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POSITIVE PERTURBATIONS OF LINEAR VOLTERRA EQUATIONS AND SINE FUNCTIONS OF OPERATORS

ABDELAZIZ RHANDI

Introduction. The purpose of this note is to study perturbations of linear Volterra equations with positive solution families and positive sine functions by positive operators.

Let E be a Banach lattice and A an unbounded closed linear operator in E with dense domain $D(A)$. We say that A is resolvent positive if there exists $w \in \mathbf{R}$ such that $(\mu - A) : D(A) \to E$ is bijective and $(\mu - A)^{-1}$ is a positive operator on E for all $\mu > w$.

Let $a : [0, \infty) \to \mathbf{R}$ be a function which is of bounded variation on each compact interval $[0, T]$, $T > 0$ and consider the linear Volterra equation

$(\mathrm{VO})_A$

$$U(t) := x + a * AU(t) = x + \int_0^t a(t-s)AU(s)\,ds, \quad t \geq 0, \quad x \in D(A).$$

We assume throughout that a is exponentially bounded, i.e., there exist $K \geq 0$, $\beta \geq 0$, such that $|a(t)| \leq K \exp(\beta t)$, $t \geq 0$. Then we can define the function $\mathrm{d}\hat{\mathrm{a}}$ by

$$\mathrm{d}\hat{\mathrm{a}}\,(\mu) = \int_0^\infty \exp(-\mu t)\,\mathrm{da}\,(t), \quad \mu > \beta.$$

We assume further that $\mathrm{d}\hat{\mathrm{a}}\,(\mu) \neq 0$, $\mu > \beta$. A strongly continuous family $(V(t))_{t \geq 0}$ of bounded linear operators on E is called a *solution family* (or a *resolvent*) for $(\mathrm{VO})_A$ if there exist $M \geq 0$, $w \geq \beta$ such that

(i) $||V(t)|| \leq M \exp(wt)$

(ii) $V(0) = 1$

(iii) $(\mu - \mathrm{d}\hat{\mathrm{a}}\,(\mu)A) : D(A) \to E$ is bijective, $\mu > w$ and

$$(\mu - \mathrm{d}\hat{\mathrm{a}}\,(\mu)A)^{-1} = \int_0^\infty \exp(-\mu t)V(t)\,dt.$$

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This notion of "solution family" is the natural extension of the concept of "C_o-semigroup" for $a(t) = 1$ and "cosine family" for the case $a(t) = t$. (M, w) is called *the type* of $v(t)$.

Necessary and sufficient conditions for the existence of a solution family for $[VO]_A$ have been considered by Da Prato and Ianneli [**4**] (see also [**12**] and [**3**]).

Assume that A generates a positive C_o-semigroup and $B : D(A) \to E$ is linear and positive such that $A + B$ is resolvent positive. Then it was shown by Desch [**5**] that $A + B$ generates a positive C_o-semigroup whenever E is a space L^1. A simple proof is given by Voigt [**16**].

In section 1 we give the analogous result when $[VO]_A$ admits a positive solution family $(V(t))_{t\geq 0}$ on L^1 and we also prove that $[VO]_{A+B}$ admits a positive solution family whenever B is a positive rank-one perturbation of A in any Banach lattice E. This result contains the result of [**2**] when A generates a positive C_o-semigroup. The proofs are based on some generalization of a perturbation result by Miyadera and Voigt (see [**11**] and [**17**]. This result is applied to the ordinary differential operator of second order: $(d/dx)^2 + b(x)(d/dx) + c(x)$.

A one-parameter family $(S(t))_{t\geq 0}$ of bounded linear operators in E is called a *sine function* with infinitesimal generator A if it satisfies the following conditions. There exist, $M, w \geq 0$ such that

(i) $S(t)$ is strongly continuous in t and exponentially bounded (i.e., $\| S(t) \| \leq M \exp(wt)(t \geq 0)$ where $M, w \geq 0$.

(ii) $(W^2, \infty) \subset \rho(A)$ and $(\mu^2 - A)^{-1} = \int_0^\infty \exp(-\mu t) S(t) dt$ for all $\mu > w$.

If A generates a sine function $(S(t))_{t\geq 0}$ on E, then we have that for $x \in D(A)$, $y \in D(A^2)$, $u(t) := S(t)x + (d/dt)S(t)y$ is a classical solution of $u''(t) = AU(t), u'(o) = x, u(o) = y$.

In Section 2 we give the same results as in Section 1 when A generates a positive sine function and an application to the Klein-Gordon equation in $L^1(\mathbf{R}^N)(N = 1, 2, 3)$ with a singular potential.

Concerning existence and positivity of the resolvent for $[VO]_A$ see [**13**].

1. Volterra equations. Let E be a Banach space, A an unbounded linear operator in E such that $[VO]_A$ admits a solution family $(V(t))_{t\geq 0}$ of type (M, w) and $B : (D(A), \| \,.\, \|_A) \to E$ a continuous linear mapping.

THEOREM 1.1. Assume that there exist constants $\mu > w$ and $\gamma \in [0, 1)$ such that

(1.1)
$$\int_0^\infty \exp(-\mu r) \parallel B \int_0^r V(r-s)x \, da(s) \parallel dr \leq \gamma \parallel x \parallel \quad (x \in D(A)).$$

Then $[VO]_{A+B}$ admits a solution family $(W(t))_{t\geq 0}$ on E and

(1.2)
$$W(t)x = V(t)x + \int_0^t W(t-r)B \int_0^R v(r-s)x \, da(s)dr \quad (x \in D(A)).$$

Proof. For $t \geq 0$ we define inductively operators $U_n(t) \in L(E)(n = 0, 1, 2, \dots)$ with the following properties:

(i) $[0, \infty) \in t \to U_n(t)$ is strongly continuous,

(ii) $\parallel U_n(t)) \parallel \leq \gamma^n M \exp(\mu t)$ $(t \geq 0)$.

$U_0(t) = V(t)$ satisfies (i) and (ii). If $U_n(.)$ is defined, we put for $x \in D(A)$

$$U_{n+1}(t)x := \int_0^t U_n(t-r)B \int_0^r V(r-s)x \, da(s)dr,$$

then $[0\infty) \in t \to U_{n+1}(t)x$ is continuous and by (ii) and (1.1)

$$\begin{aligned}\parallel U_{n+1}(t)x \parallel &\leq \gamma^n M \exp(\mu t) \int_0^t \exp(-\mu r) \parallel B \int_0^r V(r-s)x da(s) \parallel dr \\ &\leq \gamma^{n+1} M \exp(\mu t) \parallel x \parallel .\end{aligned}$$

$D(A)$ is dense, then $U_{n+1}(t)$ can be extended uniquely to an operator $U_{n+1}(t) \in L(E)$; we have (ii) and (i) for $U_{n+1}(.)$.

We define for $t \geq 0$,

$$W(t) := \sum_{n=0}^{\infty} U_n(t).$$

It follows from (i) and (ii) that $[0, \infty) \ni t \to w(t)$ is strongly continuous, $\| W(t) \| \leq (M/(1-\gamma)) \exp(\mu t)$ and

$$W(t)x = V(t)x + \int_0^t W(t-r)B \int_0^r V(r-s)x \, da(s)dr \quad (x \in D(A)).$$

So, it suffices to prove that $(\gamma - d\hat{a}(\gamma)(A+B)) : D(a) \to E$ is bijective and $(\gamma - d\hat{a}(\gamma)(A+B))^{-1} = \int_0^\infty \exp(-\gamma t)W(t)dt$ for $\gamma > \mu$. Let $\gamma > \mu$, we put

$$H(\lambda) = \int_0^\infty \exp(-\lambda t)W(t)\, dt \quad \text{and} \quad H(\lambda, A) = \int_0^\infty \exp(-\lambda t)V(t)\, dt.$$

Note that $H(\lambda)$ is a bounded linear operator because $(W(t))$ is exponentially bounded. For $x \in D(A)$,

$$\begin{aligned} H(\lambda)x - H(\lambda, A)x &= \int_0^\infty \exp(-\lambda t)(W(t)x - V(t)x)\, dt \\ &= \int_0^\infty \exp(-\lambda t)\left(\int_0^t W(t-r)B\int_0^r V(r-s)x \,\mathrm{da}\,(s)dr\right) dt \\ &= \int_0^\infty \exp(-\lambda t)\int_0^t \int_0^r W(t-r)BV(r-s)x \,\mathrm{da}\,(s)\, dr\, dt. \end{aligned}$$

Applying the Fubini theorem and changing the variable of integration, we have

$$\begin{aligned} H(\lambda)x - H(\lambda, A)x &= \int_0^\infty \left(\int_r^\infty \exp(-\lambda t)\int_0^r W(t-r)BV \right. \\ &\qquad \left. \cdot (r-s)x \,\mathrm{da}(s)\, dt\right) dr \\ &= \int_0^\infty \int_0^\infty \exp(-\lambda t)\exp(-\lambda r)\int_0^r W(t)BV \\ &\qquad \cdot (r-s)x \,\mathrm{da}(s)\, dt\, dr \end{aligned}$$

$$\begin{aligned}
&= \int_0^\infty \exp(-\lambda r)\bigg[\int_0^\infty \exp(-\lambda t) W(t) \\
&\qquad\qquad \cdot \bigg(\int_0^r BV(r-s)x\,\mathrm{da}(s)\bigg)\,dt\bigg]\,dr \\
&= \int_0^\infty \exp(-\lambda r) H(\lambda) \int_0^r BV(r-s)x\,\mathrm{da}(s)\,dr \\
&= H(\lambda)\int_0^\infty \int_s^\infty \exp(-\lambda r) BV(r-s)x\,dr\,\mathrm{da}(s) \\
&= H(\lambda)\int_0^\infty \exp(-\lambda s) B \int_0^\infty \exp(-\lambda r) V(r)x\,dr\,\mathrm{da}(s) \\
&= H(\lambda)\bigg(\int_0^\infty \exp(-\lambda s)\,\mathrm{da}(s)\bigg) BH(\lambda, A)x.
\end{aligned}$$

Since $D(A)$ is dense, we obtain $H(\lambda) - H(\lambda, A) = \mathrm{d}\hat{\mathrm{a}}\,(\lambda) H(\lambda) BH(\lambda, A)$. From the definition of $H(\lambda, A)$ and by (1.1), we have for $x \in D(A)$ and $\lambda > \mu$,

$$\begin{aligned}
||\mathrm{d}\hat{\mathrm{a}}\,(\lambda) BH(\lambda, A)x|| &= \bigg\|\int_0^\infty \exp(-\lambda r) B \int_0^r V(r-s)x\,\mathrm{da}(s)\,dr\bigg\| \\
&\le \int_0^\infty \exp(-\mu r) \bigg\| B \int_0^r V(r-s)x\,\mathrm{da}(s)\bigg\|\,dr \\
&\le \gamma ||x||.
\end{aligned}$$

Then $r(\mathrm{d}\hat{\mathrm{a}}\,(\lambda) BH(\lambda, A)) < 1$ and $H(\lambda) = H(\lambda, A)(1 - \mathrm{d}\hat{\mathrm{a}}\,(\lambda) BH(\lambda, A))^{-1} = (\lambda - \mathrm{d}\hat{\mathrm{a}}\,(\lambda)(A + B))^{-1}$.

Corollary 1.2. *If $B \in L(E)$, then* $(\mathrm{VO})_{A+B}$ *admits a solution family on E.*

Example 1.3. Let $E = C_0(\mathbf{R})$ be the space of continuous functions on $\mathbf{R}$ vanishing in infinity, with supremum norm and consider the cosine function on E defined by

$$(C(t)f)(x) = (1/2)(f(x+t) + f(x-t)). \tag{1.3}$$

Let A be the generator of $(C(t))$. Then $D(A) = \{u \in E : u'' \in E\}$ and $Au = u''$ for $u \in D(A)$. The sine function associated with a cosine function on E is a family $(S(t))$, defined by

$$(S(t)f)(x) = \int_0^t (C(s)f)(x)\,ds = (1/2)\int_{x-t}^{x+t} f(s)\,ds. \tag{1.4}$$

Let b and c belong to E. Then the operator B defined by $Bu = b(.)u' + c(.)u$ is a continuous linear mapping from $(D(A), ||.||_A)$ to E. In the following, we will prove that the operator B satisfies (1.1) when $a(t) = t$.

Let $f \in D(A)$; we have

$$\begin{aligned}(BS(t)f)(x) &= b(x)(d/dx)(S(t)f)(x) + c(x)(S(t)f)(x)\\ &= (b(x)/2)(f(x+t) - f(x-t)) + c(x)(S(t)f)(x).\end{aligned}$$

Hence, $|(BS(t)f)(x)| \le (C(t)|f|)(x)|b(x)| + |c(x)|(S(t)|f|(x)$ and

$$\begin{aligned}\int_0^\infty \exp(-\mu t)||BS(t)f||_\infty\,dt &= \int_0^\infty \exp(-\mu t)\sup_{x\in\mathbf{R}}|(b(x)/2)[f(x+t)\\ &\qquad - f(x-t)] + c(x)(S(t)f)(x)|\,dt\\ &\le \int_0^\infty \exp(-\mu t)\sup_{x\in\mathbf{R}}(|b(x)|(C(t)|f|)(x)\\ &\qquad + |c(x)|(S(t)|f|(x))\,dt\\ &\le \Big[c_1\int_0^\infty \exp(-(\mu-w)t)\,dt\\ &\qquad + c_2\int_0^\infty \exp(-(\mu-w)t)\,dt\Big]||f||_\infty\\ &= ((c_1+c_2)/(\mu-w))||f||_\infty,\end{aligned}$$

where $\mu > w$. We put $\gamma = ((c_1+c_2)/(\mu-w))$ for μ sufficiently large.

Recall that a Banach lattice E is an AL-space if $||u+v|| = ||u|| + ||v||$ whenever $u, v \in E_+$ (see [**15**]). Any space $L^1(\mu)$ is an AL-space.

Corollary 1.4. *Assume that E is an* AL-*space, $(V(t))_{t\ge 0}$ is a positive solution family and $B : D(A) \to E$ is a positive operator. If there exist $\mu > w$ such that*

$$||B((\mu/\mathrm{d}\hat{a}(\mu)) - A)^{-1}|| < 1 \tag{1.5}$$

then $(\mathrm{VO})_{A+B}$ *admits a positive solution family on* E.

Proof. By Theorem 1.1, we have only to show that (1.1) is satisfied. For $x \in D(A)_+$,

$$\begin{aligned}\int_0^\infty \exp(-\mu r)\left\| B\int_0^r V(r-s)x\,\mathrm{da}(s)\right\| dr &= \left\| \int_0^\infty \exp(-\mu r) B\int_0^r V(r-s)x\,\mathrm{da}(s)\,dr\right\| \\ &= ||B((\mu/\mathrm{d}\hat{a}\,(\mu)) - A)^{-1}x|| \\ &\le \gamma||x||\end{aligned}$$

where $\gamma = ||B((\mu/\mathrm{d}\hat{a}\,(\mu)) - A)^{-1}||$. For $x \in D(A)$, $x_{n,\pm} = n(n-A)^{-1}x_\pm$ where $n \in \Omega = \{(\xi/\mathrm{d}\hat{a}\,(\xi)) : \xi > w\}$. We have $x_{n,\pm} \in D(A)_+$, $\lim_{n\to\infty} ||(x_{n,+} - x_{n,-}) - x||_A = 0$ and $\lim_{n\to\infty} ||x_{n,\pm} - x_\pm|| = 0$ (see [**12**]). It follows that

$$\int_0^\infty \exp(-\mu r)\left\| B\int_0^r V(r-s)(x_{n,+} - x_{n,-})\,\mathrm{da}(s)\right\| dr \le \gamma(||x_{n,+}|| + ||x_{n,-}||)$$

and

$$\int_0^\infty \exp(-\mu r)\left\| B\int_0^r V(r-s)x\,\mathrm{da}(s)\right\| dr \le \gamma(||x_+|| + ||x_-||).$$

Finally, using the fact that E is an AL-space, we have that (1.1) holds. □

Theorem 1.5. *Assume that* E *is an* AL*-space,* $(V(t))_{t\ge 0}$ *is a positive family and* $B : D(A) \to E$ *is a positive operator. If* $A+B$ *is resolvent positive, then* $(\mathrm{VO})_{A+B}$ *admits a positive solution family on* E.

Before giving the proof, we apply the result to $a(t) = t$ and obtain

Corollary 1.6. *Let* A *be the generator of a positive cosine function on an* AL*-space* E *and* $B : D(A) \to E$ *a positive linear mapping such* *that $A+B$ is resolvent positive. Then $A+B$ generates a positive cosine function on E.*

Theorem 1.5 is a generalization of the following result established by Desch [**5**] (see also [**16** or **14**]).

Theorem 1.7. *Assume that E is an* AL*-space, A the generator of a positive semigroup on E and $B : D(A) \to E$ a positive operator. If $A+B$ is resolvent positive, then $A+B$ generates a positive semigroup.*

Proof of Theorem 1.5. Voigt's proof of Theorem 1.7 can be adapted to the situation considered here. Since a is exponentially bounded, we have $(\mathrm{d}\hat{a}\,(\lambda)/\lambda) \to 0$, $\lambda \to \infty$. This implies that there exists μ sufficiently large such that $(\mu/\mathrm{d}\hat{a}\,(\mu)) \in \rho(A+B)$ and $((\mu/\mathrm{d}\hat{a}\,(\mu))-A-B)^{-1} \geq 0$. By a result of Voigt [**16**], one has $r(B((\mu/\mathrm{d}\hat{a}\,(\mu))-A)^{-1}) < 1$, $r(sB((\mu/\mathrm{d}\hat{a}\,(\mu))-A)^{-1}) < 1$ for $s \in [0,1]$ and

$$\begin{aligned} ((\mu/\mathrm{d}\hat{a}\,(\mu))-A)^{-1} &\leq ((\mu/\mathrm{d}\hat{a}\,(\mu))-A-sB)^{-1} \\ &\leq ((\mu/\mathrm{d}\hat{a}\,(\mu))-A-B)^{-1} \quad \text{for } s \in [0,1]. \end{aligned} \tag{1.6}$$

Let $n \in \mathbf{N}$ be such that $||B((\mu/\mathrm{d}\hat{a}\,(\mu))-A-B)^{-1}|| < n$. As a consequence of (1.6), one has $||(1/n)B((\mu/\mathrm{d}\hat{a}\,(\mu))-A-(j/n)B)^{-1}|| < 1$, $j = 0,1,\dots,n$. For $j = 0$, $||(1/n)B((\mu/\mathrm{d}\hat{a}\,(\mu))-A)^{-1}|| < 1$, it follows from Corollary 1.4 that $(\mathrm{VO})_{A+(1/n)B}$ admits a positive solution family on E. Successively, we obtain that $(\mathrm{VO})_{A+B}$ admits a positive solution family on E. □

Example 1.8. Linear Klein-Gordon equation with a singular potential in $L^1(\mathbf{R})$. We consider the well-known class of potentials $K_N = \{V \in L^1_{\mathrm{loc}}(\mathbf{R}^N); VD(A_1) \subset L^1(\mathbf{R}^N)$ and $\lim_{\mu\to\infty} ||V(\mu - A_1)^{-1}|| = 0\}$ where A_1 is the Laplacian on $L^1(\mathbf{R}^N)$ (i.e., $D(A_1) = \{f \in L^1 : \Delta f \in L^1\}$, $A_1 f = \Delta f$). If $N = 1$, then $K_N = L^1_{\mathrm{loc,unif}}(\mathbf{R}) = \{V \in L^1_{\mathrm{loc}}(\mathbf{R}) : \sup_x \int_{|x-y|\leq 1} |V(y)|\,dy < \infty\}$ (see [**1**]). If $E = L^1(\mathbf{R})$, $A_1 f = f''$ and $D(A_1) = w^{2,1}(\mathbf{R})$, then A_1 generates a positive cosine function $(C(t))_{t\geq 0}$ on E where $(C(t)f)(x) = (1/2)(f(x+t)+f(x-t))$. So, by Corollary 1.6, $A_1 + V$ generates a positive cosine function on E whenever $0 \leq V \in L^1_{\mathrm{loc,unif}}(\mathbf{R})$.

However, we obtain perturbation results valid in any Banach lattice if we consider positive perturbations of rank-one. By $D(A)'_+$ we denote the cone of all positive linear forms on $D(A)$.

The proof of the following theorem is the same as the one of Theorem 2.2 in [**2**].

Theorem 1.9. *Suppose there exist* $\varphi \in D(A)'_+$, $g \in E_+$ *such that* $Bf := \varphi(f)g$, $f \in D(A)$. *Then* $(\mathrm{VO})_{A+B}$ *admits a positive solution family on* E.

2. Second order equation governed by a sine function. Let E be a Banach space, A be the generator of a sine function $(S_0(t))_{t\geq 0}$ of type (M, w) on E with dense domain $D(A)$. Let $B : (D(A), ||.||_A) \to E$ be a continuous linear mapping.

Theorem 2.1. *Assume that there exist constants* $\mu > w$ *and* $\gamma \in [0, 1)$ *such that*

$$\int_0^\infty \exp(-\mu t)||BS_0(t)x||\,dt \leq \gamma||x||, \quad x \in D(A). \tag{2.1}$$

Then $A + B$ *generates a sine function* $(S(t))_{t\geq 0}$ *on* E *and*

$$S(t)x = S_0(t)x + \int_0^t S(t-s)BS_0(s)x\,ds, \quad x \in D(A). \tag{2.2}$$

Proof. For $t \geq 0$, we define inductively operators $U_n(t) \in L(E)$, $n = 0, 1, 2, \ldots$, with the following properties:

(i) $[0, \infty) \ni t \to U_n(t)$ is strongly continuous.

(ii) $||U_n(t)|| \leq M\gamma^n \exp(\mu t)$, $t \geq 0$.

$U_0(t) := S_0(t)$ satisfies (i) and (ii). If $U_n(.)$ is defined, we put for $x \in D(A)$, $U_{n+1}(t)x := \int_0^t U_n(t-s)BS_0(s)x\,ds$. We can see as in the proof of Theorem 1.1 that $U_{n+1}(t) \in L(E)$ and $S(t) := \sum_0^\infty U_n(t)$ define a sine function with infinitesimal generator $A + B$. □

In the sequel we suppose that E is a Banach lattice, A is the generator of a positive sine function of type (M, w) on E such that

$$\sup_{\mu > w} ||(\mu^2 - w)(\mu^2 - A)^{-1}|| < \infty \tag{2.3}$$

and $B : D(A) \to E$ is a positive linear mapping. For example, if in addition A generates a semigroup, then (2.3) is satisfied.

We observe that, from (2.3), we have

$$\lim_{\mu \to \infty} ||\mu(\mu - A)^{-1}x - x|| = 0 \quad \text{for all } x \in E. \tag{2.4}$$

Consequently, by the same proof as the one of Corollary 1.4, we have

Corollary 2.2. *Assume that E is an* AL*-space. If there exists $\mu > w$ such that*

$$||B(\mu^2 - A)^{-1}|| < 1 \tag{2.5}$$

then $A + B$ generates a positive sine function on E.

Theorem 2.3. *Assume that E is an* AL*-space. If $A+B$ is resolvent positive, then $A + B$ generates a positive sine function on E.*

Theorem 2.4. *Suppose there exist $\varphi \in D(A)'_+$, $g \in E_+$, such that $Bf := \varphi(f)g$, $f \in D(A)$. Then $A+B$ generates a positive sine function on any Banach lattice E.*

Remark 2.5. Assumption (2.3) is essential if we want to prove that (2.5) implies (2.1) and that $\mu(\mu - A - (j/n)B)^{-1}| \to 1$, $\mu \to \infty$ strongly, in order to apply Corollary 2.2 successively for $j = 0, 1, 2, \dots$ (see the proofs of Theorem 1.5 and Theorem 1.9).

Proof. The proofs are similar to those of Theorem 1.5 and Theorem 1.9 so we shall not repeat them here. □

Example 2.6. Linear Klein-Gordon equation with a singular potential in $L^1(\mathbf{R}^N)$, $N = 2, 3$. Let $E = L^1(\mathbf{R}^N)$ and $0 \le V \in K_N$.

A_1 generates a positive sine function on E, where $D(A_1) = \{f \in E : \Delta f \in E\}$ (see [**8** or **6**]). Then, by Theorem 2.3, $A_1 + B$ generates a positive sine function on E.

Remark 2.7. So, the only perturbation result known for sine functions of operators seems to be Theorem 5.3 in [**3**] which we recall here:

Theorem. *Let A generate a sine function and $B \in L(\overline{D(A)}, E)$. Then $A + B$ generates a sine function, too.*

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REFERENCES

1. M. Aizenman and B. Simon, *Brownian motion and Harnack's inequality for Schrödinger operators*, Comm. Pure Appl. Math. **35** (1982), 209–273.

2. W. Arendt and A. Rhandi, *Perturbation of positive semigroups*, Arch. Math. **56** (1991), 107–119.

3. W. Arendt, H. Kellermann, *Integrated solutions of Volterra integro-differential equations and applications*, Proc. Conf. Volterra Integrodifferential Equations in Banach Spaces and Applications (Trento, 1987), Pitman Research Notes (1989), 190.

4. G. Da Prato, M. Iannelli, *Linear integro-differential equations in Banach space*, Rend. Sem. Math. Univ. Padova **62** (1980), 207–219.

5. W. Desch, *Perturbations of positive semigroup on* AL-*space*, preprint.

6. M. Hieber, *Integrated semigroups and differential operators in L^p*, Dissertation, Tübingen, 1989.

7. E. Hille, and R. Phillips, *Functional analysis and semigroups*, Amer. Math. Soc. Colloq. Pub. **31**, Providence, 1957.

8. V. Keyantuo, *Memoire de D.E.A.*, Besançon, Université de Franche-Comté, 1989.

9. I. Miyadera, *On Perturbation theory for semigroup of operators*, Tôhoku Math. J. **18** (1966), 299–310.

10. R. Nagel (ed.), *One-parameter semigroups of positive operators*, Springer-Verlag LN1184, Berlin, 1986.

11. N. Okazawa, T. Takenaka, *A Phillips-Miyadera type perturbation theorem for cosine functions of operators*, Tôhoku Math. J. **30** (1978), 107–115.

12. J. Prüss, *Linear Volterra equations in Banach space and applications* (book to be published).

13. ———, *Positivity and regularity of hyperbolic Volterra equations in Banach spaces*, Math. Ann. **279** (1987), 317–344.

14. A. Rhandi, *Perturbations positives des equations d'evolution et epplications*, Thèse de Doctorat de l'Université de Franche-Comté, Besançon, France, 1990.

15. H.H. Schaefer, *Banach lattice and positive operators*, Springer-Verlag, Berlin, 1974.

16. J. Voigt, *On resolvent positive operators and positive C_0-semi-groups on* AL-*space*, Semigroup Forum **38** (1989), 263–266.

17. ———, *On the perturbation theory for strongly continuous semigroups*, Math. Ann. **229** (1977), 163–171.

Laboratoire de Mathématiques, Université de Franche-Comté 25030 Besançon-Cedex, France