3. Proofs.

Proof of Lemma. a) and b). As in, e.g., [**2**, p. 321], we consider the perturbation problem

$$(18) \qquad y'' + \varepsilon H(x) y = f(x), \qquad y(0) = \alpha, y'(0) = \beta, \quad 0 \leq x \leq b,$$

and assume it has a solution of the form

$$(19) \qquad y(x) = \sum_{n=0}^{\infty} y_n(x) \varepsilon^n; \qquad y_0(0) = \alpha, y_0'(0) = \beta; \\ y_n(0) = y_n'(0) = 0, \quad n = 1, 2, 3, \dots .$$

Substituting in (18) and equating coefficients, we obtain $y_0'' = f(x)$ and $y_n'' = -H(x) y_{n-1}$, $n = 1, 2, 3, \dots$. The initial conditions in (19) imply that (10) and (11) are satisfied. Thus, an inductive argument shows that

$$(20) \qquad |y_n(x)| \leq m_0(x) H^n x^{2n}/(2n)!, \quad n = 0, 1, 2, \dots ,$$

whence

$$|y_n''(x)| \le m_0(x)H^n x^{2n-2}/(2n-2)!, \quad n = 1, 2, 3, \dots . \tag{21}$$

This shows that the series in (19), with $y_0(x)$ and $y_n(x)$ defined by (10) and (11), is indeed a solution of (18) for every ε. The conclusion now readily follows by setting $\varepsilon = 1$ in (19).

c) From (20),

$$\begin{aligned}\left|y(x) - \sum_{n=0}^{p} y_n(x)\right| = \left|\sum_{n=p+1}^{\infty} y_n(x)\right| &\le m_0(x) \sum_{n=p+1}^{\infty} H^n x^{2n}/(2n)! \\ &= m_0(x)\left[\cosh(\sqrt{H}x) - \sum_{n=0}^{p} (\sqrt{H}x)^{2n}/(2n)!\right],\end{aligned}$$

whence the conclusion readily follows.

d) From the initial conditions in (19), we know that $y_n(x) - y_n(t) = -\int_t^x \int_0^\xi H(s)y_{n-1}(s)\,ds\,d\xi$. Thus, since $m_0(x)$ is increasing, applying (20) we obtain:

$$\begin{aligned}|y_n(x) - y_n(t)| &\le \int_t^x \int_0^\xi [|H(s)|m_0(s)H^{n-1}s^{2n-2}/(2n-2)!]\,ds\,d\xi \\ &\le [H^n m_0(x)/(2n-2)!] \int_t^x \int_0^\xi s^{2n-2}\,ds\,d\xi \\ &= [H^n m_0(x)/(2n)!](x^{2n} - t^{2n}).\end{aligned}$$

Hence,

$$\begin{aligned}|y(x) - y(t)| &\le \sum_{n=0}^{\infty} |y_n(x) - y_n(t)| \\ &\le m_0(x)\left[\sum_{n=0}^{\infty} \frac{H^n x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{H^n t^{2n}}{(2n)!}\right] \\ &= m_0(x)[\cosh(\sqrt{H}x) - \cosh(\sqrt{H}t)]. \qquad \square\end{aligned}$$

Proof of Theorem 1. From (6) we have

$$e_k(x) = B(x) \int_0^x K(x,t)[v_{k-1}(x) - v_{k-1}(t)]\,dt.$$

Applying part d) of the Lemma to v_{k-1}, we conclude that

$$\begin{aligned}|e_k(x)| &\le Bm_{k-1}(x)\int_0^x |K(x,t)|[\cosh(\sqrt{H}x)-\cosh(\sqrt{H}t)]\,dt\\ &\le BPm_{k-1}(t).\end{aligned}$$

Thus, $|m_k(x)| \le BP\int_0^x\int_0^\xi |m_{k-1}(s)|\,ds\,d\xi$, $k=1,2,\dots$, and by induction we infer that

$$|m_k(x)| \le (BP)^k m_0 x^{2k}/(2k)!, \quad k=0,1,2,\dots, \tag{22}$$

and therefore that

$$|e_k(x)| \le (BP)^k m_0 x^{2k-2}/(2k-2)!, \quad k=1,2,3,\dots. \tag{23}$$

From (5) we readily see that $G_n''(x)\in C[0,b]$, and that

$$\begin{aligned}G_n''(x) &= e_{n+1}(x) - A(x)\int_0^x\int_0^t G_n''(s)\,ds\,d\xi\\ &\quad + B(x)\int_0^x K(x,t)\int_0^t\int_0^\xi G_n''(\eta)\,d\eta\,d\xi\,dt.\end{aligned}$$

If $\nu_n(x)=\sup\{|G_n''(t)|,\ 0\le t\le b\}$, $E_n(x)=\sup\{|e_n(t)|,\ 0\le t\le x\}$, and $0\le s\le x$, we have:

$$\nu_n(s)\le E_{n+1}(x)+\int_0^s (At+BKt^2/2)\nu_n(t)\,dt.$$

Thus, Gronwall's inequality (cf., e.g., [**9**]), yields

$$|G_n''(s)|\le \nu_n(s)\le E_{n+1}(x)\exp[Ax^2/2+BKx^3/6]. \tag{24}$$

Since $E_k(x)$ is bounded by the right hand side of (23), $v_n(s)\le v_n(x)$, and the initial conditions of (5) imply that $|G_n(x)|\le\int_0^x\int_0^t|G_n''(\xi)|\,d\xi\,dt$ and $|G_n'(x)|\le\int_0^x|G_n(t)|\,dt$, the conclusion follows from (24). □

Proof of Theorem 2. Part b) of the Lemma implies that $|S_n(x)|\le \cosh(\sqrt{H}x)\sum_{k=0}^n m_k$, whence from (22) we conclude that $|S_n(x)|\le$

$\cosh(\sqrt{H}x)\cosh(\sqrt{BP}x)$. This implies that $|S_{n,2}(x)| \le \cosh(\sqrt{H}x) \cdot \cosh(\sqrt{BP}x)$.

Theorem 1, when applied to $v_1(x)$ and $v_2(x)$, yields

$$|G_{n,j}(x)| \le [m_{0,j}(BP)^{n+1}b^{2n}/(2n)!]Q(x), \quad j = 1, 2.$$

Since $v_{n,2}(x) = S_{n,2}(x) + G_{n,2}(x)$, the hypotheses imply that $v_{n,2}(b) \neq 0$. A straightforward computation shows that

$$\beta - \beta_n = \frac{[S_{n,1}(b) - \gamma]G_{n,2}(b) - S_{n,2}(b)G'_{n,1}(b)}{[S_{n,2}(b) + G_{n,2}(b)]S_{n,2}(b)}$$

whence

$$|\beta - \beta_n| \le \frac{[|S_{n,1}(b) - \gamma|m_{0,2} + |S_{n,2}(b)|m_{0,1}](BP)^{n+1}b^{2n}Q(b)}{[|S_{n,2}(b)| - m_{0,2}Q(b)(BP)^{n+1}b^{2n}/(2n)!]|S_{n,2}(b)|(2n)!}.$$

Since

$$S_n(x) = V_n(x) + (\beta - \beta_n)S_{n,2}(x),$$

the conclusion follows. $\square$

Proof of Theorem 3. Clearly

$$v_k(x) = y_k(x) + Q_k(x) + R_k(x),$$

where

$$Q_k(x) = \sum_{r=0}^{r_k}[v_{kr}(x) - y_{kr}(x)]$$

and

$$R_k(x) = \sum_{r=r_k+1}^{\infty} v_{rk}.$$

Applying part c) of the Lemma to the functions $v_k(x)$, we see that

$$|R_k(x)| \le m_k(x)H^{r_k+1}\cosh(\sqrt{H}x)x^{2r_k+2}/(2r_k+2)!, \quad k=0,1,2,\dots .$$

From (22), we thus conclude that, for $0 \le x \le b$,

(25)
$$|R_k(x)| \le \frac{m_0(BP)^k H^{r_k+1}\cosh(\sqrt{H}b)b^{2(k+r_k+1)}}{(2k)!(2r_k+2)!}, \quad k = 0, 1, 2, \dots .$$

Applying (21) to the functions $v_k(x)$, we see that

$$|v''_{kr}(x)| \leq m_k H^r x^{2r-2}/(2r-2)!; \quad k, r = 1, 2, 3, \dots ,$$

and, therefore, since $v'_{kr}(0) = 0$, (cf. (14)),

$$|v'_{kr}(x)| \leq m_k H^r x^{2r-1}/(2r-1)!; \quad k, r = 1, 2, 3, \dots ,$$

then (22) implies that, for $0 \leq x \leq b$,

$$|R'_k(x)| \leq \frac{m_0 (BP)^k x^{2k} \sqrt{H}}{(2k)!} \sum_{r=r_{k+1}}^{\infty} \frac{(\sqrt{H} x)^{2r-1}}{(2r-1)!}.$$

Thus, as in the proof of part c) of the Lemma, we deduce that

$$|R'_k(x)| \leq \frac{m_0 (BP)^k H^{r_k+1} b^{2k+2r_k+1} \sinh(\sqrt{H} b)}{(2k)!(2r_k+1)!}. \tag{26}$$

Since $d_0(x) = e_0(x) = f(x)$, it is clear from their definition that $v_{0r}(x) = y_{0r}(x)$, $r = 0, 1, 2, \dots$. Thus,

$$Q_0(x) = 0 \quad \text{and} \quad Q'_0(x) = 0, 0 \leq x \leq b. \tag{27}$$

To find an estimate for $v_{kr}(x) - y_{kr}(x)$, and therefore for $Q_k(x)$, we proceed as follows:

Let $c_k(x) = e_k(x) - d_k(x)$. Clearly, $c_0(x) = 0$. Moreover, from (6) and (15), it is clear that, for $k = 1, 2, 3, \dots$,

$$c_k(x) = B(x) \int_0^x K(x,t) \{[v_{k-1}(x) - y_{k-1}(x)] - [v_{k-1}(t) - y_{k-1}(t)]\} \, dt.$$

Thus, setting

$$\begin{gathered} U_k(x) = Q_k(x) + R_k(x), \qquad U_k = \max\{|U_k(x)|, \ 0 \leq x \leq b\}, \\ U_k^1 = \max\{|U'_k(x)|, \ 0 \leq x \leq b\}, \end{gathered} \tag{28}$$

we conclude that

$$|c_k(x)| \leq 2BKxU_{k-1}; \quad k = 1, 2, 3, \dots , \tag{29}$$

and

$$|c_k'(x)| \leq BKxU_{k-1}^1. \tag{30}$$

For $k = 0, 1, 2, \ldots,$ let

$$c_{k0}(x) = -\int_0^x \int_0^\xi H(s)c_k(s)\,ds\,d\xi, \tag{31}$$

and

$$c_{kr}(x) = -\int_0^x \int_0^\xi H(s)c_{kr-1}(s)\,ds\,d\xi; \quad r = 1, 2, 3, \ldots. \tag{32}$$

From (13) and (16) it is clear that

$$c_{k0}(x) = v_{k0}(x) - y_{k0}(x).$$

Thus, by an inductive argument involving (14), (17), (29), (30), (31), and (32), we see that

$$c_{kr}(x) = v_{kr}(x) - y_{kr}(x); \quad k, r = 0, 1, 2, \ldots,$$

$$|c_{kr}(x)| \leq \frac{2BKH^{r+1}x^{2r+3}}{(2r+3)!}U_{k-1}; \quad k = 1, 2, 3, \ldots,\ r = 0, 1, 2, \ldots,$$

and

$$|c_{kr}(x)| \leq \frac{BKH^{r+1}x^{2r+3}}{(2r+3)!}U_{k-1}^1; \quad k = 1, 2, 3, \ldots,\ r = 0, 1, 2, \ldots.$$

The latter inequality, combined with (31) and (32), yields

$$|c_{kr}'(x)| \leq \frac{BKH^{r+1}x^{2r+2}}{(2r+2)!}U_{k-1}^1; \quad k = 1, 2, 3, \ldots,\ r = 0, 1, 2, \ldots.$$

Thus,

$$\begin{aligned} |Q_k(x)| &\leq 2BKxU_{k-1}\sum_{j=0}^{r_k}\frac{(\sqrt{H}x)^{2j+2}}{(2j+3)!} \\ &\leq 2BKx\cosh(\sqrt{H}x)U_{k-1}; \quad k = 1, 2, 3, \ldots, \end{aligned}$$

and, similarly,

$$|Q_k'(x)| \le BK \cosh(\sqrt{H}x) U_{k-1}^1; \quad k = 1, 2, 3, \dots .$$

Thus, from (28),

$$U_k \le TU_{k-1} + R_k; \quad k = 1, 2, 3, \dots , \tag{33}$$

and

$$U_K^1 \le T_1 U_{k-1}^1 + R_k^1; \quad k = 1, 2, 3, \dots , \tag{34}$$

where T and T_1 are defined in the statement of the theorem. Let

$$R_k = \sup\{|R_k(x)|,\ 0 \le x \le b\}.$$

Using (27) and (33), a straightforward inductive argument shows that

$$U_n \le \sum_{j=0}^{n} T^{n-j} R_j; \quad n = 0, 1, 2, \dots$$

whence, by (25),

$$\begin{aligned}
U_n &\le m_0 \cosh(\sqrt{H}b) \sum_{j=0}^{n} \frac{T^{n-j}(BP)^j H^{r_j+1} b^{2(j+r_j+1)}}{(2j)!(2rj+2)!} \\
&= m_0 T^n \cosh(\sqrt{H}b) \sum_{j=0}^{n} \frac{D^j H^{r_j+1} b^{2(j+r_j+1)}}{(2j)!(2r_j+2)!} \\
&= m_0 T^n \cosh(\sqrt{H}b) \sum_{j=0}^{n} \frac{(\sqrt{D}b)^{2j} (\sqrt{H}b)^{2j+2}}{(2j)!(2r_j+2)!} \\
&\le m_0 T^n \cosh(\sqrt{H}b) \sum_{j=0}^{\infty} \frac{(\sqrt{D}b)^{2j}}{(2j)!} \sum_{i=r+1}^{\infty} \frac{(\sqrt{H}b)^{2i}}{(2i)!} \\
&= m_0 T^n \cosh(\sqrt{H}b) \cosh(\sqrt{D}b) \left[\cosh(\sqrt{H}b) - \sum_{i=0}^{r} \frac{(\sqrt{H}b)^{2i}}{(2i)!} \right],
\end{aligned}$$

whence the assertion for U_n follows. The proof for U_n^1 is completed in a similar fashion using (26). $\square$

Proof of Theorem 4. Let $V_n(x)$ be defined as in (9), and

$$S_{n,j}(x) = T_{n,j}(x) + U_{n,j}(x), \quad j = 1, 2.$$

Since

$$\begin{aligned} v(x) - W_n(x) &= [v(x) - V_n(x)] + [V_n(x) - W_n(x)] \\ &= [v(x) - V_n(x)] + U_{n,1}(x) + \beta_n U_{n,2}(x) + (\beta_n - \gamma_n) T_{n,2}(x), \end{aligned}$$

applying Theorems 2 and 3, we have

$$\begin{aligned} |v(x) - W_n(x)| \le c_n (BP)^{n+1} b^{2n} Q(b)/(2n)! &+ (m_{0,1} + |\beta_n| m_{0,2}) p_{n,1} \\ &+ |\beta_n - \gamma_n| \, |T_{n,2}(x)|. \end{aligned}$$

From Theorem 3 we see that $c_n \le q_n$, $|\beta_n| \le \delta_n$, and $|\beta_n - \gamma_n| \le \sigma_n$, whence the conclusion follows. □

4. Examples.

Example 1. In this example we use Theorems 1 and 3 to find an approximate solution to the initial value problem (2) with $A(x) = 0.1x$, $B(x) = 0.001$, $K(x, t) = 10(x - t)$, $f(x) = 0.2$, $\alpha = 0.1$, $\beta = 0.2$, and $b = 1$, (i.e., $0 \le x \le 1$).

Clearly, $A = 0.1$, $B = 0.001$, $K = 10$, and

$$m_0(x) = \int_0^x \int_0^\xi 0.2 \, ds \, d\xi + 0.1 + 0.2x = 0.1(x + 1)^2,$$

whence $m_0 = 0.4$. Also,

$$L(1, x) = \int_0^x 10(x - t) \, dt = 5x^2;$$

thus,

$$H(x) = A(x) + B(x) L(1, x) = 0.1x + 0.005x^2,$$

whence $H = 0.105$.

Using MACSYMA, we see that

$$\begin{aligned} \int_0^x K(x, t)[\cosh(\sqrt{H}x) - \cosh(\sqrt{H}t)] \, dt = 10(x^2/2 - H^{-1}) &\cosh(\sqrt{H}x) \\ &+ 10H^{-1}, \end{aligned}$$

whence

$$P = 10(1/2 - H^{-1})\cosh(\sqrt{H}) + 10H^{-1} = 0.22090,$$

and therefore $BP = 2.209 \times 10^{-4}$. Since

$$Q(x) \leq (x^2/2)\exp(Ax^2/2 + BKx^3/6) \leq 0.5\exp(A/2 + BK/6) \leq 0.53,$$

we have:

$$|G_n(x)| \leq 4.69 \times 10^{-5}(2.21 \times 10^{-4})^n/(2n)!. \tag{35}$$

On the other hand,

$$T = 2BKb\cosh(\sqrt{H}b) \leq 0.02\cosh(\sqrt{0.105}) \leq 0.0211,$$

and

$$D = P(2Kb\cosh(\sqrt{H}b))^{-1} = 0.01049.$$

Thus,

$$\begin{aligned} U_n &\leq m_0 \cosh^2(\sqrt{H}b)\cosh(\sqrt{D}b)T^n H^{r+1}/(2r+2)! \\ &\leq 0.047(0.0105)^n(0.105)^r/(2r+2)!. \end{aligned} \tag{36}$$

Assume that we want to approximate the solution to this initial value problem with error not to exceed 10^{-3}. From (35), it is clear that $|G_0(x)| \leq 4.69 \times 10^{-5}$. This means that we need to approximate the solution $y(x)$ by means of $S_0(x)$, and $S_0(x)$ in turn by $T_0(x)$, with error not to exceed $10^{-3} - 4.69 \times 10^{-5}$, i.e., we need to find r such that $U_0 \leq 10^{-3} - 4.69 \times 10^{-5}$. This is clearly achieved by setting $r = 1$ (and $n = 0$) in (36).

Let $r_0 = 1$. Then

$$\begin{aligned} y_{00}(x) = v_{00}(x) &= \int_0^x \int_0^\xi f(s)\,ds\,d\xi + \alpha + \beta x = 0.1(x+1)^2 \\ y_{01}(x) = v_{01} &= -\int_0^x \int_0^\xi H(s)v_{00}(s)\,ds\,d\xi \\ &= -0.1\int_0^x \int_0^\xi (0.1s + 0.005s)(s+1)^2\,ds\,d\xi \\ &= -(2x^6 + 66x^5 + 205x^4 + 200x^3)/(120,000), \end{aligned}$$

and therefore

$$T_0(x) = \frac{12,000+24,000x+12,000x^2-200x^3-205x^4-66x^5-2x^6}{120,000}$$

($y_{01}(x)$ and $T_0(x)$ were computed using MACSYMA).

Example 2. We now find an approximation to the boundary value problem (2), with $A(x) = 0.1$, $B(x) = 0.001$, $K(x,t) = 10(x-t)$, $f(x) = 0$, $\alpha = 0.1$, $\gamma = 0.03$, and $b = 1$.

Clearly $m_{0,1} = 0.1$, $m_{0,2} = 1$, $K = 10$, and, as in Example 1, $L(1,x) = 5x^2$, whence $H(x) = 0.1 + 0.005x^2$, and therefore $H = 0.105$. Also, as in Example 1, $P = 0.22090$, $BP = 2.209 \times 10^{-4}$, $Q(x) \le 0.53$, $T \le 0.0211$, and $D = 0.01049$.

We now generate approximations $T_{n,1}(x)$ and $T_{n,2}(x)$ to $S_{n,1}(x)$ and $S_{n,2}(x)$:

$$d_0 = f(x) = 0; \qquad y_{00,1} = \alpha = 0.1; \qquad y_{00,2} = x.$$

$$\begin{aligned} y_{0r,1}(x) &= \int_0^x \int_0^\xi (0.1 + 0.005s^2) y_{0r-1,1}(s)\, ds\, d\xi \\ &= 0.1\left(\frac{0.1x^{2r}}{(2r)!} + \frac{0.005x^{2r+2}}{(2r+2)!}\right), \end{aligned}$$

and, similarly,

$$y_{0r,2}(x) = \frac{0.1x^{2r+1}}{(2r+1)!} + \frac{0.005x^{2r+3}}{(2r+3)!}.$$

Thus,

$$T_{0,1}(x) = y_{0,1}(x) = \sum_{j=0}^{r} y_{0j,1}(x) = 0.01 \sum_{j=0}^{r} \frac{x^{2j}}{(2j)!} + 0.0005 \sum_{j=0}^{r} \frac{x^{2j+2}}{(2j+2)!},$$

$$T_{0,2}(x) = y_{0,2}(x) = \sum_{j=0}^{r} y_{0j,2}(x) = 0.1 \sum_{j=0}^{r} \frac{x^{2j+1}}{(2j+1)!} + 0.005 \sum_{j=0}^{r} \frac{x^{2j+3}}{(2j+3)!}.$$

In particular, if $r = 1$, we obtain:

$$\begin{aligned} T_{0,1}(1) &= 0.01527, T_{0,2}(1) &&= 0.11754, \\ p_{0,1} &= 5.12 \times 10^{-4}, & p_{0,2} &= 1.167 \times 10^{-4}, \\ |T_{0,1}(1) - \gamma| &= 0.01473, \gamma_0 &&= 0.12532. \end{aligned}$$

Thus,

$$\begin{aligned} W_0(x) &= T_{0,1}(x) + \gamma_0 T_{0,2}(x) \\ &= 0.01(1 + x^2/2) + 0.0005(x^2/2 + x^4/24) \\ &\quad + 0.012532(x + x^3/6) + 0.0006266(x^3/6 + x^5/120) \\ &= 0.01 + 0.012532x + 0.00525x^2 + 0.0021931x^3 \\ &\quad + 0.0000208x^4 + 0.00000522x^5. \end{aligned}$$

We now estimate the error, assuming that $v_2(1) \neq 0$. Using the definitions, we readily obtain:

$$|T_{0,2}(1)| - p_{0,2} = 0.11742, \qquad |T_{0,2}(1)| + m_{0.2}p_{0,1} = 0.11805,$$
$$|T_{0,2}(1)| - m_{0,2}p_{0,2} = 0.11703, \qquad q_0 = 2.07271,$$
$$q_0 BP = 4.579 \times 10^{-4}, \qquad \delta_0 = 0.12630, \qquad \sigma_0 = 9.858 \times 10^{-4},$$

whence

$$(m_{0,1} + \delta_0 m_{0,2})p_{0,1} \leq 1.59 \times 10^{-4},$$
$$|\sigma_0 T_{0,2}(x)| \leq \sigma_0 T_{0,2}(1) \leq 1.159 \times 10^{-4},$$

and therefore,

$$|v(x) - W_0(x)| \leq 2.42 \times 10^{-4} + 1.16 \times 10^{-4} + 1.16 \times 10^{-4} = 4.74 \times 10^{-4}.$$

(This computation was verified using MACSYMA). In particular, this shows that we can approximate $v(x)$ by the first four terms of $W_0(x)$, i.e., a cubic polynomial, with error less than 0.0005.

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