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ON A FORCED QUASILINEAR HYPERBOLIC VOLTERRA EQUATION WITH FADING MEMORY

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ABSTRACT. In this paper we prove the global existence of a solution to a boundary initial value problem for a forced quasilinear hyperbolic Volterra equation under the assumption that the forcing term remains small and can be decomposed into a time-periodic part and a part that decays to zero as $t \to \infty$. We also show that the solution converges to a time-periodic function as $t \to \infty$; the latter is a periodic solution of a related history value problem.

1. Introduction. In this paper we consider global existence and asymptotic behavior of solutions of the problem

$$\begin{aligned} u_t &= \int_0^t a(t-\tau)\sigma(u_x)_x \, d\tau + f(t,x), && \text{for } x \in (0,1),\ t > 0, \\ u(0,x) &= u_0(x), && \text{for } x \in (0,1), \\ u(t,0) &= u(t,1) = 0, && \text{for } t \geq 0. \end{aligned} \tag{1.1}$$

Here $a : (0,\infty) \to R$, $\sigma : R \to R$ is a given smooth function, the data $f : (0,\infty) \times (0,1) \to R$ and $u_0 : (0,1) \to R$ are sufficiently smooth functions compatible with the boundary conditions.

The initial boundary value problem (1.1) has been studied by many authors. In [**7**] MacCamy established a global existence result for the problem (1.1) and showed that the problem (1.1) is related to a theory of heat flow in materials with memory. The existence of global solutions for (1.1) was also established by Dafermos and Nohel [**1**] and Staffans [**11**]. These global existence results treat the case that the initial datum u_0 is sufficiently small and the forcing term f is sufficiently small and decays to 0 as $t \to \infty$.

The purpose of this paper is to study the global existence and asymptotic behavior of solutions for (1.1) in the case that the forcing term f remains small but does not necessarily decay to zero as t tends to ∞. More precisely, we treat the case that f is sufficiently small and can be written in the form $f_1 + f_2$, where f_1 is a time periodic function and f_2 tends to 0 as $t \to \infty$. We show that problem (1.1) has a global solution which converges to time a periodic function as $t \to \infty$ (in fact, to a solution of the history value problem (1.4), (1.5) below). To study existence and asymptotic behavior of problem (1.1), we discuss existence of time periodic solutions for a certain integrodifferential equation which is closely related to the history value problem:

$$\tag{1.2} \begin{aligned} u_t &= \int_{-\infty}^{t} a(t-\tau)\sigma(u_x)_x \, d\tau + f(t,x), && \text{for } x \in (0,1),\ t \in R, \\ u(t,0) &= u(t,1) = 0, && \text{for } t \in R, \end{aligned}$$

where f is a time periodic function.

History value problems have been studied by several authors in the case that f tends to 0 as $|t|$ goes to infinity and the history v of u satisfies

$$u(t,x) = v(t,x) \quad \text{for } x \in (0,1) \text{ and } t \le 0,$$

where the function v is sufficiently smooth and satisfies the equation (1.2) for $t < 0$ [**2**].

Our approach in this paper is based on the energy method employed in [**1**].

Throughout this paper, we denote by $||\cdot||_p$ the norm of $L^p((0,1))$ $(1 \le p < \infty)$ defined by

$$||u||_p^p = \int_0^1 |u(x)|^p \, dx \ (1 \le p < \infty), \qquad ||u||_\infty = \operatorname*{ess\,sup}_{x \in (0,1)} |u(x)|.$$

We also denote the norm of $L^p(0,\infty)$ by the same symbol $||\cdot||_p$. We put $Q_T = (0,T) \times (0,1)$ and $\overline{Q}_T = (0,T) \times (0,1)$. For each function $u : (0,T) \times (0,1) \to R$, $D^k u$ represents the vector

$$D^k u = \left\{ \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial t}\right)^j u \right\}, \quad 0 \le i + j \le k.$$

We denote by $H^1(Q_T)$ the Sobolev space $\{u \in L^2(\overline{Q}_T) : ||Du||_2 < \infty\}$. For given $T > 0$, we set

$$E_T(D^k u) = \sup_{t \in (0,T)} ||D^k u(t)||_2 \quad \text{for each } k \ge 0.$$

We call a function $u \in C^2(R \times (0,1))$ T-periodic if u satisfies $u(t,x) = u(t+T,x)$ for all $x \in (0,1)$ and $t \in R$. For each $\rho > 0$, $T > 0$ and a positive integer m, we set

$$\begin{aligned} V_m(T,\rho) = \{u \in C^2((0,T) \times (0,1)) : E_T(D_u^m) \le \rho, \\ u(t,0) = u(t,1) = 0 \text{ for all } t \in (0,T), \\ D^{m-1}u(t,x) = D^{m-1}u(t+T,x) \text{ for all } t \in (0,T) \text{ and } x \in (0,1)\}. \end{aligned}$$

Let $a(\cdot) \in C^1(0,\infty)$ be a function satisfying $a'(t) \in L^1(0,\infty)$. We define a resolvent kernel $k(\cdot)$ associated with $a'(\cdot)$ by the equation

$$k(t) + \int_0^t a'(t-s)k(s)\,ds = -a'(t), \quad 0 \le t < \infty. \tag{1.3}$$

Our approach is based on an existence result for periodic solutions of the following integrodifferential equation

$$u_{tt} - \sigma(u_x)_x + k(0)u_t = \Phi(t,x) - \int_{-\infty}^t k'(t-\tau)u_t(\tau,x)\,d\tau \tag{1.4a}$$
$$t \in R,\ x \in (0,1),$$
$$u(t,0) = u(t,1) = 0, \quad t \in R, \tag{1.4b}$$

where Φ is a function periodic with respect to the variable t.

We remark that if $\lim_{t\to\infty} k(t) = 0$, then the problem (1.2) is equivalent to (1.4) with

$$\Phi = f_t(t,x) + k(0)f(t,x) + \int_{-\infty}^t k'(t-s)f(s,x)\,dt. \tag{1.5}$$

To state our main result, we impose the following assumptions on the kernel $a(\cdot) \in C^2(0,\infty)$ and the function $\sigma(\cdot)$:

$$(a_1) \quad \begin{aligned} &(i) \quad a(0) = 1; \\ &(ii) \quad a, a', a'' \in L^1(0,\infty), a \text{ is strongly positive definite;} \\ &(iii) \quad t^3 a(t), ta''(t) \in L^1(0,\infty), a''' \in L^1(0,\infty) \cap L^2(0,\infty) : \end{aligned}$$

$$(\sigma_m) \quad \sigma \in C^m(R), \qquad \sigma(0) = 0, \quad \text{and } \sigma'(0) > 0,$$

where m is a positive integer.

We also impose the following conditions on the initial data u_0 and the function $f : (0,\infty) \times (0,1) \to R$:

$$(u_0) \qquad U(u_0) = \int_0^1 (u_0^2 + u_{0x}^2 + u_{0xx}^2 + u_{0xxx}^2)\,dx < \infty.$$

(f) $\quad f = f_1 + f_2, f_1$ is a T-periodic function with $E_T(D^4 f_1) < \infty$ andf_2 is a function satisfying $\displaystyle\int_0^\infty ||D^2 f_2(t)||^2\,dt < \infty$.

For each function f satisfying (f), we put

$$F_m(f) = E_T(D^m f_1) + \int_0^\infty ||D^2 f_2(t)||^2\,dt \quad \text{for each } m \geq 1.$$

Theorem 1.1. *Let* (a_1) *and* (σ_4) *hold. Then there exist* T_0 *and* $\mu > 0$ *satisfying the following property: for each* u_0 *and* f *satisfying* (u_0), (f) *with* $T \leq T_0$, *and* $U(u_0) + F_5(f) < \mu^2$, *the problem* (1.1) *has a solution* $u \in C^2((0,\infty) \times (0,1))$ *satisfying*

$$\sup_{t\in(0,\infty)} ||D^3 u(t)|| < \infty.$$

Moreover, there exists a T*-periodic function* w *such that*

$$\lim_{t\to\infty} \int_t^{t+T} ||D^3(u-w)(s)||^2\,ds = 0 \tag{1.6}$$

holds. Here w *is the* T*-periodic solution of* (1.4) *where* Φ *is the function defined by* (1.5) *with* f *replaced by the* T*-periodic function* f_1.

For smooth solutions, (1.1) is a special case of the initial boundary value problem

$$u_{tt} = \sigma(u_x)_x + \int_0^t a'(t-\tau)\sigma(u_x)_x\,d\tau + f(t,x) \tag{1.7a}$$
$$\text{for } x \in (0,1) \text{ and } t > 0$$

$$u(0,x) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad \text{for } x \in (0,1), \tag{1.7b}$$

$$u(t,0) = u(t,1) = 0, \quad \text{for } t \geq 0. \tag{1.7c}$$

Here $a : (0,\infty) \to R$, $\sigma : R \to R$ is a given smooth function, $f : (0,\infty)\times(0,1) \to R$ and $u_0, u_1 : (0,1) \to R$ are given sufficiently smooth functions compatible with the boundary conditions; for equivalence with (1.1) f is replaced by f_t and $u_1(x) = f(0,x)$.

If $a(0) = 1$ and $a(\infty) > 0$, the initial boundary value problem (1.7) models the motion of a one-dimensional viscoelastic bar. In [**1**] Dafermos and Nohel use an energy method to establish small data global existence results for the heat flow problem (1.1) and the viscoelastic problem (1.7). Similar results were obtained by Staffans [**11**]. Our proof of Theorem 1.1 requires the assumption $a(\infty) = 0$, and we are not able to obtain a similar result for viscoelastic case $a(\infty) > 0$.

Recently, Feireisl [**3**] used techniques of compensated compactness to prove the existence of time-periodic weak solutions for a history value problem related to (1.7) in the viscoelastic case $a(\infty) > 0$ when the forcing function f is time-periodic. Closely related results for the Cauchy problem were obtained by Nohel, Dafermos and Tzavaras [**10**].

To study motions of more general viscoelastic bars, Dafermos and Nohel [**4**] obtained small data global existence and decay results for the initial boundary value problem (1.7) with (1.7a) replaced by

$$u_{tt} = \sigma(u_x)_x + \int_0^t a'(t-\tau)\psi(u_x)_x\, d\tau + f(t,x) \tag{1.7a$'$}$$

where σ, ψ are given smooth material functions. Similar results for the Cauchy problem were obtained by Hrusa and Nohel [**4**]. Unfortunately, the technique used to prove Theorem 1.1 does not extend to the viscoelastic problem (1.7) with (1.7a) replaced by (1.7a)$'$.

2. Preliminaries. Let $k(\cdot)$ be the resolvent kernel of $a'(\cdot)$. For classical solutions, it is known that the problem (1.1) can be reduced to the equivalent form

$$u_{tt} + \frac{\partial}{\partial t}\int_0^t k(t-\tau)u_t(\tau,x)\, d\tau = \sigma(u_x(t,x))_x + \Phi(t,x), \quad t \in (0,\infty),\ x \in (0,1), \tag{2.1a}$$

$$u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x) \quad x \in (0,1), \tag{2.1b}$$

$$u(t,0) = u(t,1) = 0, \quad t \in (0,\infty). \tag{2.1c}$$

In fact, we see by differentiating (1.1) with respect to t and using the resolvent equation (1.3) that problem (1.1) is equivalent to (2.1) with

$$\Phi(t,x) = f_t(t,x) + k(0)f(t,x) + \int_0^t k'(t-s)f(s,x)\,dt \tag{2.2}$$

and

$$u_1(x) = f(0,x) \quad \text{for } x \in (0,1). \tag{2.3}$$

We also note that the equation (2.1a) can be rewritten as

$$\begin{gathered} u_{tt} - \sigma(u_x)_x + k(0)u_t = \Phi(t,x) - \int_0^t k'(t-\tau)u_t(\tau,x)\,d\tau \\ t \in (0,\infty),\ x \in (0,1). \end{gathered} \tag{2.4}$$

We begin with a lemma which summarizes the properties of the resolvent $k(\cdot)$ of $a'(\cdot)$.

Lemma 2.1 [**7**]. *Suppose that assumptions* (a_1) *are satisfied. Then the resolvent kernel $k(\cdot)$ of $a'(\cdot)$ satisfies the following properties*:

(k_1)
- (i) $k(t), k'(t)$ *are bounded in* $(0,\infty)$;
- (ii) $k(t) = k_\infty + K(t), k(0), k_\infty > 0, K(t), K'(t) \in L^1(0,\infty)$;
- (iii) *there exists* $\alpha > 0$ *such that for each* $v \in L^2(0,t)$,
$$\int_0^t v(s)\left\{\frac{\partial}{\partial s}\int_0^s k(s-\tau)v(\tau)\,d\tau\right\} ds \ge \alpha \int_0^t v^2\,dt;$$
- (iv) $tk'(t) \in L^1(0,\infty), k'' \in L^2(0,\infty)$.

Assertions (ii) and (iv) are obtained by arguments given in Lemmas 2.3 and 2.4 of [**4**]. In fact, by taking the Laplace transform in (1.5), we have

$$\hat{k}(s) = \frac{1}{s\hat{a}(0)} + \hat{K}(s), \qquad \hat{K}(s) = \frac{\hat{a}(0) - \hat{a}(s)}{s\hat{a}(s)\hat{a}(0)} - \frac{1}{a(0)}.$$

Here we put $k_\infty = 1/\hat{a}(0)$. Since $t^3 a(\cdot) \in L^1(0,\infty)$, we can see by applying Proposition 4.3 of [**6**] with $\rho = 1 + |t|$ that $\hat{K}(t)$ is locally analytic. Then, since $\hat{K}(\infty) = 0$, it follows by Lemma 2.3 of [**6**] that $tK(t) \in L^1(0,\infty)$. By a similar argument, it follows that $K(t) \in L^1(0,\infty)$. On the other hand, we have by differentiating (1.3) and multiplying by t that

$$tk'(t) + ta'(t)k_\infty + \int_0^t (t-s)a''(t-s)K(s)\,ds + \int_0^t a''(t-s)(sK(s))\,ds = -ta''(t).$$

Then noting that $k'(t) = K'(t)$ and $ta''(t) \in L^1(0,\infty)$, we see that $tk'(t) \in L^1(0,\infty)$. It also follows from (1.3) that $K'(t) \in L^1(0,\infty)$ [**4,7**]. From (1.3) and the assumption that $a''' \in L^1(0,\infty) \cap L^2(0,\infty)$, (i) and the second part of (iv) follow. For the proof of (iii), we refer the reader to [**7**, Lemma 3.1].

3. Existence of periodic solutions for the problem (1.4). In this section we give an existence result for the problem (1.4).

Theorem 3.1. *Let m be a positive integer with $m \geq 2$. Suppose that (σ_m) holds, and that the kernel $k(\cdot)$ satisfies $k' \in L^1(0,\infty)$. Then, for given $\rho > 0$, there exist $T_0 > 0$ and $\tilde{\rho} > 0$ such that for each T-periodic function Φ with $T < T_0$ and $E_T(D^m\Phi) \leq \tilde{\rho}$, the problem* (1.4) *has a T-periodic solution u with $E_T(D^{m+1}u) < \rho$.*

The following existence result due to Matumura [**9**] is crucial for our argument.

Theorem A. *Let $\alpha > 0$. Then, for given $\rho > 0$, there exists $\tilde{\rho} > 0$ such that for each T-periodic function h with $E_T(D^m h) < \tilde{\rho}$, the problem*

$$\begin{aligned} u_{tt} - \sigma(u_x)_x + \alpha u_t = h, &\quad \text{for } t \in R \text{ and } x \in (0,1), \\ u(t,0) = u(t,1) = 0, &\quad \text{for } t \in R, \end{aligned} \tag{3.1}$$

possesses a unique T-periodic solution u satisfying

$$E_T(D^{m+1}u) \leq \rho.$$

Throughout the rest of this section, we fix a positive integer $m \geq 2$. We also fix positive numbers ρ_0, ρ_1 such that for each T-periodic function h with $E_T(D^m h) \leq \rho_1$, there is a unique T-periodic solution u of (3.1) satisfying $E_T(D^{m+1}u) \leq \rho_0$. For each $T > 0$, we define the mapping $L : V_m(T, \rho_1) \to V_{m+1}(T, \rho_0)$ by $u = Lh$ where u is the T-periodic solution of the problem (3.1) with $\alpha = k(0)$. We fix a T-periodic function Φ with $E_T(D^m \Phi) < \infty$ and define the mapping K by

(3.2)

$$Ku = -\int_0^\infty k'(s) u_t(t-s, x)\, ds + \Phi(t, x) \text{ for each } u \in V_{m+1}(T, \rho_0).$$

Then we have

Lemma 3.2. (1) *L is a continuous mapping from $V_m(T, \rho_1)$ into $V_{m+1}(T, \rho_0)$.*

(2) *K is a continuous mapping from $V_{m+1}(T, \rho_0)$ into $H^1(Q_T)$. Here $V_{m+1}(T, \rho_0)$ and $V_m(T, \rho_1)$ are endowed with the relative topology of the Sobolev space $H^2(Q_T)$.*

Proof. (1). We first note that, from the definition of the set $V_m(T, \rho_1)$, we may identify each element of $V_m(T, \rho_1)$ with a T-periodic function. Let $\{h_n\}$ be a sequence in $V_m(T, \rho_1)$ such that $h_n \to h$ strongly in $H^1(Q_T)$. Then, since $V_{m+1}(T, \rho_0)$ is compact in $H^1(Q_T)$, there exists a subsequence $\{Lh_i\}$ of $\{Lh_n\}$ such that Lh_i converges to a point $u \in V_{m+1}(T, \rho_0)$. Here we put $u_i = Lh_i$ for each i. Since $\{u_i\} \subset V_{m+1}(T, \rho_0)$, we may assume that u_{itt} and u_{ixx} converge weakly to u_{tt} and u_{xx} in $L^2(\overline{Q}_T)$. Then it is easy to see that u is the solution of the problem (3.1). Since the solution of the problem (3.1) is unique, we have that any convergent subsequence of $\{Lh_n\}$ converges to u and that $u = Lh$. Thus, we have shown that L is a continuous mapping from a compact convex subset $V_m(T, \rho_1)$ of $H^1(Q_T)$ into $V_{m+1}(T, \rho_0)$.
(2). The assertion of (2) can be proved by a parallel argument as in the proof of (1). $\square$

Proof of Theorem 3.1. From the above argument, we deduce that if

$$K(V_{m+1}(T, \rho_0)) \subset (V_m(T, \rho_1)), \tag{3.3}$$

then the product LK of mapping L and K is well defined and LK is a continuous mapping from $V_{m+1}(T,\rho_0))$ into itself. Then since $V_{m+1}(T,\rho_0))$ is a compact convex subset of $H^1(Q_T)$, there exists a fixed point of LK. It is obvious that the fixed point of LK is a solution of the problem (1.4). We now show that there exists $T_0 > 0$ such that for each T-periodic function Φ with $T < T_0$ and $E_T(D^{m+1}\Phi) < \rho_0/4$, (3.3) holds.

We define a step function $\tilde{k}$ by

$$\tilde{k} = \sum_{n=1}^{\infty} k'(nT)\chi_{nT},$$

where $\chi_{nT}(x) = 1$ for $x \in ((n-1)T, nT)$ and $\chi_{nT}(x) = 0$ otherwise. From the definition of T-periodic functions, we see that

$$\int_{nT}^{(n+1)T} u_t(s,x)\,ds = 0 \quad \text{for each } n \geq 1 \text{ and } x \in (0,1).$$

Then we find

$$g(t) = \int_0^{\infty} k'(s)u_t(t-s,x)\,ds = \int_0^{\infty} (k'(s) - \tilde{k}(s))u_t(t-s,x)\,ds.$$

If $T \to 0$, then it follows that $\int_0^{\infty} |k' - \tilde{k}|\,ds \to 0$. This implies that there exists $T_0 > 0$ such that for each $T < T_0$,

$$E_T(D^m g) \leq \frac{\rho_1}{2} \quad \text{for each } u \in V_{m+1}(T,\rho_0).$$

Let Φ satisfy $E_T(D^m\Phi) < \rho_1/2$. Then it follows that $Ku \in V_m(T,\rho_1)$ for all $u \in V_{m+1}(T,\rho_0)$. This completes the proof. $\square$

4. Proof of Theorem 1.1. In the following the symbols $C, C_0, C_1, \dots$ stand for constants which depend only on σ and k. Throughout this section, we assume that (σ_4) holds. Also, we assume for simplicity that

$$1 < \sigma'(0) < 2, \quad |\sigma''(0)|, \quad |\sigma'''(0)|, \quad |\sigma''''(t)| < 2.$$

We also assume temporarily that

$$(\sigma_*) \qquad 1 \le \sigma'(t) < 2, \quad |\sigma''(t)|, \quad |\sigma'''(t)|, \quad |\sigma''''(t)| < 2$$

for all $t \in R$. There is no restriction in imposing the assumption (σ_*) because we will show the existence of a solution u of (1.1) such that $\sup\{|u_x(t,x)| : t \ge 0, t \in (0,1)\}$ is so small that

$$|\sigma^{(i)}(u(t,x))| < 2 \quad \text{for all } (t,x) \in (0,\infty) \times (0,1),\ i = 1,2,3,4.$$

For each function f satisfying (f), f_1 and f_2 denote the corresponding functions defined in (f). We first show that if v is a solution of (1.1) on $(0,T')$ with $T' > T$ and if v is sufficiently close to a periodic solution w of (1.4) on $(0,T)$, then v remains close to w on $(0,T')$.

Lemma 4.1. *Let* (a_1) *hold. Let* $T > 0$*, and let* f *be a function satisfying* (f)*, and let* w *be a* T*-periodic solution of* (1.4) *where* Φ *is given by* (1.5) *with* f *replaced by* f_1*. Let* v_0 *be a function on* $(0,1)$*. Suppose that the problem* (1.1) *has a solution* v *on* $(0,T')$ $(T' > T)$ *with* $v(0,x) = v_0(x)$ *on* $(0,1)$*. Then there exist positive constants* $\rho_0 < 1$ *and* C_0 *such that, if*

$$(*) \qquad \begin{aligned} &0 < \rho \le \rho_0, \\ &E_T(D^4 w) < \rho,\ E_T(D^3 v) < \rho,\ F_3(f) < \rho^2 \ \textit{and} \\ &E_{T'}(D^3 v) < \rho_0, \end{aligned}$$

then

$$E_{T'}(D^2 v) < C_0 \rho,\ \textit{and} \tag{4.1}$$

$$\int_0^{T'} ||D^2(v-w)(t)||^2 \, d\tau \le C_0 \rho^2. \tag{4.2}$$

Proof. Let f, w and v satisfy the hypotheses of Lemma 4.1. Let $\rho, \rho_0 > 0$ with $\rho \le \rho_0 < 0$. We suppose that ρ, ρ_0 satisfy $(*)$. We put $u = v - w$. We first see from the boundary conditions (1.4b) and (2.1c) that

$$\begin{aligned} &\sup\{u^2(t,x), u_t^2(t,x), u_x^2(t,x) : t \in (0,T'), x \in (0,1)\} \le 4\rho_0^2, \\ &\int_0^1 u_t^2(t,x)\,dx \le \int_0^1 u_{tx}^2(t,x)\,dx \quad \text{and} \\ &\int_0^1 u^2(t,x)\,dx \le \int_0^1 u_x^2(t,x)\,dx \le \int_0^1 u_{xx}^2(t,x)\,dx \end{aligned} \tag{4.3}$$

for $t \in (0, T')$. Since v and w are solutions of (2.1) and (1.4) with Φ defined by (2.2) and (1.5), respectively, we have

$$u_{tt} - \sigma'(v_x)u_{xx} + k(0)u_t = h, \tag{4.4}$$

where $h = h_1 + h_2 + h_3 + h_4 + h_5$,

$$\begin{aligned}
h_1 &= -\int_0^t k'(t-s)u_t(s,x)\,ds, \\
h_2 &= (\sigma'(v_x) - \sigma'(w_x))w_{xx}, \\
h_3 &= \int_{-\infty}^0 k'(t-s)w_t(s,x)\,ds, \\
h_4 &= -\int_{-\infty}^0 k'(t-s)f_1(s,x)\,dt, \\
h_5 &= f_{2t}(t,x) + k(0)f_2(t,x) + \int_0^t k'(t-s)f_2(s,x)\,dt.
\end{aligned}$$

It is easy to see that (4.4) is equivalent to the equation

$$u_{tt} - \sigma'(v_x)u_{xx} + \frac{\partial}{\partial t}\int_0^t k(t-s)u_t(s)\,ds = h_0, \tag{4.5}$$

where $h_0 = h_2 + h_3 + h_4 + h_5$.

Define

$$F_2(u;s) = \int_0^1 \{u_{tt}^2 + (1+\sigma'(v_x))u_{tx}^2 + \sigma'(v_x)u_{xx}^2\}(s,x)\,dx. \tag{4.6}$$

We will show that if ρ_0 is sufficiently small, then

$$F_2(u;s) + C\int_0^s F_2(u;\tau)\,d\tau \le F_2(u;0) + C_1\rho^2, \quad 0 \le s \le T', \tag{4.7}$$

holds for some $C, C_1 > 0$. We first differentiate (4.4) with respect to t. Then we have

$$u_{ttt} - \sigma'(v_x)u_{xxt} - \sigma''(v_x)v_{tx}u_{xx} + k(0)u_{tt} = h_t. \tag{4.8}$$

Multiplying (4.8) by $u_{tt}(t,x)$ and integrating over $(0,s)\times(0,1)$, we find
(4.9)
$$\begin{aligned}\frac{1}{2}\int_0^1 &\{u_{tt}^2+\sigma'(v_x)u_{tx}^2\}(s,x)\,dx+k(0)\int_0^s\int_0^1 u_{tt}^2\,dx\,dt\\ &=\frac{1}{2}\int_0^1\{u_{tt}^2+\sigma'(v_x)u_{tx}^2\}(0,x)\,dx+\int_0^s\int_0^1\frac{1}{2}\sigma''(v_x)v_{tx}u_{tx}^2\,dx\,dt\\ &\quad+\int_0^s\int_0^1(\sigma''(v_x)v_{tx}u_{tt}u_{xx}-\sigma''(v_x)v_{xx}u_{tt}u_{tx}+h_tu_{tt})\,dx\,dt\end{aligned}$$

We next differentiate (4.4) with respect to x and multiply by $u_{tx}(t,x)$. Then we find

$$\begin{aligned}\frac{1}{2}\int_0^1 &(u_{tx}^2+\sigma'(v_x)u_{xx}^2\}(s,x)\,dx+k(0)\int_0^s\int_0^1 u_{tx}^2\,dx\,dt\\ &=\frac{1}{2}\int_0^1\{u_{tx}^2+\sigma'(v_x)u_{xx}^2\}(0,x)\,dx\\ &\quad+\int_0^s\int_0^1\left(\frac{1}{2}\sigma''(v_x)v_{tx}u_{xx}^2+h_xu_{tx}\right)\,dx\,dt.\end{aligned}\tag{4.10}$$

Then, from (4.9) and (4.10), we have
(4.11)
$$\begin{aligned}F_2(u;s)&+2k(0)\int_0^s\int_0^1(u_{tt}^2+u_{tx}^2)\,dx\,dt\\ &\le F_2(u;0)+\int_0^s\int_0^1(\sigma''(v_x)v_{tx}u_{tx}^2+2\sigma''(v_x)v_{tx}u_{tt}u_{xx}\\ &\qquad-2\sigma''(v_x)v_{xx}u_{tt}u_{tx})\,dx\,dt+\int_0^s\int_0^1\sigma''(v_x)v_{tx}u_{xx}^2\,dx\,dt\\ &\qquad+2\int_0^s\int_0^1(h_tu_{tt}+h_xu_{tx})\,dx\,dt.\end{aligned}$$

Since $E_{T'}(D^3u)<2\rho_0$, we can see from (4.11) that

$$\begin{aligned}&F_2(u;s)+2k(0)\int_0^s\int_0^1(u_{tt}^2+u_{tx}^2)\,dx\,dt\\ &\le F_2(u;0)+C_2\left(\rho_0\int_0^s F_2(u;\tau)\,d\tau+\int_0^s\int_0^1(h_tu_{tt}+h_xu_{tx}\right)\,dx\,dt.\end{aligned}\tag{4.12}$$

Here we observe from (iii) of (k_1) that

(4.13)

$$
\begin{aligned}
k(0)\int_0^s\int_0^1 u_{tt}^2\,dx\,dt &- \int_0^s\int_0^1 h_{1t}u_{tt}\,dx\,dt \\
&= \int_0^1\bigg\{\int_0^s u_{tt}(\tau)\bigg(\frac{\partial}{\partial\tau}\int_0^\tau k(\tau-t)u_{tt}(t,x)\,dt + k'(\tau)u_t(0)\bigg)d\tau\bigg\}\,dx \\
&\geq \alpha\int_0^s\int_0^1 u_{tt}^2\,dx\,dt - \frac{1}{2}\bigg(\varepsilon\int_0^s\int_0^1 u_{tt}^2\,dx\,dt + \frac{1}{\varepsilon}||k'||_2^2\rho^2\bigg),
\end{aligned}
$$

where ε is a positive number satisfying $\varepsilon < \alpha$. Similarly, we have

$$
\begin{aligned}
k(0)\int_0^s\int_0^1 u_{tx}^2\,dx\,dt &- \int_0^s\int_0^1 h_{1x}u_{tx}\,dx\,dt \\
&= \int_0^1\bigg\{\int_0^s u_{tx}(\tau)\frac{\partial}{\partial t}\int_0^\tau k(\tau-t)u_{tx}(t,x)\,dt\bigg\}\,dx \\
&\geq \alpha\int_0^s\int_0^1 u_{tx}^2\,dx\,dt.
\end{aligned}
\tag{4.14}
$$

Combining (4.13) and (4.14) with (4.12), we find that

(4.15)

$$
\begin{aligned}
F_2(u;s) &+ C_3\int_0^s\int_0^1 (u_{tt}^2+u_{tx}^2)\,dx\,dt \\
&\leq F_2(u;0) + C_4\bigg(\rho_0\int_0^s F_2(u;\tau)\,d\tau + \int_0^s\int_0^1 (h_{0t}^2+h_{0t}^2)\,dx\,dt + \rho^2\bigg).
\end{aligned}
$$

Hence, we multiply (4.4) by u_{xx}. Then, by (4.3), we find that

$$
\begin{aligned}
\int_0^s\int_0^1 u_{xx}^2(t,x)\,dx\,dt &\leq C_5\int_0^s\int_0^1 (u_{tt}^2+u_t^2+h_0^2)\,dx\,dt \\
&\leq C_5\int_0^2\int_0^1 (u_{tt}^2+u_{tx}^2+h_0^2)\,dx\,dt.
\end{aligned}
\tag{4.16}
$$

From (4.16) and (4.15), it follows that

$$
\begin{aligned}
F_2(u;s) &+ C_6(1-\rho_0)\int_0^s F_2(u;\tau)\,d\tau \\
&\leq F_2(u;0) + C_7\bigg(\int_0^s\int_0^1 (h_0^2+h_{0t}^2+h_{0t}^2)\,dx\,dt + \rho^2\bigg).
\end{aligned}
\tag{4.17}
$$

We next show that

$$\int_0^s \int_0^1 (h_0^2 + h_{0t}^2 + h_{0x}^2)\, dx\, dt \le C_8\biggl(\rho_0 \int_0^s F_2(u;\tau)\, d\tau + \rho^2\biggr). \tag{4.18}$$

We first observe that

$$\begin{aligned}\int_0^s \int_0^1 &((\sigma'(v_x) - \sigma'(w_x))w_{xx})^2\, dx\, dt \\ &\le 4\rho_0 \int_0^s \int_0^1 |v_x - w_x|^2\, dx\, dt \\ &\le 4\rho_0 \int_0^s \int_0^1 |v_{xx} - w_{xx}|^2\, dx\, dt \le 4\rho_0 \int_0^s F_2(u;\tau)\, d\tau.\end{aligned} \tag{4.19}$$

Noting that

$$\begin{aligned}&((\sigma'(v_x) - \sigma'(w_x))w_{xx})_t \\ &\quad = ((\sigma''(v_x)v_{tx} - \sigma''(w_x)w_{tx})w_{xx} + (\sigma'(v_x) - \sigma'(w_x))w_{txx}) \\ &\quad = ((\sigma''(v_x) - \sigma''(w_x))v_{tx} + (\sigma''(w_x)(v_{tx} - w_{tx}))w_{xx} \\ &\qquad + (\sigma'(v_x) - \sigma'(w_x))w_{txx}),\end{aligned} \tag{4.20}$$

we find from the assumption that

$$\begin{aligned}&\int_0^s \int_0^1 \{((\sigma'(v_x) - \sigma'(w_x))w_{xx})_t\}^2\, dx\, dt \\ &\qquad \le C_9\rho_0\biggl\{\int_0^s \int_0^1 |v_x - w_x|^2\, dx\, dt + \int_0^s \int_0^1 |v_{tx} - w_{tx}|^2\, dx\, dt\biggr\} \\ &\qquad \le 2C_9\rho_0 \int_0^s F_2(u;\tau)\, d\tau.\end{aligned} \tag{4.21}$$

Similarly, we obtain

$$\int_0^s \int_0^1 \{((\sigma'(v_x) - \sigma'(w_x))w_{xx})_x\}^2\, dx\, dt \le 2C_9\rho_0 \int_0^s F_2(u;\tau)\, d\tau.$$

Thus, we find that

$$\int_0^s \int_0^1 (h_2^2 + h_{2t}^2 + h_{2x}^2)\, dx\, dt \le C_{10}\rho_0 \int_0^s F_2(u;\tau)\, d\tau. \tag{4.22}$$

We next show that

$$\int_0^s \int_0^1 (h_3^2 + h_{3t}^2 + h_{3x}^2)\, dx\, dt \le C_{11}\rho_0 \int_0^s F_2(u;\tau)\, d\tau. \tag{4.23}$$

Since $E_T(D^4 w) \le \rho$, we have

$$\int_0^s \int_0^1 \left\{ \left(\int_{-\infty}^0 k'(t-\tau) w_t(\tau, x)\, d\tau \right)_x \right\}^2 dx\, dt \\ \le 4\rho^2 \int_0^s \left(\int_{-\infty}^0 |k'(t-s)|\, ds \right)^2 dt.$$

Since $k'(t)$, $tk'(t) \in L^1(0,\infty)$, we obtain that

$$\int_0^s \int_0^1 \left\{ \left(\int_{-\infty}^0 k'(t-\tau) w_t(\tau, x)\, d\tau \right)_x \right\}^2 dx\, dt \le C_{12}\rho^2. \tag{4.24}$$

After similar calculations, we obtain (4.23). It is also easy to see from the assumption $F_3(f) < \rho^2$ that

$$\int_0^s \int_0^1 (h_4^2 + h_{4t}^2 + h_{4x}^2)\, dx\, dt \le C_{13}\rho^2. \tag{4.25}$$

In fact, we can see, for example,

$$\int_0^s \int_0^1 \left(\int_{-\infty}^0 k'(t-s) f_1(s, x)\, dt \right)^2 dx\, dt \\ \le 2\rho^2 \int_0^s \left(\int_t^\infty |k(\tau)|\, d\tau \right)^2 dt \le C_{14}\rho^2. \tag{4.26}$$

From the assumption, we have

$$\int_0^s \int_0^1 (h_5^2 + h_{5t}^2 + h_{5x}^2)\, dx\, dt \le C_{14} F_2(f)^2 \le C_{14}\rho^2.$$

Then, combining (4.22), (4.23) and (4.25) with the inequality above, we obtain the inequality (4.17). Then (4.7) follows from (4.17) by choosing ρ_0 sufficiently small. Then, noting that

$$F_2(u;0) \le (E_T(D^2 v) + E_T(D^2 w))^2 \le 4\rho^2,$$

we obtain

$$||D^2v(t)||^2 \le 4||D^2(v-w)(t)||^2 \le 8F_2(u;t) \le C_0\rho^2,\ 0 \le t \le T'$$

and

$$\int_0^{T'} ||D^2(v-w)(t)||^2\,d\tau \le C_0\rho^2. \qquad \square$$

Lemma 4.2. *Let* (a_1) *hold. Let* f, w *and* v *be as in Lemma* 4.1. *Then there exist positive constants* $\rho_2 < \rho_0$ *and* C_1 *such that, if*

$$(**)\qquad \begin{aligned} &0 < \rho \le \rho_2,\\ &E_T(D^4w) < \rho,\ E_T(D^3v) < \rho,\ F_5(f) < \rho \text{ and}\\ &E_{T'}(D^3v) < \rho_2, \end{aligned}$$

then

$$E_{T'}(D^3v) < C_1\rho \tag{4.27}$$

$$\int_0^{T'} ||D^3(v-w)(t)||^2\,d\tau \le C_1\rho^2. \tag{4.28}$$

Proof. Let f, v and w satisfy the assumption and u be as in Lemma 4.1. Let $\rho, \rho_2 > 0$ which satisfy $(**)$. We first take the second derivative of (4.4) with respect to t and multiply by $u_{ttt}(t,x)$. Then, by integrating over $(0,s)\times(0,1)$, we have

$$\begin{aligned} &\frac{1}{2}\int_0^1 \{u_{ttt}^2 + \sigma'(v_x)u_{ttx}^2\}(s,x)\,dx + k(0)\int_0^s\int_0^1 u_{ttt}^2\,dx\,dt\\ &\quad = \frac{1}{2}\int_0^1 \{u_{ttt}^2 + \sigma'(v_x)u_{ttx}^2\}(0,x)\,dx\\ &\qquad + \int_0^s\int_0^1 \bigg(\frac{1}{2}\sigma''(v_x)v_{tx}u_{ttx}^2 - \sigma''(v_x)v_{xx}u_{ttt}u_{ttx}\\ &\qquad\qquad + 2\sigma''(v_x)v_{tx}u_{ttt}u_{txx} + \sigma''(v_x)v_{ttx}u_{xx}u_{ttt}\\ &\qquad\qquad + \sigma'''(v_x)v_{tx}^2u_{xx}u_{ttt} + h_{tt}u_{ttt}\bigg)\,dx\,dt. \end{aligned} \tag{4.29}$$

We next take the second derivative of (4.4) with respect to x and t and multiply by $u_{ttx}(t,x)$. Then, we have

$$
\begin{aligned}
&\frac{1}{2}\int_0^1 \{u_{ttx}^2 + \sigma'(v_x)u_{txx}^2\}(s,x)\,dx + k(0)\int_0^s\int_0^1 u_{ttx}^2\,dx\,dt \\
&\quad = \frac{1}{2}\int_0^1 \{u_{ttx}^2 + \sigma'(v_x)u_{txx}^2\}(0,x)\,dx \\
&\qquad + \int_0^s\int_0^1 \bigg(\frac{1}{2}\sigma''(v_x)v_{tx}u_{txx}^2 + \sigma''(v_x)v_{tx}u_{ttx}u_{xxx} \\
&\qquad\qquad + \sigma''(v_x)v_{txx}u_{xx}u_{ttx} + \sigma'''(v_x)v_{tx}v_{xx}u_{xx}u_{ttx} \\
&\qquad\qquad + h_{tx}u_{ttx}\bigg)\,dx\,dt.
\end{aligned} \tag{4.30}
$$

Similarly, by taking the derivative of (4.4) with respect to x and multiplying the equation by $u_{txxx}(t,x)$, we have

$$
\begin{aligned}
&\frac{1}{2}\int_0^1 \{u_{txx}^2 + \sigma'(v_x)u_{xxx}^2\}(s,x)\,dx + k(0)\int_0^s\int_0^1 u_{txx}^2\,dx\,dt \\
&\quad = \frac{1}{2}\int_0^1 \{u_{txx}^2 + \sigma'(v_x)u_{xxx}^2\}(s,x)\,dx \\
&\qquad + \int_0^s\int_0^1 \bigg(-\frac{1}{2}\sigma''(v_x)v_{tx}u_{xxx}^2 + \sigma'''(v_x)v_{xx}^2u_{xx}u_{txx} \\
&\qquad\qquad + \sigma''(v_x)v_{xx}u_{txx}u_{xxx} + \sigma''(v_x)v_{xxx}u_{xx}u_{txx} \\
&\qquad\qquad + h_{xx}u_{txx}\bigg)\,dx\,dt.
\end{aligned} \tag{4.31}
$$

Define

$$
\begin{aligned}
F_3(u;s) = \int_0^1 \{&u_{ttt}^2 + (1+\sigma'(v_x))u_{ttx}^2 \\
&+ (1+\sigma'(v_x))u_{txx}^2 + \sigma'(v_x)u_{xxx}^2\}(s,x)\,dx.
\end{aligned} \tag{4.32}
$$

Then, from (4.29), (4.30) and (4.31), we obtain
(4.33)

$$\begin{aligned} F_3(u;s) &+ 2k(0)\int_0^s\int_0^1 (u_{ttt}^2 + u_{ttx}^2 + u_{txx}^2)\,dx\,dt \\ &\le F_3(u;0) + C_2\bigg(\rho_2\int_0^s F_3(u;\tau)\,d\tau + \rho^2 \\ &\qquad + \int_0^s\int_0^1 (h_{tt}u_{ttt} + h_{tx}u_{ttx} + h_{xx}u_{txx})\,dx\,dt\bigg). \end{aligned}$$

In order to assist the reader to see how the estimate (4.33) follows from (4.29)–(4.31), we give a sample calculation:
(4.34)

$$\begin{aligned} \int_0^s\int_0^1 &|\sigma''(v_x)v_{ttx}u_{xx}u_{ttt}|\,dx\,dt \\ &\le 2\int_0^s \left\{(\sup_{x\in(0,1)}|u_{xx}(t,x)|)\int_0^1 |v_{ttx}|\,|u_{ttt}|\,dx\right\}dt \\ &\le 2\int_0^s \left\{(\sup_x|u_{xx}(t,x)|)\left\{\int_0^1 |v_{ttx}|^2\,dx\cdot\int_0^1 |u_{ttt}|^2\,dx\right\}^{\frac12}\right\}dt \\ &\le \left(\sup_t\int_0^1 v_{ttx}^2(t,x)\,dx\right)^{\frac12}\int_0^s\int_0^1 (u_{ttt}^2 + \sup_{y\in(0,1)} u_{xx}^2(t,y))\,dx\,dt. \end{aligned}$$

Here we note that

$$(\sup_{y\in(0,1)}|u_{xx}(t,y)|^2) \le \int_0^1 (u_{xxx}^2(t,y) + u_{xx}^2(t,y))\,dy + u_{xx}^2(t,x)$$

for each $t \in (0,\infty)$ and $x \in (0,1)$. From Lemma 4.1, we have $\int_0^{T'}\int_0^1 u_{xx}^2(t,x)\,dx \le C_0\rho^2$. Then it follows that

$$\int_0^s\int_0^1 (\sup_{y\in(0,1)}|u_{xx}(t,y)|^2)\,dx\,dt \le \rho_2\left(\int_0^s F_3(u;\tau)\right)d\tau + C_0\rho^2 + \rho^2.$$

Thus, we obtain

$$\text{(4.35)}\quad \int_0^s\int_0^1 |\sigma''(v_x)v_{ttx}u_{xx}u_{ttt}|\,dx\,dt \le C_2\left(\rho_2\int_0^s F_3(u;\tau)\right)d\tau + \rho.$$

We next observe from equation (4.4) that

$$\int_0^s \int_0^1 u_{xxx}^2 \, dx \, dt \le 4 \int_0^s \int_0^1 (u_{ttx}^2 + h_x^2) \, dx \, dt + C_3(\rho_2 + 1) \int_0^s F_2(u; \tau) \, d\tau.$$

Then, from (4.33) and the inequality above, we find

$$\begin{aligned} F_3(u; s) &+ C_4(1 - \rho_2) \int_0^s F_3(u; \tau) \, d\tau \\ &\le F_3(u; 0) + C_5 \biggl(\rho^2 + \int_0^s \int_0^1 (h_x^2 + h_{tt}^2 + h_{tx}^2 + h_{xx}^2) \, dx \, dt \biggr). \end{aligned} \tag{4.36}$$

After a long calculation, we deduce

$$\int_0^s \int_0^1 (h_x^2 + h_{tt}^2 + h_{tx}^2 + h_{xx}^2) \, dx \, dt \le C_6 \biggl(\rho_2 \int_0^s F_3(u; \tau) \, d\tau + \rho^2 \biggr). \tag{4.37}$$

We give a calculation to show the roles of assumptions $(**)$ and (σ_*) to deduce (4.35). From the definition of h and (4.20), we find that h_{tt} contains the terms

$$(\sigma'''(v_x) - \sigma'''(w_x)) v_{tx} w_{tx} w_{xx}, \qquad (\sigma'(v_x) - \sigma'(w_x)) w_{ttxx}.$$

We can see from (σ_*),

$$\begin{aligned} &\int_0^s \int_0^1 |(\sigma'''(v_x) - \sigma'''(w_x)) v_{tx} w_{tx} w_{xx}|^2 \, dx \, dt \\ &\qquad \le 4 \int_0^s \int_0^1 |v_x - w_x|^2 |v_{tx} w_{tx} w_{xx}|^2 \, dx \, dt \\ &\qquad \le C_7 \rho_2^6 \int_0^s \int_0^1 u^2 \, dx \, dt \le C_7 \rho_2 \int_0^s F_3(u; \tau) \, d\tau. \end{aligned}$$

We also see from $E_T(D^5 w) < \rho$ and Lemma 4.1 that

$$\begin{aligned} &\int_0^s \int_0^1 |(\sigma'(v_x) - \sigma'(w_x)) w_{ttxx}|^2 \, dx \, dt \\ &\qquad \le 4 \int_0^s \int_0^1 |v_x - w_x|^2 |w_{ttxx}|^2 \, dx \, dt \\ &\qquad \le 4 \sup_{(t,x) \in Rx(0,1)} |w_{ttxx}(t, x)|^2 \int_0^s \int_0^1 u_x^2 \, dx \, dt \\ &\qquad \le 8 \rho_2 \int_0^s F_2(u; \tau) \, d\tau \le C_7 \rho_2 \rho^2. \end{aligned}$$

We also note that the condition $k'' \in L^2(0,\infty) \cap L^1(0,\infty)$ is needed for the estimate (4.37). From (4.36) and (4.37), we find that if we choose ρ_2 sufficiently small, there are constants $C_8, C_9 > 0$ satisfying

$$F_3(u;s) + C_8 \int_0^s F_3(u;\tau)\,d\tau \le F_3(u;0) + C_9\rho^2 \tag{4.38}$$

for all $s \in (0,T')$. Then, since $F_3(u;0) \le 2\rho^2$, the assertion of Lemma 4.2 follows. □

Remark 4.1. It follows from Lemma 4.2 that the periodic solution w of (1.4) is unique. In fact, if w and v are periodic solutions of (1.4), then by Lemma 4.2, $v - w$ converges to 0. Since v, w are periodic, it implies that $v = w$.

To prove Theorem 1.1, we need the following existence theorem which is a direct consequence of Theorem 2 of [**11**] (see also [**6**]).

Theorem B. *Let* (a$_1$) *hold. Then, for given $T, M > 0$, there is a constant $\mu_M > 0$ with the following property. For each u_0 and f satisfying* (u_0)*,* (f)*, and $U(u_0) + F_3(f) < \mu_M^2$, the initial value problem* (1.1) *has a unique solution $u \in C^2((0,T)\times(0,1))$ satisfying $E_T(D^3u) < M$.*

Proof of Theorem 1.1. We first consider the problem

(4.39a)

$$\begin{aligned} v_{tt} - \sigma(v_x)_x + k(0)v_t + \int_0^t k'(t-s)v_t(s,x)\,ds \\ = \Phi(t+nT) - \int_0^{nT} k'(t+nT-s)u_t(s,x)\,ds \quad \text{on } (0,T)\times(0,1), \end{aligned}$$

(4.39b)

$$v(0,x) = u(nT,x), \qquad v_t(0,x) = u_t(nT,x) \quad \text{for } x \in (0,1),$$

(4.39c)

$$v(t,0) = v(t,1) = 0 \quad \text{for } t \in (0,T),$$

where u is the solution of (1.1) on $(0,nT)$ and Φ is the function defined by (2.2).

It follows that if we define $\tilde{u} : (0,(n+1)T) \times (0,1) \to R$ by

$$\tilde{u}(t,x) = \begin{cases} u(t,x) & \text{for } (t,x) \in (0,nT) \times (0,1) \\ v(t-nT,x) & \text{for } (t,x) \in (nT,(n+1)T) \times (0,1), \end{cases}$$

then $\tilde{u}$ is a solution of (1.1) on $(0,(n+1)T)$. Here, we put

(4.40)
$$\Phi_n(t,x) = \Phi(t+nT) - \int_0^{nT} k(t+nT-s)u_t(s,x)\,ds \text{ for each } n \geq 1.$$

Then we see that

(4.41)
$$E_T(D^2\Phi_n) \leq E_T(D^2\Phi) + C_2E_{nT}(D^3u) \leq C_3(F_3(f) + E_{nT}(D^3(u)).$$

From (4.39b) and (4.41), we have by using Theorem B that for given $\rho > 0$, there exists a positive number $\varepsilon < \rho$ such that if

(4.42)
$$F_3(f) < \varepsilon^2, \quad E_{nT}(D^3u) < \varepsilon,$$

then u can be extended to the interval $(0,(n+1)T)$ satisfying

(4.43)
$$E_{(n+1)T}(D^3u) < \rho.$$

On the other hand, we have by using Lemma 4.2 that if ρ is sufficiently small, and the T-periodic solution w of (1.4) with f replaced by f_1 satisfies

(4.44)
$$E_T(D^5w) < \varepsilon,$$

we obtain that

$$E_{(n+1)T}(D^3u) < \varepsilon.$$

Thus, the cycle is closed. That is, u can be extended to $(0,\infty)$ by repeating the argument above in case that (4.44) is satisfied. It follows from Theorem 3.1 that if T and $E_T(D^5f_1)$ $(\leq F(f))$ are sufficiently small, then there exists a periodic solution w satisfying (4.44). Thus, we have show the existence of global solution for (1.1). From the argument above, the inequality (4.28) holds for any $T' > 0$. Then we have that the inequality (1.6) is satisfied. $\square$

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