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ON THE PIECEWISE CONSTANT COLLOCATION METHOD FOR MULTIDIMENSIONAL WEAKLY SINGULAR INTEGRAL EQUATIONS

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ABSTRACT. Convergence rates of the piecewise constant collocation method (PCCM) and related methods for weakly singular integral equations on an open bounded set $G \subset \mathbf{R}^n$ are investigated in [**3, 7–10**]. The main purpose of this paper is to show how the l_h^2 elements of the system of PCCM can be evaluated in $\mathcal{O}(l_h^2)$ arithmetical operations with an accuracy preserving the convergence rate of the basic PCCM.

1. Integral equation. In this paper, we shall deal with an integral equation

$$u(x) = \int_G K(x,y)u(y)\,dy + f(x), \quad x \in G, \tag{1.1}$$

where $G \subset \mathbf{R}^n$ is an open bounded set with a piecewise smooth boundary ∂G. The following assumptions (A1)–(A4) are made.

(A1) The kernel $K(x,y)$ is twice continuously differentiable on $(G \times G)\backslash\{x=y\}$ and there exists a real number ν ($\nu < n$) such that, for any $x,y \in G$, $x \neq y$, and any multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ with $|\alpha| + |\beta| \le 2$,

(1.2)
$$\left| \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_n} + \frac{\partial}{\partial y_n}\right)^{\beta_n} K(x,y) \right|$$
$$\le b \begin{cases} 1, & \nu + |\alpha| < 0 \\ 1 + |\log|x-y||, & \nu + |\alpha| = 0\,, \\ |x-y|^{-\nu-|\alpha|}, & \nu + |\alpha| > 0 \end{cases} \quad b = \text{constant}.$$

Here the following usual conventions are adopted:

$$|\alpha| = \alpha_1 + \cdots + \alpha_n \quad \text{for } \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}_+^n,$$
$$|x| = (x_1^2 + \cdots + x_n^2)^{1/2} \quad \text{for } x = (x_1, \dots, x_n) \in \mathbf{R}^n.$$

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Note that from (1.2) a similar estimate for $|D_y^\alpha D_{x+y}^\beta D(x,y)|$ follows.

(A2) The homogeneous integral equation corresponding to (1.1) has in $L(G)$ only the trivial solution.

(A3) $f \in C^{2,\nu}(G)$, i.e., f is twice continuously differentiable on G and, for any $x \in G$ and any multi-index $\alpha \in \mathbf{Z}_+^n$ with $|\alpha| \le 2$,

$$|D^\alpha f(x)| \le c_f \begin{cases} 1, & |\alpha| < n-\nu \\ 1 + |\log \rho(x)|, & |\alpha| = n-\nu \\ \rho(x)^{n-\nu-|\alpha|}, & |\alpha| > n-\nu \end{cases}, \quad c_f = \text{constant},$$

where $\rho(x) = \inf_{y\in\partial G}|x-y|$ is the distance from x to ∂G.

(A4) For any $x^1, x^2 \in G$,

$$|f(x^1) - f(x^2)| \le c'_f \begin{cases} d_G(x^1,x^2), & \nu < n-1 \\ d_G(x^1,x^2)[1 + |\log d_G(x^1,x^2)|], & \nu = n-1 \\ d_G(x^1,x^2)^{n-\nu}, & \nu > n-1 \end{cases}$$

where $d_G(x^1,x^2)$ is defined as the infimum of lengths of polygonal paths in G joining points x^1 and x^2; if x^1 and x^2 belong to different connectivity components of G, define $d_G(x^1,x^2) = \infty$.

In many cases (A4) is a consequence of (A3), e.g., if $\nu < n-1$ or if G satisfies the cone condition [**5**].

From (A1)–(A3) it follows that equation (1.1) is uniquely solvable in $C^{2,\nu}(G)$ (see [**6**]).

Note that the kernels $K(x,y) = a(x,y)|x-y|^{-\nu}$ $(0 < \nu < n)$ and $K(x,y) = a(x,y)\log|x-y|$ $(\nu = 0)$ satisfy (A1) if $a(x,y)$ is twice continuously differentiable on $(G\times G)\backslash\{x=y\}$ and its derivatives are bounded or, more generally, e.g., in case $0 < \nu < n$,

$$\left|\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_n} + \frac{\partial}{\partial y_n}\right)^{\beta_n} a(x,y)\right| \\ \le b'|x-y|^{-|\alpha|}, \quad |\alpha| + |\beta| \le 2, \ b' = \text{constant}.$$

A further example of a kernel satisfying (A1) derives from radiation transfer theory and is known as Peierls kernel

$$K(x,y) = \frac{1}{4\pi} e^{-\tau(x,y)} |x-y|^{-2} \sigma_s(y), \quad n=3,\ \nu=2,$$

where

$$\tau(x,y) = |x-y| \int_0^1 \sigma(tx+(1-t)y)\,dt$$

is the optical distance between points $x, y \in G$ (the set G is assumed to be convex in this example); the extinction coefficient $\sigma : \overline{G} \to \mathbf{R}$ and the scattering coefficients $\sigma_s : \overline{G} \to \mathbf{R}$ are assumed to be twice continuously differentiable.

A more general example of a kernel satisfying (A1) is given by

$$K(x,y) = \kappa(x,y,|x-y|)$$

where $\kappa : G \times G \times \mathbf{R}_+ \to \mathbf{R}$ is a twice continuously differentiable function such that, for $|\alpha| + |\beta| + k \le 2$,

$$\left| D_x^\alpha D_y^\beta \frac{\partial^k}{\partial r^k} \kappa(x,y,r) \right| \le b'' r^{-\nu-k}, \quad 0 < \nu < n,\ b'' = \text{constant}.$$

2. Subdivisions of G. Let us denote, for a set $G' \subset G$,

$$d_G\text{-diam}\, G' = \sup_{x,y \in G'} d_G(x,y).$$

Denote by H_c the collection of $h = (h_1, \dots, h_n) \in \mathbf{R}^n$ with $h_i > 0$, $|h|/h_i \le c$ $(i = 1, \dots, n)$. For any $h \in H_c$, divide $\mathbf{R}^n$ into rectangular boxes

$$B_{\lambda,h} = \{x \in \mathbf{R}^n : (\lambda_i - 1/2)h_i \le x_i < (\lambda_i + 1/2)h_i,\ i = 1, \dots, n\},$$
$$\lambda = (\lambda_i, \dots, \lambda_n) \in \mathbf{Z}^n.$$

Let $\Lambda_h \subset \mathbf{Z}^n$ be the subset of those $\lambda \in \mathbf{Z}^n$ that $G \cap B_{\lambda,h}$ is nonvoid. Then, by

$$G_{\lambda,h} = G \cap B_{\lambda,h}, \quad \lambda \in \Lambda_h \subset \mathbf{Z}^n, \tag{2.1}$$

is defined a subdivision of G. We make the following assumption about the regularity of the boundary ∂G:

(A5) For all $h \in H_c$ with sufficiently small $|h|$, the sets $G_{\lambda,h}$, $\lambda \in \Lambda_h$ are connected and d_G-diam $G_{\lambda,h} \le \text{const} \cdot |h|$ where the constant does not depend on h.

Note that d_G-diam $G_{\lambda,h} = |h|$ for inner boxes $G_{\lambda,h} = B_{\lambda,h} \subset G$; thus, this assumption concerns only subsets $G_{\lambda,h}$ such that $\partial G \cap B_{\lambda,h} \neq \varnothing$. Note also that (A5) is trivially fulfilled for a convex set G and const $= 1$ in this case.

Further, in any $G_{\lambda,h}$ choose a point $\xi_{\lambda,h}$ as follows:

$$\begin{cases} \xi_{\lambda,h} = (\lambda_1 h_1, \dots, \lambda_n h_n) \text{ is the center of } G_{\lambda,h} & \text{in case } B_{\lambda,h} \subset G; \\ \xi_{\lambda,h} \in G_{\lambda,h} \text{ is arbitrary} & \text{in case } \partial G \cap B_{\lambda,h} \neq \varnothing. \end{cases} \tag{2.2}$$

These points will be used as collocation points in the PCCM and as nodes in the cubature formula

$$\int_{G_{\lambda,h}} v(y)\,dy \approx v(\xi_{\lambda,h}) w_{\lambda,h}. \tag{2.3}$$

Here weights $w_{\lambda,h}$ are considered as approximations to meas $G_{\lambda,h}$: it is assumed that

$$\begin{cases} w_{\lambda,h} = h_1 \cdot \ldots \cdot h_n & \text{in case } B_{\lambda,h} \subset G, \\ |w_{\lambda,h} - \operatorname{meas} G_{\lambda,h}| \le \text{const}\,|h| h_1 \cdot \ldots \cdot h_n & \text{in case } \partial G \cap B_{\lambda,h} \neq \varnothing, \end{cases} \tag{2.4}$$

where the constant does not depend on $h \in H_c$. Note that in the case of a piecewise C^2-smooth boundary ∂G we can put, e.g.,

$$w_{\lambda,h} = \operatorname{meas} \tilde{G}_{\lambda,h}$$

where $\tilde{G}_{\lambda,h}$ is obtained from $G_{\lambda,h}$ approximating ∂G, inside a box $G_{\lambda,h}$, by means of tangent or secant planes. This procedure can be performed in $\mathcal{O}(1)$ arithmetical operations, but it is not the purpose of this paper to develop these procedures in detail. Instead, we make our last assumption:

(A6) Every weight $w_{\lambda,h}$, $\lambda \in \Lambda_h$, corresponding to a $G_{\lambda,h}$ with a nonvoid $B_{\lambda,h} \cap \partial G$ and satisfying (2.4) can be found in const $|h|^{-1}$ arithmetical operations where the constant does not depend on $h \in H_c$.

Note that the number of $B_{\lambda,h}$ intersecting ∂G is $\mathcal{O}(|h|^{-n+1})$; thus all weights $w_{\lambda,h}$, $\lambda \in \Lambda_h$, satisfying (2.4) can be found in $\mathcal{O}(|h|^{-n})$ arithmetical operations. Note also that $\mathrm{card}\Lambda_h$ is of the order $|h|^{-n}$.

3. Piecewise constant collocation method (PCCM) and the related cubature formula method (CFM). Represent an approximate solution to equation (1.1) in the form $\bar{u}_h = \sum_{\lambda' \in \Lambda_h} u_{\lambda',h}\chi_{\lambda',h}$ where $\chi_{\lambda',h}$ is the characteristic function of $G_{\lambda',h}$ (see (2.1)) and $u_{\lambda',h}$ is an approximate value to the exact solution of (1.1) at the point $\xi_{\lambda',h}$ (see (2.2)). Substituting $\bar{u}_h$ into equation (1.1) and collocating at points $\xi_{\lambda,h}$ we obtain the following PCCM-system of equations with respect to $u_{\lambda,h}$, $\lambda \in \Lambda_h$:

$$u_{\lambda,h} = \sum_{\lambda' \in \Lambda_h} \int_{G_{\lambda',h}} K(\xi_{\lambda,h}, y)\, dy u_{\lambda',h} + f(\xi_{\lambda,h}), \quad \lambda \in \Lambda_h. \tag{3.1}$$

Now, using the cubature formula (2.3) we obtain a related cubature formula method (CFM)

$$u_{\lambda,h} = \sum_{\{\lambda' \in \Lambda_h : |\xi_{\lambda,h} - \xi_{\lambda',h}| \geq c_1 |h|\}} K(\xi_{\lambda,h}, \xi_{\lambda',h}) w_{\lambda',h} u_{\lambda',h} + f(\xi_{\lambda,h}), \qquad \lambda \in \Lambda_h. \tag{3.2}$$

We omitted the terms where the arguments of $K(x,y)$ were too close to one another; c_1 is a positive constant not depending on h.

Theorem 1. *Let assumptions* (A1)–(A5) *hold. Then there exists a* $\delta_0 > 0$ *such that, for all* $h \in H_c$ *with* $|h| < \delta_0$, *systems* (3.1) *and* (3.2) *are uniquely solvable and the following error estimates hold.*

(a) *For* PCCM (3.1),

$$\max_{\lambda \in \Lambda_h} |u_{\lambda,h} - u(\xi_{\lambda,h})| \leq \mathrm{const}\, (\varepsilon_{\nu,h})^2, \tag{3.3}$$

$$\sup_{x \in G} |u_h(x) - u(x)| \leq \mathrm{const}\, (\varepsilon_{\nu,h})^2 \tag{3.4}$$

where u *is the solution to* (1.1),

$$u_h(x) = \sum_{\lambda' \in \Lambda_h} \int_{G_{\lambda',h}} K(x,y)\, dy u_{\lambda',h} + f(x), \quad x \in G, \tag{3.5}$$

and

$$\varepsilon_{\nu,h} = \begin{cases} |h|, & \nu < n-1 \\ |h|(1+|\log|h||), & \nu = n-1 \\ |h|^{n-\nu}, & \nu > n-1 \end{cases}. \tag{3.6}$$

(b) *For* CFM (3.2),

$$\max_{\lambda\in\Lambda_h} |u_{\lambda,h} - u(\xi_{\lambda,h})| \le \text{const}\,\varepsilon'_{\nu,h}, \tag{3.7}$$

$$\sup_{x\in G} |u_h(x) - u(x)| \le \text{const}\,\varepsilon'_{\nu,h} \tag{3.8}$$

where

$$u_h(x) = \sum_{\{\lambda'\in\Lambda_h:\text{dist}\,(x,G_{\lambda',h})\ge c_1|h|\}} K(x,\xi_{\lambda',h}) w_{\lambda',h} u_{\lambda',h} + f(x), \quad x \in G, \tag{3.9}$$

and

$$\varepsilon'_{\nu,h} = \begin{cases} |h|^2, & \nu < n-2 \\ |h|^2(1+|\log|h||), & \nu = n-2 \\ |h|^{n-\nu}, & \nu > n-2 \end{cases}. \tag{3.10}$$

For the proof of this Theorem, we refer to [**9**]. The case of PCCM (3.1) is considered also in [**7**]. In these papers, more general (approximate) subdivisions of G are used.

We see that, for $\nu < n-2$ method (3.2) achieves the accuracy of method (3.1); for $\nu \ge n-2$, method (3.1) is more precise than method (3.2).

4. Refined algorithms for the evaluation of coefficients. To evaluate the integrals (coefficients of system (3.1))

$$t_{\lambda,\lambda',h} = \int_{G_{\lambda',h}} K(\xi_{\lambda,h}, y)\,dy, \quad \lambda,\lambda' \in \Lambda_h \tag{4.1}$$

we shall use cubature formula (2.3) and its composite version:

$$\tilde{t}_{\lambda,\lambda',h} = K(\xi_{\lambda,h}, \xi_{\lambda',h}) w_{\lambda',h}, \tag{4.2}$$

$$\tilde{t}_{\lambda,\lambda',h} = \sum_{\{\mu \in \Lambda_{N^{-1}h} : \xi_{\mu,N^{-1}h} \in G_{\lambda',h}\}} K(\xi_{\lambda,h}, \xi_{\mu,N^{-1}h}) w_{\mu,N^{-1}h}, \tag{4.3}$$

again omitting terms where the arguments of $K(x,y)$ are too near to one another. Here N is an integer which will be chosen depending on $|\xi_{\lambda,h} - \xi_{\lambda',h}|$. More precisely, the following algorithms are proposed.

Algorithm 1. *Fix numbers $c_0 > 0$ and $c_1 > 0$, find an integer $p = p(h)$ such that*

$$2^{-p-1} c_0 < |h| \leq 2^{-p} c_0 \tag{4.4}$$

and

(i) *use formula* (4.2) *if* $|\xi_{\lambda,h} - \xi_{\lambda',h}| \geq c_0$;

(ii) *use formula* (4.3) *with* $N = 2^k$ *if* $2^{-k} c_o \leq |\xi_{\lambda,h} - \xi_{\lambda',h}| < 2^{-k+1} c_0,\ 1 \leq k \leq p-1$;

(iii) *use formula* (4.3) *with* $N = 2^p$ *if* $|\xi_{\lambda,h} - \xi_{\lambda',h}| < 2^{-p+1} c_0$ *omitting terms where* $|\xi_{\mu,N^{-1}h} - \xi_{\lambda,h}| < c_1 2^{-p} |h|$.

Algorithm 2. *Differs from Algorithm* 1 *only in prescription* (ii) *which now has the form*:

(ii′) *use formula* (4.3) *with* $N = 2^{k-\sigma_k}$ *if* $2^{-k} c_0 \leq |\xi_{\lambda,h} - \xi_{\lambda',h}| < 2^{-k+1} c_0,\ 1 \leq k \leq p-1$, *where* $\sigma_k = [s \log_2 k]$ *is the integer part of* $s \log_2 k$ *and* $s > 1/n$ *is a further parameter.*

Due to (4.4) the smallest mesh size used in Algorithms 1 and 2 is of order $|h|^2$. Let us denote by $l_h = \mathrm{card}\Lambda_h$ the number of unknowns in system (3.1). It is a quantity of order $\mathcal{O}(|h|^{-n})$.

Proposition 1. *Let* (A6) *be satisfied. Then the amount of work to evaluate l_h^2 integrals* (4.1) *by means of Algorithm* 1 *or* 2 *is, respectively,* $\mathcal{O}(l_h^2 \log_2 l_h)$ *and* $\mathcal{O}(l_h^2)$ *arithmetical operations.*

Proof. First we estimate the amount of the work needed to compute nonstandard weights $w_{\mu,2^{-k}h}$, $\mu \in \Lambda_{2^{-k}h}$, $k = 0, 1, \dots, p$. According to (A6), the calculation of $w_{\mu,2^{-k}h}$, $\mu \in \Lambda_{2^{-k}h}$, with k fixed, requires $\le c|2^{-k}h|^{-n} = c2^{kn}|h|^{-n}$ arithmetical operations. All weights used in Algorithm 1 do not require more than

$$c|h|^{-n}\sum_{k=0}^{p} 2^{kn} \le c'|h|^{-n}2^{pn} \le c''|h|^{-2n} \le c'''l_h^2$$

arithmetical operations (we exploited (4.4) here). Algorithm 2 requires slightly less work, but still $\mathcal{O}(l_h^2)$ arithmetical operations to evaluate the weights.

Let us consider the case of Algorithm 1. It is sufficient to show that, for any *fixed* $\lambda \in \Lambda_h$, elements $\tilde{t}_{\lambda,\lambda',h}$, $\lambda' \in \Lambda_h$ can be calculated in $\mathcal{O}(l_h \log_2 l_h)$ operations. It is clear that the calculations via (4.2) take $\mathcal{O}(l_h)$ operations. Further, every application of (4.3) with $N = 2^k$, $1 \le k \le p-1$, costs $\le 2 \cdot 2^{kn}$ operations (here 2^{kn} is the number of nodes used in (4.3)). Thereby, formula (4.3) with $N = 2^k$ is used not more than

$$\operatorname{meas}\{y \in \mathbf{R}^n : 2^{-k}c_0 - |h| \le |\xi_{\lambda,h} - y| \le 2^{-k+1}c_0 + |h|\}/\operatorname{meas} B_{\lambda,h} \equiv \tau_{k,h}$$

times, and this quantity can be estimated as follows:

$$\tau_{k,h} \le c|h|^{-n}\operatorname{meas}\{y \in \mathbf{R}^n : |y| \le 2^{-k+2}c_0\} \le c'|h|^{-n}2^{-kn}.$$

Thus, the total work with (4.3) with $N = 2^k$ is $c|h|^{-n}$ arithmetical operations and, for all $k = 1, \dots, p-1$, this number is $c'|h|^{-n}|\log_2|h||$ while, due to (4.4), $p \le |\log_2|h|| + \log_2 c_0$. Quantities $|h|^{-n}|\log_2|h||$ and $l_h \log_2 l_h$ are of the same order. It remains to estimate the amount of work with $N = 2^p$. Due to (4.4), condition $|\xi_{\lambda,h} - \xi_{\lambda',h}| < 2^{-p+1}c_0$ implies inequality $|\xi_{\lambda,h} - \xi_{\lambda',h}| < 4|h|$, therefore the number of $\tilde{t}_{\lambda,\lambda',h}$ calculated via $\{(4.3),\ N = 2^p\}$ remains bounded as $|h| \to 0$. One evaluation by $\{(4.3),\ N = 2^p\}$ costs no more than $2 \cdot 2^{pn} \le 2(c_0/|h|)^n \le cl_h$ arithmetical operations, thus the total amount of work is $\mathcal{O}(l_h)$ arithmetical operations. This completes the proof of Proposition 1 for Algorithm 1.

Let us consider the case of Algorithm 2. Let $\lambda \in \Lambda_h$ be fixed again. The number of arithmetical operations on every application of (4.3) with $N = 2^{k-\sigma_k}$ is $2 \cdot 2^{(k-\sigma_k)n} \le 2 \cdot 2^{kn} \cdot (2k^{-2})^n$; the number of evaluations was estimated by $c'|h|^{-n}2^{-kn}$. Thus, all applications of (4.3) with $N = 2^{k-\sigma_k}$ for $k = 1, \dots, p-1$, but fixed $\lambda \in \Lambda_h$, are done in

$$c|h|^{-n} \sum_{k=1}^{p-1} k^{-ns} \le c'|h|^{-n} \le c'' l_h$$

arithmetical operations. This completes the proof of Proposition 1 for Algorithm 2. □

5. Error analysis (preliminaries). Here we examine the preciseness of cubature formula (4.3) for a single coefficient $t_{\lambda,\lambda',h}$.

Lemma 1. *Let* (A1) *with* $\nu > 0$ *and* (A5) *be satisfied. Let* $|\xi_{\lambda,h} - \xi_{\lambda',h}| \ge 2|h|$. *Then, for cubature formula* (4.3),

(5.1)
$$|t_{\lambda,\lambda',h} - \tilde{t}_{\lambda,\lambda',h}| \le \text{const}\, N^{-2}|h|^2 \int_{G_{\lambda',h}} |\xi_{\lambda,h} - y|^{-\nu-2}\, dy + \text{const}\, N^{-1}|h| \int_{\{y \in B_{\lambda',h} : \rho(y) < N^{-1}|h|\}} |\xi_{\lambda,h} - y|^{-\nu-1}\, dy, \quad \lambda, \lambda' \in \Lambda_h$$

(*if* $\partial G \cap B_{\lambda',h} = \varnothing$, *then the second term in the right hand side can be cancelled*).

Proof. We have, due to (2.4),

(5.2)
$$\begin{aligned} &|t_{\lambda,\lambda',h} - \tilde{t}_{\lambda,\lambda',h}| \\ &\le \sum_{\{\mu \in \Lambda_{N^{-1}h} : \xi_{\mu,N^{-1}h} \in G_{\lambda',h}\}} \left| \int_{G_{\mu,N^{-1}h}} [K(\xi_{\lambda,h}, y) - K(\xi_{\lambda,h}, \xi_{\mu,N^{-1}h})]\, dy \right| \\ &\quad + \sum_{\{\mu \in \Lambda_{N^{-1}} : \xi_{\mu,N^{-1}h} \in G_{\lambda',h}, B_{\mu,N^{-1}h} \not\subset G\}} \left| K(\xi_{\lambda,h}, \xi_{\mu,N^{-1}h}) \right| \\ &\qquad \cdot \left| w_{\mu,N^{-1}h} - \operatorname{meas} G_{\mu,N^{-1}h} \right|. \end{aligned}$$

For inner boxes $G_{\mu,N^{-1}h} = B_{\mu,N^{-1}h} \subset G$, (2.2) implies

$$\int_{G_{\mu,N^{-1}h}} (y - \xi_{\mu,N^{-1}h})\, dy = 0;$$

therefore, for those $G_{\mu,N^{-1}h}$,

$$\begin{aligned}
&\left| \int_{G_{\mu,N^{-1}h}} [K(\xi_{\lambda,h}, y) - K(\xi_{\lambda,h}, \xi_{\mu,N^{-1}h})]\, dy \right| \\
&\qquad = \left| \int_{G_{\mu,N^{-1}h}} \Big[K(\xi_{\lambda,h}, y) - K(\xi_{\lambda,h}, \xi_{\mu,N^{-1}h}) \right. \\
&\qquad\qquad\qquad \left. - \frac{\partial K(\xi_{\lambda,h}, \xi_{\mu,N^{-1}h})}{\partial y} (y - \xi_{\mu,N^{-1}h}) \Big]\, dy \right| \\
&\qquad \le \frac{1}{2} \int_{G_{\mu,N^{-1}h}} \max_{0 \le t \le 1} \left| \frac{\partial^2 K(\xi_{\lambda,h}, ty + (1-t)\xi_{\mu,N^{-1}h})}{\partial y^2} \right| \\
&\qquad\quad \cdot |y - \xi_{\mu,N^{-1}h}|^2\, dy
\end{aligned}$$

where the derivatives are understood in Frechet sense. Here $|y - \xi_{\mu,N^{-1}h}|^2 \le N^{-2}|h|^2/4$ for $y \in G_{\mu,N^{-1}h}$ and, as a consequence of (1.2),

$$\begin{aligned}
\left| \frac{\partial^2 K(\xi_{\lambda,h}, ty + (1-t)\xi_{\mu,N^{-1}h})}{\partial y^2} \right| &\le c|\xi_{\lambda,h} - (ty + (1-t)\xi_{\mu,N^{-1}h})|^{-\nu-2} \\
&\le c'|\xi_{\lambda,h} - y|^{-\nu-2}.
\end{aligned}$$

Summing up over μ we obtain the first term in the right hand side of estimate (5.1).

Now consider boxes $B_{\mu,N^{-1}h}$ intersecting ∂G. For these we use a more simple estimate

$$\begin{aligned}
&\left| \int_{G_{\mu,N^{-1}h}} [K(\xi_{\lambda,h}, y) - K(\xi_{\lambda,h}, \xi_{\mu,N^{-1}h})]\, dy \right| \\
&\qquad \le cN^{-1}|h| \int_{G_{\mu,N^{-1}h}} |\xi_{\lambda,h} - y|^{-\nu-1}\, dy
\end{aligned}$$

which is a corollary of (1.2) and (A5). Summing up over these μ, we obtain the second term in the right hand side of estimate (5.1).

Further, due to (2.4), the second sum in (5.2) can be bounded by quantity

$$cN^{-1}|h| \sum_{\substack{\{\mu\in\Lambda_{N^{-1}h}:\xi_{\mu,N^{-1}h}\in G_{\lambda',h},\\ B_{\mu,N^{-1}h}\not\subset G\}}} \int_{B_{\mu,N^{-1}h}} |K(\xi_{\lambda,h},\xi_{\mu,N^{-1}h})|\,dy.$$

Estimating $|K(x,y)|$ by $c|x-y|^{-\nu-1}$ (not by $b|x-y|^{-\nu}$ which would also be possible) we represent this quantity also in the form of second term in the right hand side of (5.1). We thereby exploit the fact that the $|\xi_{\lambda,h}-\xi_{\mu,N^{-1}h}|$ and $|\xi_{\lambda,h}-y|$ are of the same order if $y\in B_{\mu,N^{-1}h}$ and $|\xi_{\lambda,h}-\xi_{\mu,N^{-1}h}|\geq 2N^{-1}|h|$. The proof of Lemma 1 is completed. □

Now consider the case where $\xi_{\lambda,h}$ and $\xi_{\lambda',h}$ may be close to one another, i.e., $\lambda=\lambda'$.

Lemma 2. *Let* (A1) *with* $\nu>0$ *and* (A5) *be satisfied. Omit from* (4.3) *the terms with* $|\xi_{\mu,N^{-1}h}-\xi_{\lambda,h}|<c_1N^{-1}|h|$. *Then*

(5.3)

$$\begin{aligned}|t_{\lambda,\lambda',h}-\tilde t_{\lambda,\lambda',h}| &\leq \mathrm{const}\,(N^{-1}|h|)^{n-\nu}\\ &\quad+\mathrm{const}\,N^{-2}|h|^2\int_{\substack{\{y\in G_{\lambda',h}:\\ |\xi_{\lambda,h}-y|>N^{-1}|h|\}}}|\xi_{\lambda,h}-y|^{-\nu-2}\,dy\\ &\quad+\mathrm{const}\,N^{-1}|h|\int_{\substack{\{y\in B_{\lambda',h}:\\ \rho(y)<N^{-1}|h|,\\ |\xi_{\lambda,h}-y|>N^{-1}|h|\}}}|\xi_{\lambda,h}-y|^{-\nu-1}\,dy.\end{aligned}$$

Proof. We estimate the integrals and their cubature approximations in a rough manner if arguments of $K(x,y)$ are too near to one another:

$$\begin{aligned}\Bigg|\sum_{\{\mu\in\Lambda_{N^{-1}h}:\xi_{\mu,N^{-1}h}\in G_{\lambda',h},|\xi_{\lambda,h}-\xi_{\mu,N^{-1}h}|<2N^{-1}|h|\}}\int_{G_{\mu,N^{-1}h}}K(\xi_{\lambda,h},y)\,dy\Bigg|\\ \leq b\int_{\{y\in\mathbf{R}^n:|\xi_{\lambda,h}-y|<3N^{-1}|h|\}}|\xi_{y,h}-y|^{-\nu}\,dy\\ \leq \mathrm{const}\,(N^{-1}|h|)^{n-\nu},\end{aligned}$$

and

$$\Bigg| \sum_{\{\mu\in\Lambda_{N^{-1}h}:\xi_{\mu,N^{-1}h}\in G_{\lambda',h},c_1N^{-1}|h|\le|\xi_{\lambda,h}-\xi_{\mu,N^{-1}h}|<2N^{-1}|h|\}} \cdot K(\xi_{\lambda,h},\xi_{\mu,N^{-1}h})w_{\mu,N^{-1}h}\Bigg|$$
$$\le b(c_1N^{-1}|h|)^{-\nu}\mathrm{meas}\,\{y\in\mathbf{R}^n : |\xi_{\lambda,h}-y|<3N^{-1}|h|\}$$
$$\le \mathrm{const}\,\{N^{-1}|h|)^{n-\nu}$$

(the last sum occurs in (4.3) only in case $c_1 < 2$). After this the remaining terms can be treated in a similar way as in the proof of Lemma 1, and the result is (5.3). □

6. Error analysis of Algorithms 1 and 2. Introduce matrices

$$T_h = (t_{\lambda,\lambda',h})_{\lambda,\lambda'\in\Lambda_h}, \qquad \tilde{T} = (\tilde{t}_{\lambda,\lambda',h})_{\lambda,\lambda'\in\Lambda_h}$$

where $\tilde{t}_{\lambda,\lambda',h}$, $\lambda,\lambda'\in\Lambda_h$ are computed by means of Algorithm 1 or 2. We shall estimate the norm

$$||T_h-\tilde{T}_h|| = \max_{\lambda\in\Lambda_h}\sum_{\lambda'\in\Lambda_h}|t_{\lambda,\lambda',h}-\tilde{t}_{\lambda,\lambda',h}|.$$

Lemma 3. *Let* (A1) *and* (A5) *be satisfied. Then, for Algorithms* 1 *and* 2*,*

$$||T_h-\tilde{T}_h|| \le \mathrm{const}\,(|h|^2+|h|^{2(n-\nu)}). \tag{6.1}$$

Proof. It suffices to prove (6.1) in case $n-1\le\nu<n$. Indeed, if (1.2) is fulfilled with a $\nu<n-1$, then also with $\nu=n-1$, and (6.1) with $\nu=n-1$ provides $||T_h-\tilde{T}_h||\le\mathrm{const}\,|h|^2$. Thus, let $n-1\le\nu<n$ hold.

Let us prove (6.1) for Algorithm 1. We have

$$\begin{aligned}||T_h-\tilde{T}_h|| &= \max_{\lambda\in\Lambda_h}\sum_{\lambda'\in\Lambda_h}|t_{\lambda,\lambda',h}-\tilde{t}_{\lambda,\lambda',h}|\\ &= \max_{\lambda\in\Lambda_h}\sum_{k=0}^{p}\tau^{(k)}_{\lambda,h}\end{aligned}$$

where the terms

$$
\begin{aligned}
\tau_{\lambda,h}^{(0)} &= \sum_{\{\lambda' \in \Lambda_h : |\xi_{\lambda,h} - \xi_{\lambda',h}| \geq c_0\}} |t_{\lambda,\lambda',h} - \tilde{t}_{\lambda,\lambda',h}|, \\
\tau_{\lambda,h}^{(k)} &= \sum_{\{\lambda' \in \Lambda_h : 2^{-k} c_0 \leq |\xi_{\lambda,h} - \xi_{\lambda',h}| < 2^{-k+1} c_0\}} |t_{\lambda,\lambda',h} - \tilde{t}_{\lambda,\lambda',h}|, \\
&\qquad k = 1, \dots, p-1, \\
\tau_{\lambda,h}^{(p)} &= \sum_{\{\lambda' \in \Lambda : |\xi_{\lambda,h} - \xi_{\lambda',h}| < 2^{-p+1} c_0\}} |t_{\lambda,\lambda',h} - \tilde{t}_{\lambda,\lambda',h}|
\end{aligned}
$$

correspond to different definitions of $\tilde{t}_{\lambda,\lambda',h}$, see (i), (ii) and (iii) in Algorithm 1. Denote by d the diameter of G. Using Lemmas 1 and 2, we estimate

$$
\begin{aligned}
\tau_{\lambda,h}^{(0)} &\leq c|h|^2 \int_{\{y \in \mathbf{R}^n : c_0 - |h| \leq |\xi_{\lambda,h} - y| < d\}} |\xi_{\lambda,h} - y|^{-\nu-2}\, dy \\
&\qquad + c|h| \int_{\substack{\{y \in \mathbf{R}^n : c_0 - |h| < |\xi_{\lambda,h} - y| < d, \\ \rho(y) < |h|\}}} |\xi_{\lambda,h} - y|^{-\nu-1} \leq c'|h|^2, \\
\tau_{\lambda,h}^{(k)} &\leq c 2^{-2k} |h|^2 \int_{\{y \in \mathbf{R}^n : 2^{-k-1} c_0 < |\xi_{\lambda,h} - y| < 2^{-k+2} c_0\}} |\xi_{\lambda,h} - y|^{-\nu-2}\, dy \\
&\qquad + c 2^{-k} |h| \int_{\substack{\{y \in \mathbf{R}^n : 2^{-k-1} c_0 < |\xi_{\lambda,h} - y| < 2^{-k+2} c_0, \\ \rho(y) < 2^{-k}|h|\}}} |\xi_{\lambda,h} - y|^{-\nu-1}\, dy \\
&\leq c' 2^{-k(n-\nu)} |h|^2, \quad k = 1, \dots, p-1,
\end{aligned} \tag{6.2}
$$

and

$$
\begin{aligned}
\tau_{\lambda,h}^{(p)} &\leq c(2^{-p}|h|)^{n-\nu} \\
&\qquad + c 2^{-2p} |h|^2 \int_{\{y \in \mathbf{R}^n : 2^{-p}|h| < |\xi_{\lambda,h} - y| < 2^{-p+2} c_0\}} |\xi_{\lambda,h} - y|^{-\nu-2} \\
&\qquad + c 2^{-p} |h| \int_{\substack{\{y \in \mathbf{R}^n : 2^{-p}|h| < |\xi_{\lambda,h} - y| < 2^{-p+2} c_0, \\ \rho(y) < 2^{-p}|h|\}}} |\xi_{\lambda,h} - y|^{-\nu-1}\, dy \\
&\leq c' |h|^{2(n-\nu)}.
\end{aligned}
$$

Here we took into account that, due to (4.4), $2^{-p} \le 2c_0^{-1}|h|$, and for $\nu \ge n-1$,

$$\int_{\{y\in\mathbf{R}^n : r<|y|<d\}} |y|^{-\nu-2}\,dy \le \mathrm{const}\, r^{n-\nu-2},$$

$$\int_{\{y'\in\mathbf{R}^{n-1} : r<|y'|<d\}} |y'|^{-\nu-1}\,dy' \le \mathrm{const}\, r^{n-\nu-2};$$

we also have

$$\int_{\{y\in\mathbf{R}^n : r<|\xi_{\lambda,h}-y|<r_1,\,\rho(y)<\delta\}} |\xi_{\lambda,h}-y|^{-\nu-1}\,dy \le c\delta \int_{\{y'\in\mathbf{R}^{n-1} : r<|y'|<r_1\}} |y'|^{-\nu-1}\,dy'$$

(this inequality can be established by arguments using the rectification of boundary ∂G).

Summing up, we obtain

$$\begin{aligned}||T_h - \tilde{T}_h|| &\le c'(|h|^2 + \sum_{k=1}^{p-1} 2^{-k(n-\nu)}|h|^2 + |h|^{2(n-\nu)}) \\ &\le c|h|^2 + c'|h|^{2(n-\nu)}.\end{aligned}$$

This completes the proof in case of Algorithm 1.

In the case of Algorithm 2, only inequalities (6.2) must be overlooked. Now, instead of 2^{-2k} and 2^{-k}, multipliers $2^{-2(k-\sigma_k)}$ and $2^{-(k-\sigma_k)}$ arise in front of the integrals in (6.2), and the result is

$$\tau_{\lambda,h}^{(k)} \le c' k^{2s} 2^{-k(n-\nu)}|h|^2, \quad k = 1, \dots, p-1.$$

Consequently, (6.1) holds again. The proof of Lemma 3 is completed. □

7. Error estimates for approximate solutions. We are interested in the behavior of the solution of system (3.1) with approximated coefficients:

$$\tilde{u}_{\lambda,h} = \sum_{\lambda'\in\Lambda_h} \tilde{\tau}_{\lambda,\lambda',h}\tilde{u}_{\lambda',h} + f(\xi_{\lambda,h}), \quad \lambda \in \Lambda_h. \tag{7.1}$$

Solving this system we can define an approximate solution to (1.1) for all $x \in G$ in a similar way as in (3.5):

$$\tilde{u}_h(x) = \sum_{\lambda' \in \Lambda_h} \int_{G_{\lambda',h}} K(x,y)\,dy\tilde{u}_{\lambda',h} + f(x), \quad x \in G. \tag{7.2}$$

The integrals in (7.2) can be calculated using an extension of Algorithm 1 or 2 which we obtain by substituting $\xi_{\lambda,h}$ for x in (4.2), (4.3) and (i)–(iii). Thus, we design an approximation

$$\tilde{v}_h(x) = \sum_{\lambda' \in \Lambda_h} \tilde{t}_{\lambda',h}(x)\tilde{u}_{\lambda',h} + f(x), \quad x \in G. \tag{7.3}$$

Every evaluation of $\tilde{v}_h(x)$ at a point $x \in G$, $x \notin \Xi_h$, costs $\mathcal{O}(l_h)$ or $\mathcal{O}(l_h \log_2 l_h)$ arithmetical operations if Algorithm 1, or, respectively, Algorithm 2 is used. For $x = \xi_{\lambda,h}$, we have $\tilde{v}_h(\xi_{\lambda,h}) = \tilde{u}_{\lambda,h}$, $\lambda \in \Lambda_h$.

Theorem 2. *Let assumptions* (A1)–(A5) *hold. Let coefficients* $\tilde{t}_{\lambda,\lambda',h}$ *be calculated by means of Algorithm* 1 *or Algorithm* 2. *Then there exists a* $\delta > 0$ *such that, for all* $h \in H_c$ *with* $|h| < \delta$, *systems* (3.1) *and* (7.1) *are uniquely solvable, and*

$$\max_{\lambda \in \Lambda_h} |u_{\lambda,h} - \tilde{u}_{\lambda,h}| \le \text{const}\,(|h|^2 + |h|^{2(n-\nu)}), \tag{7.4}$$

$$\sup_{x \in G} |u_h(x) - \tilde{u}_h(x)| \le \text{const}\,(|h|^2 + |h|^{2(n-\nu)}), \tag{7.5}$$

$$\sup_{x \in G} |\tilde{u}_h(x) - \tilde{v}_h(x)| \le \text{const}\,(|h|^2 + |h|^{2(n-\nu)}). \tag{7.6}$$

where $\{u_{\lambda,h}\}$ *and* $\{\tilde{u}_{\lambda,h}\}$ *are solutions to systems* (3.1) *and* (7.1), *respectively,* $u_h(x)$ *is defined in* (3.5), $\tilde{u}_h(x)$ *is defined in* (7.2) *and* $\tilde{v}_h(x)$ *is obtained from* $\tilde{u}_h(x)$ *approximating the integrals in* (7.2) *by means of the extension of Algorithm* 1 *or* 2.

Proof. Introduce the space E_h of grid functions $u_h : \Xi \to \mathbf{R}$ where $\Xi_h = \{\xi_{\lambda,h}\}_{\lambda \in \Lambda}$, and equip it with norm

$$||u_h|| = \max_{\lambda \in \Lambda_h} |u_h(\xi_{\lambda,h})|.$$

Systems (3.1) and (7.1) can be represented as equations in E_h, $u_h = T_h u_h + p_h f$ and $\tilde{u}_h = \tilde{T}_h \tilde{u}_h + p_h f$, respectively, where $p_h f$ is the restriction of f to grid Ξ_h. In [**7,9**], it is proved that, under assumptions (A1)–(A5), operators $I_h - T_h$ are for sufficiently small $|h|$ invertible, and the inverse operators are uniformly bounded in h:

$$||(I_h - T_h)^{-1}||_{L(E_h,E_h)} \le \text{const}\,, \quad h \in H_c,\ |h| < \delta.$$

According to Lemma 3, we have

$$||T_h - \tilde{T}_h||_{L(E_h,E_h)} \le \text{const}\,(|h|^2 + |h|^{2(n-\nu)})$$

(note that we estimated namely this operator norm).

The last two inequalities immediately imply (7.4). Estimation (7.5) is a direct consequence of (7.4). Repeating the arguments of the proof of Lemma 3, we see that

$$\sup_{x\in G} \sum_{\lambda'\in\Lambda_h} \left| \int_{G_{\lambda',h}} K(x,y)\,dy - \tilde{t}_{\lambda',h}(x) \right| \le \text{const}\,(|h|^2 + |h|^{2(n-\nu)}).$$

This together with the uniform boundedness of $\{\tilde{u}_{\lambda,h}\}$ as $|h| \to 0$ implies (7.6). The proof of Theorem 2 is complete. $\square$

Corollary of Theorems 1 and 2. *Under assumptions* (A1)–(A5)*,*

$$\max_{\lambda\in\Lambda_h} |\tilde{u}_{\lambda,h} - u(\xi_{\lambda,h})| \le \text{const}\,(\varepsilon_{\nu,h})^2,$$
$$\sup_{x\in G} |\tilde{u}_h(x) - u(x)| \le \text{const}\,(\varepsilon_{\nu,h})^2,$$
$$\sup_{x\in G} |\tilde{v}_h(x) - u(x)| \le \text{const}\,(\varepsilon_{\nu,h})^2$$

where u is the solution to integral equation (1.1)*, $\{\tilde{u}_{\lambda,h}\}$ is the solution to system* (7.1) *with coefficients $\tilde{t}_{\lambda,\lambda',h}$ evaluated by means of Algorithm* 1 *or* 2*, $\tilde{u}_h(x)$ is defined by* (7.2) *and $\tilde{v}_h(x)$ by* (7.3)*.*

In other words, Algorithms 1 and 2 are sufficiently precise to preserve the convergence rate of the basic method (3.1).

8. Some further algorithms. Cubature formulae Algorithms 1 and 2 are universal in the sense that they do not depend on ν, the strengthness of the singularity of the kernel. In some sense, they are most properly adapted to the case $\nu = n-1$. Here we give some further modifications of Algorithm 1 depending on ν. In Algorithms 3 and 4, prescription (ii) allows the use of cubature formula (4.3) with an essentially smaller N than $N = 2^k$ as in the case of Algorithm 1. On the other hand, we have not so much succeeded in prescription (iii).

Algorithm 3 (for $n-1 \le \nu < n$). *Fix* $c_0 > 0$, $c_1 > 0$, $c_2 > 0$, $a \in (1-(n-\nu)/2, 1]$, *find* p *such that*

$$2^{-p-1}c_0 < |h|^{1-\Theta} \le 2^{-p}c_0 \quad \text{with} \quad \Theta = \frac{2-2(n-\nu)}{2-(n-\nu)}$$

and

(i) *use* (4.2) *if* $|\xi_{\lambda,h} - \xi_{\lambda',h}| \ge c_0|h|^{\Theta}$;

(ii) *use* (4.3) *with* $N = 2^{[ak]}$ *if* $2^{-k}c_0h^{\Theta} \le |\xi_{\lambda,h} - \xi_{\lambda',h}| < 2^{-k+1}c_0h^{\Theta}$, $1 \le k \le p-1$;

(iii) *use* (4.3) *with* $N = [c_1|h|^{-1}]$ *if* $|\xi_{\lambda,h} - \xi_{\lambda',h}| < 2^{-p+1}c_0h^{\Theta}$ *omitting the terms where* $|\xi_{\mu,N^{-1}h} - \xi_{\lambda,h}| < c_2|h|^2$.

Algorithm 4 (for $n-2 < \nu \le n-1$). *Fix* $c_0 > 0$, $c_1 > 0$, $c_2 > 0$, $a \in (1-(n-\nu)/2, 1]$; *find* p *such that*

$$2^{-p-1}c_0 < |h| \le 2^{-p}c_0$$

and

(i) *use* (4.2) *if* $|\xi_{\lambda,h} - \xi_{\lambda',h}| \ge c_0$;

(ii) *use* (4.3) *with* $N = 2^{[ak]}$ *if* $2^{-k}c_0 \le |\xi_{\lambda,h} - \xi_{\lambda',h}| < 2^{-k+1}c_0$, $1 \le k \le p-1$;

(iii) *use* (4.3) *with* $N = [c_1|h|^{1-2/(n-\nu)}]$ *if* $|\xi_{\lambda,h} - \xi_{\lambda',h}| < 2^{-p+1}c_0$ *omitting the terms where* $|\xi_{\mu,N^{-1}h} - \xi_{\lambda,h}| < c_2|h|^{2/(n-\nu)}$.

Algorithm 5 (for $\nu = n-2$). *Fix* $c_0 > 0$, $c_1 > 0$; *find* p *such that*

$$2^{-p-1}c_0 < |h| < 2^{-p}c_0$$

and

(i) *use* (4.2) *if* $|\xi_{\lambda,h} - \xi_{\lambda',h}| \geq c_0$;

(ii) *use* (4.3) *with* $N = k$ *if* $2^{-k}c_0 \leq |\xi_{\lambda,h} - \xi_{\lambda',h}| < 2^{-k+1}c_0$, $1 \leq k \leq p-1$;

(iii) *use* (4.3) *with* $N = p$ *if* $|\xi_{\lambda,h} - \xi_{\lambda',h}| < 2^{-p+1}c_0$ *omitting the terms where* $|\xi_{\mu,N^{-1}h} - \xi_{\lambda,h}| < c_1|h|\,|\log|h||^{-1/2}$ *or, in case* $n \geq 3$, *simply put* $\tilde{t}_{\lambda,\lambda',h} = 0$.

Note that in the case $\nu < n-1$, the simplest cubature formula method (3.2) is of accuracy $\mathcal{O}(|h|^2)$, and no further algorithms are needed.

Proposition 2. *Let* (A6) *be satisfied. Then the number of arithmetical operations to evaluate the* l_h^2 *integral* (4.1) *is as follows*:

$\mathcal{O}(l_h^2)$ *for Algorithm* 3 *if* $\nu > n-1$ *or* $a < 1$ *and* $\mathcal{O}(l_h^2 \log_2 l_h)$ *if* $\nu = n-1$, $a = 1$;

$\mathcal{O}(l_h^2)$ *for Algorithm* 4 *if* $a < 1$ *and* $\mathcal{O}(l_h^2 \log_2 l_h)$ *if* $a = 1$;

$\mathcal{O}(l_h^2)$ *for Algorithm* 5.

Lemma 4. *Let* (A1) *and* (A5) *be satisfied. Then*:

$||T_h - \tilde{T}_h|| \leq \mathrm{const}\,|h|^{2(n-\nu)}$ *if* $n-1 \leq \nu < n$ *and Algorithm* 3 *is applied*;

$||T_h - \tilde{T}_h|| \leq \mathrm{const}\,|h|^2$ *if* $n-2 < \nu \leq n-1$ *and Algorithm* 4 *is applied*;

$||T_h - \tilde{T}_h|| \leq \mathrm{const}\,|h|^2$ *if* $\nu = n-2$ *and Algorithm* 5 *is applied.*

The proofs of these assertions are similar to the proofs of Proposition 1 and Lemma 3.

From Lemma 4 it follows again that Algorithms 3, 4 and 5 preserve the convergence rate of the basic method (3.1) for respective ν.

9. Concluding remarks. The results of the paper remain valid if, instead of exact subdivisions of G, approximate partitions of G are used where tangent or secant planes are constructed to approximate

∂G inside the boxes which intersect the boundary; the weights $w_{\lambda,h} = \operatorname{meas} G_{\lambda,h}$ can be found exactly in this case. The results can be extended to other sufficiently regular approximate or sharp partitions of G, e.g., for simplex partitions.

Using two grid methods (see [**2, 4, 8, 11**]) system (3.1) and its approximations by Algorithms 1–5 can be solved with an accuracy $\mathcal{O}((\varepsilon_{\nu h})^2)$ in $\mathcal{O}(l_h^2)$ arithmetical operations. Details and proofs are given in [**12**].

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Added in proof:

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