JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 3, Number 1, Winter 1991

ANALYSIS OF A CHARACTERISTIC EQUATION

DUN-YUAN HAO AND FRED BRAUER

To John Nohel—colleague, collaborator, and friend—in birthday celebration.

ABSTRACT. The characteristic equation

$$z+s+a+c\left[\frac{1-e^{-(z+s)}}{z+s}\right]=0,$$

with $s \geq 0$, arises in the analysis of stability of equilibria of some integrodifferential equations which model the spread of infectious diseases. We obtain some results giving conditions on the parameters a and c for which all roots have negative real part, thus implying stability of an equilibrium.

1. The characteristic equation

$$z+a+c\left(\frac{1-e^{-z}}{z}\right)=0 \tag{1}$$

analyzed in [**2, 6, 7**] has arisen in a variety of epidemic models which are formulated as integrodifferential equations. Recently, it has arisen in an S-I-R-S model with a nonlinear incidence rate of the form $\beta I^p S$, a recovery rate γI, and temporary immunity to reinfection for a fixed period ω [**6**]. For this model, there is always a disease-free equilibrium; the number of nontrivial equilibria depends on the values of p and $\sigma = \beta/\gamma$. More specifically, if $p < 1$, there is one nontrivial equilibrium. If $p = 1$, there is no nontrivial equilibrium if $\sigma \leq 1$ and one nontrivial equilibrium if $\sigma > 1$. If $p > 1$, there is a critical value σ^* such that there is no nontrivial equilibrium if $\sigma < \sigma^*$, one nontrivial equilibrium if $\sigma = \sigma^*$, and two nontrivial equilibria if $\sigma > \sigma^*$ [**8, 9**].

If births and deaths are introduced in the above model, with birth rate μ and constant total population size, similar results hold for the

AMS *Subject Classification:* 34K20, 92A15.
Key words. Characteristic equations, stability, epidemic models

number of equilibria, except that now $\sigma = \beta/(y+\mu)$. The characteristic equation determining the stability of each nontrivial equilibrium is now

$$z + s + a + c\left[\frac{1 - e^{-(z+s)}}{z+s}\right] = 0 \tag{2}$$

with $s > 0$. The behavior of the roots of (2) is considerably more complicated than in the special case (1) corresponding to $s = 0$. By writing (2) in the form

$$z + s + a = -c\left[\frac{1 - e^{-(z+s)}}{z+s}\right]$$

and using the estimates

$$\left|-c\frac{1 - e^{-(z+s)}}{z+s}\right| \le |c|\left(\frac{1 - e^{-s}}{s}\right),$$
$$|z + s + a| \ge s + a$$

if $a \ge 0$, $Rz \ge 0$, it is easy to show that, if $a > 0$ and

$$|c|\left(\frac{1 - e^{-s}}{s}\right) < s + a, \tag{3}$$

the roots of (2) must lie in the left-half plane $Rz < 0$. However, in the characteristic equation for the equilibria of the model of [**8**], the parameter a is not necessarily positive, and a more general result is needed.

2. The problem of finding conditions under which all zeros of a given transcendental function have negative real part has been studied by many authors using a variety of approaches, see, for example, [**1, 2, 3, 4, 10**]. We shall use a geometric approach to analyze equation (2). Our goal is to find the largest region in the $a - c$ plane such that all roots of equation (2) have negative real part for (a, c) in the region.

If we consider $s > 0$ as a parameter, equation (2) defines a mapping from the complex z-plane, with $z = x + iy$, into the $a - c$ plane:

$$\begin{aligned} a(x,y) &= -2(x+s) + \{[y^2 + (x+s)^2]e^{-(x+s)}\sin y\}/\Delta(x,y) \\ c(x,y) &= -y\{y^2 + (x+s)^2\}/\Delta(x,y), \end{aligned} \tag{4}$$

where

$$\Delta(x,y) = (x+s)e^{-(x+s)}\sin y - y\{1 - e^{-(x+s)}\cos y\} \tag{5}$$

When $x+s=0$, the mapping (4) reduces to

$$a = -\frac{y\sin y}{1-\cos y}, \quad c = \frac{y^2}{1-\cos y}. \tag{6}$$

To find the imaginary root curve (the curve in the $a-c$ plane along with (2) has pure imaginary roots), we let $x=0$ in (4), and we obtain

$$\begin{aligned} a(0,y) &= -2s + \frac{(y^2+s^2)e^{-s}\sin y}{se^{-s}\sin y - y(1-e^{-s}\cos y)} \\ c(0,y) &= \frac{-y(y^2+s^2)}{se^{-s}\sin y - y(1-e^{-s}\cos y)}. \end{aligned} \tag{7}$$

The function $\Delta(x,y)$ has the properties $\Delta(x,y) = -\Delta(x,-y)$ and $\Delta(x,0)=0$, but $\Delta(x,y)\neq 0$ for $y \neq 0$. In fact, if $y\neq 0$,

$$\begin{aligned} |\Delta(x,y)| &= |y|\left|(x+s)e^{-(x+s)}\frac{\sin y}{y} - (1-e^{-(x+s)}\cos y)\right| \\ &\geq |y||1-e^{-(x+s)}\cos y| - |y|\left|(x+s)e^{-(x+s)}\frac{\sin y}{y}\right| \\ &\geq |y|\left[1-e^{-(x+s)} - (x+s)e^{-(x+s)}\right] \\ &= |y|e^{-(x+s)}\left[e^{(x+s)} - 1 - (x+s)\right] > 0. \end{aligned}$$

The imaginary root curve is given parametrically by the functions $a(0,y), c(0,y)$. Because $a(0,y)+2s = a(0,-y)+2s$ and $c(0,y) = c(0,-y) > 0$, the imaginary root curve is traversed twice as y varies from $-\infty$ to $+\infty$, once as y varies from $-\infty$ to 0 and once as y varies from 0 to ∞.

We also have $a(0,k\pi)+2s=0$ and, for sufficiently large k, $c(0,(2k+2)\pi) > c(0,2k\pi) > c(0,(2k+1)\pi)$ because

$$\frac{(2k+2)^2\pi^2+s^2}{1-e^{-s}} > \frac{(2k)^2\pi^2+s^2}{1-e^{-s}} > \frac{(2k+1)^2\pi^2+s^2}{1+e^{-s}}.$$

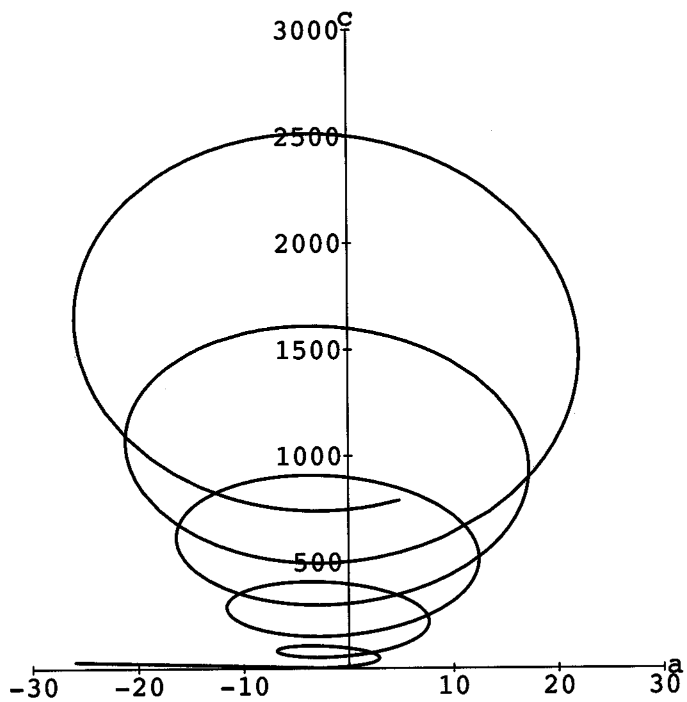

FIGURE 1.

Thus, the imaginary root curve intersects itself and "spirals upward" as shown in Figure 1.

In order to determine the orientation of the mapping (4), we must calculate its Jacobian. We write equation (2) with $z = x + iy$ in the form

$$V(x, y, a, c) + iW(x, y, a, c) = 0,$$

with

$$V(x, y, a, c) = a(x + s) + c(1 - e^{-(x+s)} \cos y) + (x + s)^2 - y^2,$$
$$W(x, y, a, c) = 2y(x + s) + ay + ce^{-(x+s)} \sin y.$$

By the Cauchy-Riemann equations for the analytic function $V + iW$, we have

$$\frac{\partial(a,c)}{\partial(x,y)} = \frac{1}{\Delta(x,y)}\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

If $\partial v/\partial x = \partial v/\partial y = 0$, then

$$y = \tan y, \quad \cos y = \frac{1}{e^s + \sqrt{e^{2s} - (1+s)^2}}.$$

If either of these equations is not satisfied, then $\partial(a,c)/\partial(x,y) \neq 0$. Because $\Delta(x,y) < 0$, if $y > 0$ and $\Delta(x,y) > 0$, if $y < 0$, we see that $\partial(a,c)/\partial(x,y)$ is negative if $y > 0$ and positive if $y < 0$. Thus, if we give an orientation to the imaginary root curve corresponding to the direction of increasing y as y varies from 0 to $+\infty$, since $\partial(a,c)/\partial(x,y)|_{x=0,y>0} < 0$, there are roots of equation (2) with positive real part to the left of the imaginary root curve.

The zero root curve for (2) is the curve in the $a - c$ plane on which $z = 0$ is a root of (2). It is obtained by substituting $z = 0$ in (2), which gives the line

$$s^2 + as + c(1 - e^{-s}) = 0. \tag{8}$$

Because

$$a(0,0) = \lim_{y\to 0} a(0,y) = -2s + \frac{s^2 e^{-s}}{se^{-s} - 1 + e^{-s}} = -2s - \frac{s^2}{e^s - 1 - s},$$
$$c(0,0) = \lim_{y\to 0} c(0,y) = -\frac{s^2}{se^{-s} - 1 + e^{-s}} = \frac{s^2 e^s}{e^s - 1 - s},$$

the point $(a(0,0), c(0,0))$ is on the zero root line (8) as well as on the imaginary root curve (7).

Lemma 1. *The imaginary root curve* (7) *lies above the zero root line* (8) *except for an intersection at* $(a(0,0), c(0,0))$.

PROOF. The region above the zero root line is characterized by $s^2 + as + c(1 - e^{-s}) > 0$. We calculate the quantity

$$\begin{aligned}
D &= s^2 + sa(0,y) + c(0,y)(1-e^{-s}) \\
&= s^2 + s\left[-2s + \frac{e^{-s}(y^2+s^2)\sin y}{\Delta(0,y)}\right] + (1-e^{-s})\frac{-y(y^2+s^2)}{\Delta(0,y)} \\
&= \frac{1}{\Delta(0,y)}\left[-s^2\Delta(0,y) = se^{-s}(y^2+s^2)\sin y - (1-e^{-s})(y^3+ys^2)\right] \\
&= \frac{1}{\Delta(0,y)}\left[-y^3(1-e^{-s}) + s^2e^{-s}y(1-\cos y) + se^{-s}y^2\sin y\right] \\
&= \frac{-y^3}{\Delta(0,y)}\left[1 - e^{-s} - s^2e^{-s}\frac{1-\cos y}{y^2} - se^{-s}\frac{\sin y}{y}\right].
\end{aligned}$$

Because

$$\begin{aligned}
1-e^{-s} - s^2e^{-s}\frac{1-\cos y}{y^2} - se^{-s}\frac{\sin y}{y} &\geq 1 - e^{-s} - se^{-s} - \frac{1}{2}s^2e^{-s} \\
&= e^{-s}(e^s - 1 - s - \frac{1}{2}s^2) > 0
\end{aligned}$$

and $-y^3/\Delta(0,y) > 0$, we have $D > 0$ for $s > 0$, $0 < y < \infty$. Thus, the imaginary root curve lies above the zero root line. □

Using the calculations of this section, we may now divide the $a - c$ plane into four parts, as follows:

(i) the region below the zero root line,

(ii) the region above the zero root line and to the left of the imaginary root curve,

(iii) the region surrounded by the imaginary root curve, and

(iv) the region above the zero root line and to the right of the imaginary root curve.

It is well known that, for a pathwise connected region G in the $a - c$ plane such that neither the zero root line nor the imaginary root curve passes through any point of G, if equation (2) has a root with positive real part at one point (a_0, c_0) of G, then equation (2) has a root with positive real part at every point of G.

We have already seen that if (a, c) is in the region surrounded by the imaginary root curve, there are roots of equation (2) with positive real part. If (a, c) is below the zero root line, the function

$$f(z) = z^2 + (2s + a)z + as + s^2 + c[1 - e^{-(z+s)}]$$

is negative for $z = 0$ and $\lim_{t\to\infty} f(z) = +\infty$. Thus, there is a positive real root of f, and, since $f(z) = 0$ is equivalent to (2) for $z \neq -s$, there is a positive real root of (2) if (a, c) is below the zero root line. The line

$$(x + s)^2 + a(x + S) + c[1 - e^{-(x+s)}] = 0 \tag{4}$$

in the $a - c$ plane has slope

$$-\frac{x + s}{1 - e^{-(x+s)}} < -\frac{s}{1 - e^{-s}}$$

and intersects the a-axis at $-(x+s) < -s$ for every $x > 0$. For suitable $x > 0$, this line contains points above the zero root line and to the left of the imaginary root curve. The line (9) is the locus of points in the $a - c$ plane for which $x > 0$ is a root of (2) and enters the region above the zero root line and to the left of the imaginary root curve. This region cannot contain points for which all roots of (2) have negative real part because it is possible to go from a point where all roots of (2) have negative real part to a point where (2) has positive roots only by crossing either the zero root line or the imaginary root curve.

The point $(0, 0)$ is above the zero root line and to the right of the imaginary root curve. For $a = 0$, $c = 0$, equation (2) has only the root $z = -s < 0$. Thus, the region of the $a - c$ plane in which all roots of equation (2) have negative real part is the set of points which can be reached from the origin without crossing either the zero root line or the imaginary root curve. We have now established the following result.

THEOREM 1. *All roots of equation* (2) *have negative real part if and only if the point* (a, c) *lies above the zero root line and to the right of the imaginary root curve.*

3. As we may see from Figure 1, the imaginary root curve is rather complicated. In order to examine the stability of nontrivial equilibria of the infectious disease model studied in [**6**], it is necessary for us to obtain a more explicit characterization of a region in which all roots of (2) have negative real part.

The condition that $z = iy$ is a root of (2) is $(iy+s)^2 + iay + as + c - ce^{-(iy+s)} = 0$. Separation into real and imaginary parts gives

$$\begin{aligned} ce^{-s}\cos y &= s^2 + as + c - y^2, \\ ce^{-s}\sin y &= -2sy - ay. \end{aligned} \tag{10}$$

Squaring and adding the two equations of (10) gives the condition

$$y^4 + y^2[a^2 + 2as + 2s^2 - 2c] + [(s^2 + as + c)^2 - c^2e^{-2}e^{-2s}] = 0. \tag{11}$$

This quadratic equation for y^2 must have a positive real root in order for (2) to have a pure imaginary root. Thus, a region of the $a-c$ plane in which (11) does not have a positive real root for y^2 does not contain any points of the imaginary root curve.

If

$$a^2 + 2as + 2s^2 - 2c > 0, \quad (s^2 + as + c)^2 - c^2e^{-2s} > 0, \tag{12}$$

the roots for y^2 are real and negative. If

$$2c < (a+s)^2 + s^2, \quad s^2 + as + c(1 - e^{-s}) > 0,$$

that is above the zero root line (8) and below the parabola,

$$2c = (a+s)^2 + s^2, \tag{13}$$

the conditions (12) are both satisfied. The parabola (13) and the zero root line intersect for

$$\begin{aligned} a = a_1 &= -2s + \frac{s}{e^s - 1}\left(\sqrt{2e^s - 1} - 1\right), \\ a = a_2 &= -2s - \frac{s}{e^s - 1}\left(\sqrt{2e^s - 1} - 1\right), \end{aligned}$$

and

$$a_2 < a(0,0) < a_1.$$

From this, we see that, in the region above the zero root line (8) and below the parabola (13) lying to the right of $a = a_1$, the roots of (2) all have negative real part.

If $2c \geq (a+s)^2 + s^2$, but

$$(s^2 + as + c)^2 - c^2 e^{-2s} - \frac{1}{4}[(a+s)^2 + s^2 - 2c]^2 > 0, \tag{14}$$

the root for y^2 of (11) are complex, and, again, there can be no pure imaginary roots of (2). The inequality (14) is equivalent to

$$(a+2s)^2 \left(c - \frac{a^2}{4}\right) > c^2 e^{-2s}. \tag{15}$$

The quadratic equation for c

$$(a+2s)^2 \left(c - \frac{a^2}{4}\right) = c^2 e^{-2s} \tag{16}$$

has real roots if $(a+2s)^2 e^{2s} - a^2 \geq 0$, which is true if

$$a \geq a_3 = -2s + \frac{2s}{e^s + 1} = -\frac{2se^s}{e^2 + 1}. \tag{17}$$

If (17) is satisfied, the roots of (16) are

$$c = \frac{1}{2}\left[(a+2s)^2 e^{2s} \pm e^s |a+2s| \sqrt{(a+2s)^2 e^{2s} - a^2}\right]. \tag{18}$$

The curves described by (18) have left extreme points on the left (a_3, c_3) with

$$c_3 = \frac{2s^2 e^{2s}}{(e^s + 1)^2}.$$

The point (a_3, c_3) is above the parabola (13) and $a(0,0) < a_3 < a_1$. The region described by (14) is above this parabola and is described by

$$a \geq a_3,\ c < c_2 = \frac{1}{2}\left[(a+2s)^2 e^{2s} + e^s |a+2s| \sqrt{(a+2s)^2 e^{2s} - a^2}\right].$$

In this region, the roots for y^2 of (11) are complex, and, thus, the roots of (2) have negative real part.

By combining the considerations of this section, we obtain the following explicit characterizations of some regions in the $a-c$ plane for which the roots of (2) all have negative real part.

THEOREM 2. *All roots of equation* (2) *have negative real part if the coefficients* a *and* c *satisfy either*

$$a > a_1, \quad s^2 + as + c(1-e^{-s}) > 0, \quad 2c < (a+s)^2 + s^2$$

or

$$a > a_3, \quad c < c_2,\ s^2 + as + c(1-e^{-s}) > 0, \quad 2c > (a+s)^2 + s^2.$$

As this result covers negative values of a, it extends the simple estimate (3) of the stability region for (2). In the special case $s = 0$ of (2), studied in [**7**], $a_1 = a_3 = 0$ and the parabola (13) reduces to $c = a^2/2$. Thus, for $s = 0$, Theorem 2 reduces to a result of [**7**].

On the imaginary root curve given by (7), we have

$$\frac{a+2s}{c} = -e^{-s}\frac{\sin y}{y} = f(y).$$

We let $y_k = (2k+1)\pi + \hat{y}_k$, with $0 < \hat{y}_k < \pi/2$, be the zeros of $f'(y)$ for which $f(y)$ is a relative maximum. Because $y_k = \tan y_k$,

$$f(y_k) = -e^{-s}\frac{\sin y_k}{y_k} = -e^{-s}\cos y_k = e^{-s}\cos \hat{y}_k.$$

Also, $\{\hat{y}_k\}$ is an increasing sequence, and therefore, the maximum of $f(y)$ is attained at $y_0 = \pi + \hat{y}_0 = 4.493409$. We may then estimate

$$\max_{0\le y<\infty} f(y) = e^{-s}\cos \hat{y}_0 = 0.217233e^{-s}.$$

Thus, on the imaginary root curve,

$$\frac{a+2s}{c} \le 0.217233e^{-s}$$

and the imaginary root curve is above the line

$$c = 4.60334e^s(a + 2s). \tag{19}$$

Another line which lies below the imaginary root curve is the tangent line to this curve at $(a(0,\pi), c(0,\pi))$, as $y = \pi$ is the first value for which $a(0,y) = -2s$. It is easy to calculate

$$a(0,\pi) = -2s, \quad c(0,\pi) = \frac{\pi^2 + s^2}{1 + e^{-s}}.$$

The slope of the tangent line at this point is $c'(0,\pi)/a'(0,\pi))$, and a routine calculation gives

$$\frac{c'(0,\pi)}{a'(0,\pi)} = e^s \left[\frac{2\pi^2}{\pi^2 + s^2} - \frac{se^{-s}}{1 + e^{-s}}\right].$$

Thus, the tangent line to the imaginary root curve at $(a(0,\pi), c(0,\pi))$ is

$$c = \frac{\pi^2 + s^2}{1 + e^{-s}} + e^s \left[\frac{2\pi^2}{\pi^2 + s^2} - \frac{se^{-s}}{1 + e^{-s}}\right](a + 2s). \tag{20}$$

We obtain two further estimates for the stability region of (2).

THEOREM 3. *All roots of equation* (2) *have negative real part if*

$$s^2 + as + c(1 - e^{-s}) > 0$$

and either

$$\text{(i)} \quad c < 4.60334e^s(a + 2s)$$

or

$$\text{(ii)} \quad c < \frac{\pi^2 + s^2}{1 + e^{-s}} + e^s \left[\frac{2\pi^2}{\pi^2 + s^2} - \frac{se^{-s}}{1 + e^{-s}}\right](a + 2s).$$

In each case, the result is valid for a to the right of the intersection of the zero root line and the line (19) or (20), respectively.

For $s = 0$, the lines (19) and (20) are

$$c = 4.60334a, \quad c = \frac{\pi^2}{2} + 2a,$$

respectively, and both parts of Theorem 3 are valid for $a > 0$. Theorems 2 and 3 overlap; their regions of applicability in the $a-c$ plane intersect, but their relation depends on the value of s. In general, Theorem 3 is more useful if a is negative or positive and small, while Theorem 2 is more useful if a is large.

4. The infectious disease model formulated in [**6**] is

$$I'(t) = -(\gamma+\mu)I(t)+\beta I^p(t)\left[1-I(t)-\gamma\int_{t-\omega}^{t} I(x)e^{-\mu(t-x)}\,dx\right]. \tag{21}$$

In this model it is assumed that there is a nonlinear rate $\beta I^p S$ of incidence of the disease, a recovery rate γI and a period ω of immunity to reinfection. The total population size is a constant normalized to 1 with a birth rate μ of susceptibles and a corresponding death rate divided proportionally among susceptible, infective, and removed members of the population. The special case $\mu = 0$, $p = 1$ of a closed population is studied in [**6**], while population dynamics are added in work now in progress by the authors of [**6**].

The equilibria of (21) satisfy the equation

$$I^{p-1} - (1+r)I^p = \frac{\gamma+\mu}{\beta}, \tag{22}$$

where

$$r = \gamma(1-e^{-\omega\mu})/\mu \tag{23}$$

and corresponding susceptible equilibrium $S = 1-(1+r)I > 0$. The function $f(I) = I^{p-1}-(1+r)I^p$ has a maximum at $I_0 = (p-1)/p(1+r)$ if $p > 1$, and from this it follows that there is a single nonzero equilibrium I_1 if $p < 1$, two equilibria I_1, I_2 with $I_1 < I_0 < I_2$ if $p > 1$, and $f(I_0) > (\gamma+\mu)/\beta$, one equilibrium I_0 if $f(I_0) = (\gamma+\mu)/\beta$, and no nonzero equilibrium of $p > 1$ and $f(I_0) < (\gamma+\mu)/\beta$. At each nonzero equilibrium, I, the characteristic equation is (2) with

$$\begin{aligned} a &= \omega[\gamma - p(\gamma+\mu) + \beta I^p], \\ c &= \omega^2[\beta\gamma I^p, \\ s &= \omega\mu. \end{aligned} \tag{24}$$

From (22), we see that

$$c = \omega\gamma(a + 2s) - \omega\gamma[(1 - p)\omega\gamma + (2 - p)s]. \tag{25}$$

THEOREM 4. *If $p < 1$ and $\omega\gamma \leq 4.60334$, or if*

$$p \leq 1 - \frac{\gamma - 2\mu^{2s}}{2e^{2s}(\gamma + \mu)}, \tag{26}$$

the nonzero equilibrium is asymptotically stable.

PROOF. We have $a + s = (1 - p)\omega\gamma + (1 - p)s + \beta\omega I^p > 0$, so that $a + 2s > 0$ and $s^2 + as + c(1 - e^{-s}) > 0$. If $\omega\gamma \leq 4.60334$, $c < 4.60334(a + 2s)$ and the equilibrium is asymptotically stable by Theorem 3. The condition (26) implies

$$\omega\gamma(a + 2s) - \omega\gamma[(1 - p)\omega\gamma + (2 - p)s] \leq e^{2s}(a + 2s)^2/2 < c_2$$

so that $c_2 < e^{2s}(a + 2s)^2$, and, since $a + 2s > s > (2s)/(e^s + 1)$, we have $a > a_3$. Then the conditions of Theorem 2 are satisfied, and the asymptotic stability of the nonzero equilibrium follows. □

If $\mu = 0$, the condition (26) reduces to $p \leq 1/2$ and the result is Theorem 5.2 of [**6**]. Thus, the introduction of births and deaths, or increasing μ, tends to stabilize the equilibrium. The other part of Theorem 4 is a refinement of the results of [**6**], even for $\mu = 0$.

THEOREM 5. *If $p > 1$ and $f(I_0) > (\gamma + \mu)/\beta$, the smaller nonzero equilibrium, I_1, is unstable.*

PROOF. Using (22) and (24) obtains

$$\begin{aligned} s^2 + as + c(1 - e^{-s} &= \omega^2\mu\beta\left[I_1^{p-1} - p\left(\frac{\gamma + \mu}{\beta}\right)\right] \\ &= \omega^2\mu\beta p\left[\frac{I_1^{p-1}}{p} - \frac{\gamma + \mu}{\beta}\right]. \end{aligned}$$

Because $I_1 < I_0$,

$$f'(I_1) = I_1^{p-2}[(p-1) - (1+r)pI_1] > 0,$$

and, therefore, $(p-1) - p(I+r)I_1 > 0$. From (22),

$$\begin{aligned}\frac{I^{p-1}}{p} - \frac{\gamma+\mu}{\beta} &= \frac{I^{p-1}}{p} - [I_1^{p-1} - (1+r)I_1^P] \\ &= -\frac{1}{p}I_1^{p-1}[(p-1) - (1+r)I_1] < 0.\end{aligned}$$

Thus, $s^2 + as + c(1 - e^{-s}) < 0$, and the equilibrium I_1 is unstable. □

If $p > 1$ and $f(I_0) > (\gamma+\mu)/\beta$, the stability of the larger equilibrium I_2 is more complicated. For this equilibrium, $s^2 + as + c(1 - e^{-s}) > 0$ by reasoning analogous to the proof of Theorem 5 but using $f'(I_2) < 0$. However, $(a + 2s)$ may be negative. The following result gives a sufficient condition for stability.

THEOREM 6. *If $p > 1$ and $f(I_0) > (\gamma+\mu)/\beta$, the larger equilibrium I_2 is locally asymptotically stable if*

$$\omega\gamma < e^s\left[\frac{2\pi^2}{\pi^2+s^2} - \frac{se^{-s}}{1+e^{-s}}\right], \tag{27}$$

$$-\omega\gamma[\omega\gamma(1-p) + (2-p)s] < \frac{\pi^2+s^2}{1+e^{-s}}. \tag{28}$$

PROOF. From (25), we have

$$-\omega\gamma[\omega\gamma(1-p) + (2-p)s] = c - \omega\gamma(a+2s),$$

and the result follows directly from Theorem 3(ii). □

In the case $\mu = 0$, $r = \omega\gamma$ and $s = 0$. The conditions (27) and (28) become $r < 2$ and $r^2(p-1) < \pi^2/2$, respectively, and Theorem 6 reduces to Theorem 5.4 of [**6**].

As a and c vary, the point (a, c) may cross the imaginary root curve. Because, as we have seen,

$$\left.\frac{\partial(a,c)}{\partial(x,y)}\right|_{x=0,y>0} \neq 0,$$

there is a local homeomorphism between the $x-y$ and $a-c$ planes in a neighborhood of each point $(a(0,y), c(0,y))$. Thus, the real parts of the roots of (2) vary from negative to positive. According to the theorem on Hopf bifurcation [**5**], Hopf bifurcation may occur and periodic solutions may arise as the imaginary root curve is crossed.

5. The infectious disease model (21) has not yet been analyzed completely. Theorems 4 and 6 give convenient estimates of the stability region and extend earlier results. It would be of interest to examine the analogous model with nonlinear population dynamics and/or non-exponential recovery rates. It is not known whether the behavior of such models is the same as the behavior of model (21).

REFERENCES

1. R.E. Bellman and K.L. Cooke, *Differential-Difference equations*, Academic Press, New York, 1963.

2. F. Brauer and M. Zhien, *Stability of stage-structured population models*, J. Math. Anal. Appl. **126** (1987), 301–315.

3. N.G. Čebotarev and N.N. Meĭman, *The Routh-Hurwitz problem for polynomials and entire functions* (Russian), Trudy Mat. Inst. Steklov., Vol. 26 (1949).

4. K.L. Cooke and P. van den Driessche, *On zeros of some transcendental functions*, Funkcial. Ekvac. **29** (1986), 77–90.

5. S.-N. Chow and J. Mallet-Paret, *Integral averaging and bifurcation*, J. Differential Equations **26** (1977), 112–159.

6. H.W. Hethcote, M.A. Lewis and P. van den Driessche, *An epidemiological model with a delay and a nonlinear incidence rate*, J. Math. Biol. **27** (1989), 49–64.

7. —— and ——, *Stability analysis for models of diseases without immunity*, J. Math. Biol. **13** (1981), 185–198.

8. W.M. Liu, S.A. Levin and Y. Iwasa, *Influence of nonlinear incidence rate upon the behavior of SIRS epidemiological models*, J. Math. Biol. **23** (1986), 187–204.

9. W.M. Liu, H.W. Hethcote and S.A. Levin, *Dynamical behavior of epidemiological models with nonlinear incidence rates*, J. Math. Biol. **25** (1987), 359–380.

10. L.S. Pontrjagin, *On the zeros of some elementary transcendental functions*, Izv. Adak. Nauk. SSSR, Ser. Mat. **6** (1942), 115–134; English translation, Amer. Math. Soc. Transl. Ser. 2 **1** (1955), 95–110.

Department of Mathematics, Inner Mongolia University, Huhehot, Inner Mongolia, China

Department of Mathematics, University of Wisconsin, Madison, WI 53706