JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 3, Number 1, Winter 1991

GENERIC HOPF BIFURCATION IN A CLASS OF INTEGRO-DIFFERENTIAL EQUATIONS

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Dedicated to John Nohel in commemoration of his 65^{th} birthday.

ABSTRACT. The Hopf bifurcation structure for a class of scalar functional differential equations is examined. It is shown that, within some classes of nonlinear perturbations, points of potentially nongeneric bifurcation can be identified by the orientation of the neutral stability curve associated with the linearized problem. A general "normal form" equation is derived which effectively determines the behavior of generic bifurcations. Practical computational issues are addressed and illustrated with a specific application first considered by Levin and Nohel [**6**].

I. Introduction. Consider the linear scalar equation

$$\dot{x}(t) = \alpha x + \beta \int_{-\infty}^{0} x(t+s)\, d\eta(s) \tag{1.1}$$

and the nonlinear perturbation

(1.2)

$$\begin{aligned}\dot{x}(t) = \alpha x &+ \beta \int_{-\infty}^{0} x(t+s)\, d\eta(s)\\ &+ a_2 x^2(t) + a_2 x^3(t) + b_2 x(t)\beta \int_{-\infty}^{0} x(t+s)\, d\eta(s)\\ &+ b_3 x^2(t)\beta \int_{-\infty}^{0} x(t+s)\, d\eta(s) + c_2\beta \int_{-\infty}^{0} x^2(t+s)\, d\eta(s)\\ &+ c_3 x(t)\beta \int_{-\infty}^{0} x^2(t+s)\, d\eta(s) + d_3\beta \int_{-\infty}^{0} x^3(t+s)\, d\eta(s) \dots ,\end{aligned}$$

where the measure $d\eta$ is a suitably normalized positive measure. It is well known that, under certain natural assumptions about the measure $d\eta$, the asymptotic behavior of solutions to (1.1) can be determined by locating the roots of the associated characteristic equation $\Delta(\alpha, \beta; \lambda) = 0$, where

$$\Delta(\alpha, \beta; \lambda) = \lambda - \alpha - \beta \int_{-\infty}^{0} e^{-\lambda s} d\eta(s). \tag{1.3}$$

For example, given that, for some $\delta > 0$, one has $\int_{-\infty}^{0} e^{\delta s} d\eta(s) < \infty$ — a condition satisfied by all "finite delay" problems—the zero solution to (1.1) is asymptotically stable when all roots to (1.3) have negative real parts. Under such conditions, the zero solution to (1.2) is locally asymptotically stable. In general, one can subdivide the (α, β) plane into regions within which the characteristic roots associated with (1.1) either all have real parts negative, there exist roots with positive real parts, or there are imaginary (or zero) characteristic roots. At "critical" parameter values at which there are purely imaginary roots $\lambda = \pm\omega i$, the coefficients of the higher order terms in (1.2) determine (generically) the stability of the zero solution, and one expects that, for certain nearby parameter values, that (1.2) has small nontrivial solutions with period near $2\pi/\omega$.

Apart from this general intuition, we seek here to address the following specific issues:

• Under the above general assumptions on $d\eta$, what conclusions can be made about the partition of the (α, β) plane into regions of stability and instability?

• At criticality, which of the higher order terms in (1.2) tend to stabilize the equilibrium? Which tend to destabilize it?

Closely related to the previous question:

• How are the "direction" and stability of Hopf bifurcations dependent on the higher order terms in (1.2)?

Since, in practical applications, the coefficients in (1.2) may be complicated functions of more natural "model" parameters, one must also consider the question of how to treat actual applications with a minimal amount of computation.

The remainder of this paper is subdivided as follows: Section 2 contains the specific assumptions made of $d\eta$ and introduces the phase space setting within which (1.1) and (1.2) will be considered. The first of the problems cited above will then be considered. Section 3 introduces a framework (derived originally in [**9**]) for resolving the two nonlinear problems above. A scalar "normal form" equation is derived which can be used to resolve specific bifurcation problems. Moreover, for certain of the higher order terms in (1.2), it is shown that their stabilizing/destabilizing influences can be easily predicted by considering the orientation of the neutral stability curve near the critical value of interest. Both Sections 2 and 3 present illustrative examples.

II. Linearized analysis. We begin by more precisely stating the technical assumptions made of equations (1.1) and (1.2). For $\rho > 0$, we define $X_\rho = \{\psi : (-\infty, 0] \to \mathbf{R} \,|\, |\phi(s)|e^{-\rho s}$ is bounded and uniformly continuous on $(-\infty, 0]\}$, $\rho < 0$, which, with norm $||\phi|| = \sup_{(-\infty,0]} |\phi(s)|e^{-\rho s}$, is a Banach space. For the measure $d\eta$ we make two assumptions:

(H1) The function $\eta : (-\infty, 0] \to \mathbf{R}$ is a nondecreasing function normalized to be left continuous on $(-\infty, 0]$ such that, for some $\delta > 0$, one has $\int_{-\infty}^{0} e^{\delta s} d\eta(s) < \infty$.

For all $\rho > \delta$ sufficiently small, we may take X_ρ as phase space for consideration of the initial value problems for (1.1) and (1.2). In particular, the usual notions of stability of equilibria and periodic orbits will be understood relative to the norm on X_ρ. See [**9**] for details. Note that the measure $d\eta$ is allowed to have an atom at $s = 0$. However, for purposes that will become clear later, we choose not to include the term αx in (1.1) with the integral contribution. It is convenient to introduce the notation $L\phi = \int_{-\infty}^{0} \phi(s)\, d\eta(s)$.

The full nonlinear problem under consideration is

$$\dot{x}(t) = \alpha x(t) + \beta L x_t + H(x_t), \tag{2.1}$$

where $x_t \in X_\rho$ is defined by $x_t(s) = x(t+s)$; $s \le 0$,

(H2) H is assumed to be C^5 on X_ρ with the expansion $H(\phi) = H_2(\phi, \phi) + H_3(\phi, \phi, \phi) + \cdots$, $\phi \in X_\rho$, and H_2, H_3 are the continuous (under appropriate selection of ρ relative to δ), symmetric, bilinear and trilinear forms

(2.2)
$$H_2(\phi_1,\phi_2) = a_2\phi_1(0)\phi_2(0) + b_2[\phi_1(0)L\phi_2 + \phi_2(0)L\phi_1]/2 + c_2L(\phi_1\phi_2)$$

(2.3)
$$\begin{aligned} H_3(\phi_1,\phi_2,\phi_3) = {} & a_3\phi_1(0)\phi_2(0)\phi_3(0) \\ & + b_3[\phi_1(0)\phi_2(0)L\phi_3 + \phi_1(0)\phi_3(0)L\phi_2 \\ & + \phi_2(0)\phi_3(0)L\phi_1]/3 \\ & + c_3[\phi_1(0)L(\phi_2\phi_3) + \phi_2(0)L(\phi_1\phi_3) + \phi_3(0)L(\phi_1\phi_2)]/3 \\ & + d_3L(\phi_1\phi_2\phi_3). \end{aligned}$$

The functional differential equation (2.1) now reduces to (1.2). The degree of generality assumed here allows for direct application of the results below to a wide variety of equations from the literature. Specific examples will be cited below.

Under the assumptions above, the stability of the zero solution of (2.1) can be analyzed in terms of the roots of the characteristic equation $\Delta(\alpha,\beta;\lambda) = 0$ of Section 1. In particular, the zero solution of (2.1) is locally asymptotically stable if and only if $(\alpha,\beta) \in \Omega_- \equiv \{(\alpha,\beta) \mid \Delta(\alpha,\beta;\lambda) = 0 \Rightarrow \operatorname{Re}\lambda < 0\} \subseteq \mathbf{R}^2$. Clearly, $\lambda = 0$ is a characteristic root if and only if

$$\alpha + \beta m_0 = 0, \tag{2.4}$$

where here (and later) we use the notation

$$m_j = \int_{-\infty}^{0} s^j \, d\eta(s)$$

for $j = 0,1,2,\dots$. Moreover, the real and imaginary parts of $\Delta(\alpha,\beta;i\omega) = 0$ imply $\lambda = \pm i\omega$; $\omega > 0$ is an imaginary root (pair) if and only if $(\alpha,\beta) = (\tilde{\alpha}(\omega),\tilde{\beta}(\omega))$, where

$$\tilde{\beta}(\omega) = \omega \Big/ \int_{-\infty}^{0} \sin(\omega s)\, d\eta(s) \tag{2.5}$$

$$\tilde{\alpha}(\omega) = -\tilde{\beta}(\omega) \int_{-\infty}^{0} \cos(\omega s)\, d\eta(s). \tag{2.6}$$

As we shall see, the location of the set Ω_- in the (α, β)-plane is, in a sense, typified by the special case

$$\dot{x}(t) = \alpha x(t) + \beta x(t-1) \tag{2.7}$$

(see [**3**, p. 108].) In particular, Ω_- contains an open angular sector including the negative half-axis $\beta = 0$, $\alpha < 0$, and is bounded above by points (α, β) at which $\lambda = 0$ is a characteristic root and elsewhere by points $(\tilde{\alpha}, \tilde{\beta})$ associated with purely imaginary root pairs.

LEMMA 2.1. *If either* $\alpha + |\beta| m_0 < 0$ *or* $1/m_1 < \beta < -\alpha/m_0$, *then* $(\alpha, \beta) \in \Omega_-$.

PROOF. The function Δ is (for fixed α, β) analytic for $\operatorname{Re} \lambda > -\delta$. Since $\operatorname{Re} \lambda \geq 0$ implies that $|\lambda| \leq |\alpha| + |\beta| m_0 < 0$, it is clear that Ω_- is bounded by points (α, β) at which $\Delta(\alpha, \beta; \lambda) = 0$ either has real (i.e., zero) or imaginary roots. If $\lambda = \mu + i\omega$, $\mu \geq 0$, solves $\Delta(\alpha, \beta; \lambda) = 0$, then

$$\begin{aligned} \mu &= \alpha + \beta \int_{-\infty}^{0} e^{\mu s} \cos(\omega s)\, d\eta(s) \\ &\leq \alpha + |\beta| m_0, \end{aligned}$$

from which the first condition follows. The second is proved in [**7**]. □

It is clear that the imaginary root curve $(\alpha, \beta) = (\tilde{\alpha}(\omega), \tilde{\beta}(\omega))$ is defined on the intervals $(0, \omega_1), (\omega_1, \omega_2), \ldots$, where either $\omega_1 = \infty$ or $\int_{-\infty}^{0} \sin(\omega_1 s)\, d\eta(s) = 0$. By direct calculation, one finds

$$\begin{aligned} \tilde{\beta}(\omega) &= 1/m_1 + (m_3 \omega^2)/(6 m_1^2) + \cdots \\ \tilde{\alpha}(\omega) &= -m_0/m_1 - (m_0 m_3 - 3 m_1 m_2)\omega^2/(6 m_1^2) + \cdots \end{aligned}$$

as $\omega \to 0^+$. Under the second condition of the previous lemma, the imaginary part of the equation $\Delta(\alpha, \beta; i\omega) = 0$ shows there can be no purely imaginary characteristic roots. Moreover, whether or not ω_1 is finite, $\tilde{\beta}(\omega) \to -\infty$ as $\omega \to \omega_1^-$. Thus, the imaginary root curve $(\tilde{\alpha}(\omega), \tilde{\beta}(\omega))$ on $0 < \omega < \omega_1$ creates a bound for Ω_- connecting the zero root curve (2.4) as $\omega \to 0$ to $\beta = -\infty$. (See [**8**] where, in the finite delay case, qualitative conditions on $d\eta$ are derived which imply ω_1 is finite).

As the example below shows, the actual nature of the imaginary root curve can be quite complicated.

EXAMPLE 2.2. Consider the parameterized family, for $0 \le \varepsilon \le 1$,

$$\dot{x}(t) = \alpha x(t) + \beta \int_{-1}^{0} x(t+s)(\varepsilon s + 1)\, ds. \tag{2.8}$$

One readily computes that

$$\tilde{\alpha}(\varepsilon, \omega) = \frac{\omega((\varepsilon - 1)\omega \sin(\omega) + \varepsilon(\cos(\omega) - 1))}{\varepsilon(\sin(\omega) - \omega\cos(\omega)) + \omega(\cos(\omega) - 1)} \tag{2.9}$$

$$\tilde{\beta}(\varepsilon, \omega) = \frac{\omega^3}{\varepsilon(\sin(\omega) - \omega\cos(\omega)) + \omega(\cos(\omega) - 1)} \tag{2.10}$$

Case 1. ($\varepsilon = 1$). The above expressions for the imaginary root curve simplify considerably to $\tilde{\alpha}(1,\omega) = \omega(\cos(\omega)-1)/(\sin(\omega)-\omega)$ and $\tilde{\beta}(1,\omega) = \omega^3/(\sin(\omega) - \omega)$. Clearly $\omega_1 = +\infty$ and $\tilde{\beta}(1,\omega) \to -\infty$ as $\omega \to \infty$. Note that $\tilde{\alpha}(1,\omega)$ is strictly positive except at multiples of 2π (see Figure 2.1). See [**4**] for generalizations and a precise classification of measures $d\eta(s) = a(s)\, ds$, a nonnegative, monotone increasing, and concave upwards for which $\tilde{\alpha}(\omega) \ge 0$ for all $\omega > 0$. Relevant, as well, is the discussion in [**11**] concerning qualitative conditions on $P(s)$ (P nonnegative, increasing) for which $\int_{-\infty}^{0} \cos(\omega s)P(s)\, ds$ is always positive, or changes sign.

Case 2. ($0 < \varepsilon < 1$). Rearranging the denominator of $\tilde{\beta}$, one sees that $\omega_1 = \infty$ as in Case 1. However, $\tilde{\alpha}(\varepsilon,\omega) \approx (\omega\sin(\omega)(\varepsilon - 1))/((1 - \varepsilon)\cos(\omega) - 1)$ is unbounded as $\omega \to \infty$, in contrast to the situation of Case 1. Convergence of these imaginary root curves as $\varepsilon \to 1$ takes place uniformly on *compact* subsets of the (α, β) plane. See Figures 2.2.

Case 3. ($\varepsilon = 0$). Here, $\tilde{\alpha}(0,\omega) = -\omega\sin(\omega)/(\cos(\omega) - 1)$ and $\tilde{\beta}(0,\omega) = \omega^2/(\cos(\omega)-1)$ are defined except at multiples of 2π, $\omega_1 = 2\pi$. Between multiples of 2π, $\tilde{\alpha}$ is easily seen to be monotone decreasing, and $\lim_{\omega\to n2\pi} \tilde{\beta}(\omega) = -\infty$. See Figure 2.3. Again, convergence of the

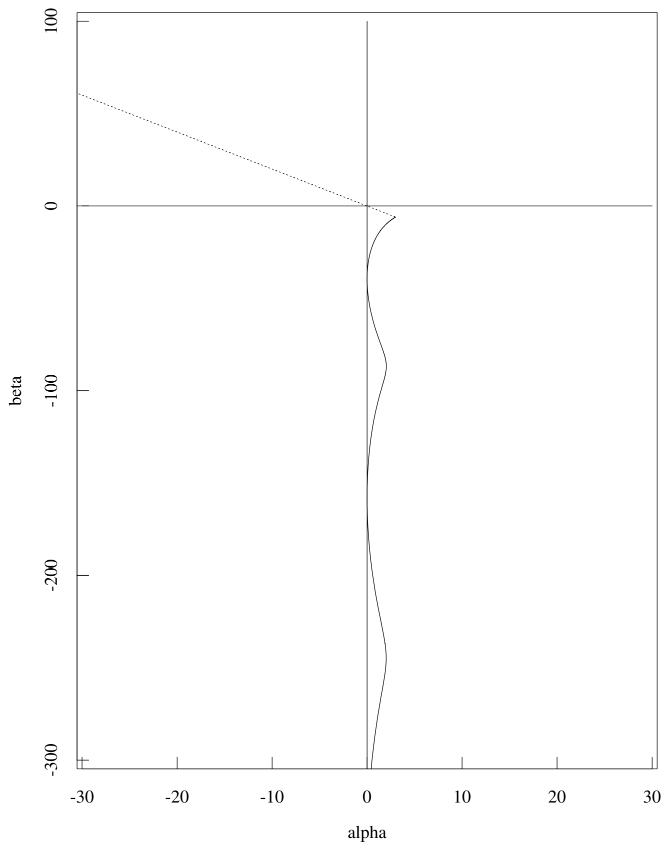

FIGURE 2.1. Linear stability curves (epsilon = 1).

imaginary root curves in Case 2 as $\varepsilon \to 0$ occurs only on compact subsets. See [**12**], where the special case $\tilde{\alpha} = 0$ is considered.

III. The effects of nonlinearities. The classical hypotheses of the Hopf bifurcation involve the assumption of the existence of a unique, simple imaginary root $\lambda = \pm i\omega$ which cross the imaginary axis transversally as the bifurcation parameter under use is varied past the critical value. In order to make conclusions about the stability of bifurcating periodic orbits, one must additionally assume that all other

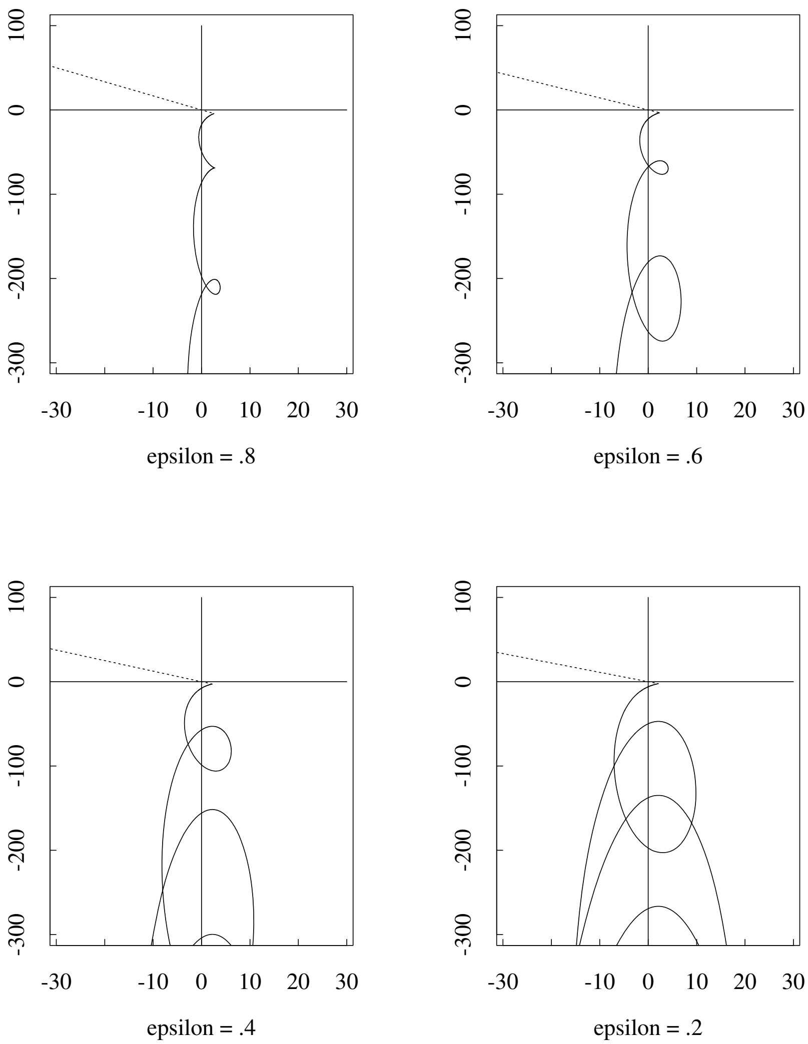

FIGURE 2.2. Linear stability curves.

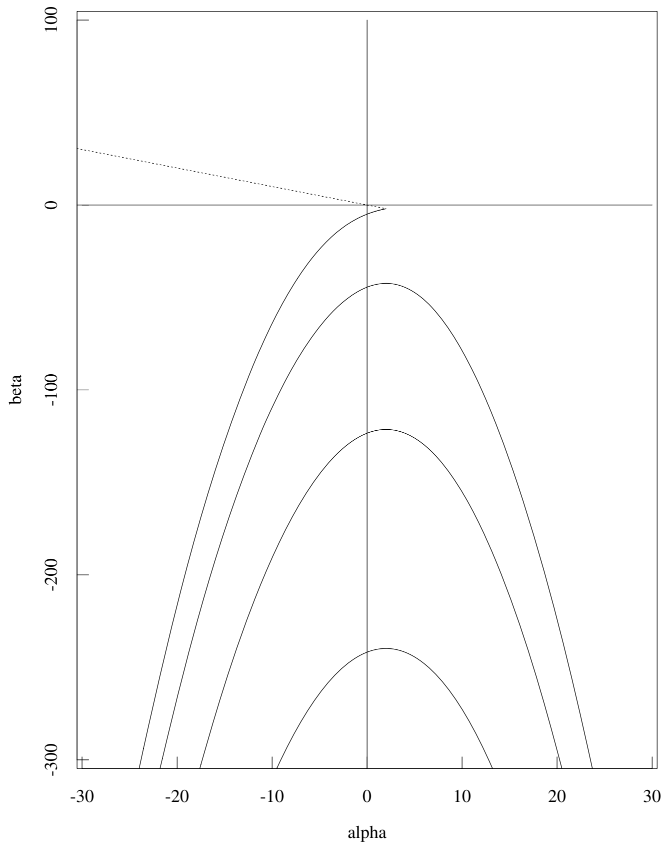

FIGURE 2.3. Linear stability curves (epsilon = 0).

characteristic roots have negative real parts at criticality. The previous example shows that equations of the type (1.2) can support a variety of nongeneric situations, including simultaneous imaginary root pairs and multiple root pairs.

We proceed under the assumption that $\lambda = \pm i\omega$, $\omega > 0$, is a simple root pair (i.e., $\Delta' \equiv \partial\Delta/\partial\lambda \neq 0$ at $\lambda = \pm i\omega$), all other characteristic roots having negative real parts. Observe that the equation $\Delta(\alpha, \beta; \lambda) = 0$ and its conjugate can be viewed as a system of two (linear!) equations in the unknowns (α, β). This system is easily seen to be nonsingular for $\lambda \approx i\omega$, allowing one to solve $\alpha = \alpha(\lambda)$, $\beta = \beta(\lambda)$ for λ near the imaginary root curve. Transversality is seen to hold under our simplicity assumption. For (α, β) near the critical value $(\tilde{\alpha}(\omega), \tilde{\beta}(\omega))$, we define $\lambda = \lambda(\alpha, \beta) = \mu(\alpha, \beta) + i\omega(\alpha, \beta)$ to be the unique characteristic root family such that $\lambda(\tilde{\alpha}(\omega), \tilde{\beta}(\omega)) = i\omega$.

Our goal is to understand the effects the higher order terms in (1.2) have on the stability of $x = 0$ for (α, β) on the imaginary root curve, as well as their effects on associated Hopf bifurcations. This study is simplified by the following Lemma. As it is justified by straightforward calculation, we omit the details of its proof.

LEMMA 3.1. *Under the change of variables* $x = y + h_2 y^2 + h_3 y^3 + \cdots$, *equation* (1.2) *takes the form*
(3.1)

$$\begin{aligned}\dot{y} &= \alpha y + \beta L y_t + a_2' y^2 + a_3' y^3 + b_2' y \beta L y_t \\ &\quad + b_3' y^2 \beta L y_t + c_2' \beta L(y_t^2) + c_3' y \beta L(y_t^2) + d_3' \beta L(y_t^3) + O(\|y_t^4\|),\end{aligned}$$

where

$$\begin{aligned}
a_2' &= a_2 - \alpha h_2 & a_3' &= a_3 + 2\alpha h_2^2 - 2\alpha h_3 \\
b_2' &= b_2 - 2h_2 & b_3' &= b_3 - b_2 h_2 + 4h_2^2 - 3h_3 \\
c_2' &= c_2 + h_2 & c_3' &= c_3 - (2c_2 - b_2)h_2 - 2h_2^2 \\
d_3' &= d_3 + 2c_2 h_2 + h_3. & &
\end{aligned}$$

The following Proposition is a consequence of the Lyapunov-Schmidt based decomposition derived in [**9**]. It asserts the existence of a computable real scalar function $g(\alpha, \beta; c)$ whose zeros near $c = 0$ are in one-to-one correspondence with small periodic solutions of (1.2).

PROPOSITION 3.2. *There exists a function g defined and C^5 in a neighborhood of $(\tilde{\alpha}(\omega), \tilde{\beta}(\omega), 0) \in \mathbf{R}^3$ of the form*

$$g(\alpha, \beta; c) = \mu(\alpha, \beta)c + K_3(\alpha, \beta; \omega)c^3 + O(c^4)$$

such that the zeros of g correspond in a one-to-one fashion with the small periodic solutions of (1.2) *with period near $2\pi/\omega$. Under this* *correspondence, the periodic solution of* (1.2) *associated with a root* c *of* g *has the form*

$$x(t) = 2c\cos(\omega t) + O(c^2)$$

(*up to phase shift*). *Moreover,* $x(t)$ *is orbitally asymptotically stable* (*unstable*) *if and only if* c *is stable* (*unstable*) *when viewed as an equilibrium point of the scalar equation* $\dot{c} = g(\alpha, \beta; c)$. *Finally,* K_3 *above is the real part of* $N_3(\omega)/\Delta'(i\omega)$, *where*

$$\begin{aligned} N_3(\omega) = {} & [4/\Delta(0) + 2/\Delta(2i\omega)](a_2 - \tilde{\alpha}b_2/2)^2 \\ & + [4(L(1) + L(e^{i\omega\cdot}))/\Delta(0) \\ & + 2(L(e^{i\omega\cdot}) + L(e^{2i\omega\cdot}))/\Delta(2i\omega)]\tilde{\beta}(\omega)(a_2 - \tilde{\alpha}b_2/2)(c_2 + b_2/2) \\ & + [4L(1)/\Delta(0) + 2L(e^{2i\omega\cdot})/\Delta(2i\omega)]L(e^{i\omega\cdot})\tilde{\beta}^2(\omega)(c_2 + b_2/2)^2 \\ & + 3(a_3 - 2\tilde{\alpha}b_3/3 + \tilde{\alpha}b_2^2/6) + [L(1) + L(e^{2i\omega\cdot})]\tilde{\beta}(\omega)(c_3 - b_2c_2) \\ & + 3L(e^{i\omega\cdot})\tilde{\beta}(\omega)(d_3 + b_2c_2 + b_3/3 + b_2^2/6). \end{aligned}$$

PROOF. Only the computation of K_3 will be addressed, as the other assertions follow from Theorems 2.1 and 2.2 of [**9**]. Indeed, it follows directly from the first of those theorems that

$$\begin{aligned} K_3(\omega) = {} & c_{a_2a_2}(\omega)a_2^2 + c_{a_2b_2}(\omega)a_2b_2 + c_{b_2b_2}(\omega)b_2^2 \\ & + c_{a_2c_2}(\omega)a_2c_2 + c_{c_2c_2}(\omega)c_2^2 + c_{b_2c_2}(\omega)b_2c_2 \\ & + c_{a_3}(\omega)a_3 + c_{b_3}(\omega)b_3 + c_{c_3}(\omega)c_3 + c_{d_3}(\omega)d_3. \end{aligned} \tag{3.2}$$

Since the dynamics and scale of (1.2) near $x = 0$ are invariant under a change of variables of the type in Lemma 3.1, we do so and select $h_2 = b_2/2$ and $h_3 = (2b_3 + b_2^2)/6$ so that the coefficients b_2' and b_3' in the transformed equation are zero:

$$\begin{aligned} \dot{y} = {} & \alpha y + \beta L y_t + (a_2 - \alpha b_2/2)y^2 + (a_3 - 2\alpha b_3/3)y^3 + (c_2 - b_2/2)\beta L y_t^2 \\ & + (c_3 - b_2c_2)y\beta L y_t^2 + (d_3 + b_2c_2 + b_3/3 + b_2^2/6)\beta L y_t^3 + O(||y_t||^4). \end{aligned} \tag{3.3}$$

The invariance of scale implies that the coefficient K_3 associated with the transformed equation must match that of (1.2). The value of K_3 is then computed from (3.3) by the algorithm of [**9**]. □

REMARK . Since, at criticality, $\mu(\tilde{\alpha}, \tilde{\beta}) = 0$, the stability of the zero solution for (1.2) can be determined from K_3, provided that value is nonzero; $K_3 < 0$ ($K_3 > 0$) implies that $x = 0$ is locally asymptotically stable (unstable). Similarly, $K_3 < 0$ ($K_3 > 0$) corresponds to supercritical (subcritical), stable (unstable) Hopf bifurcations. (Recall, for example, that subcritical bifurcations are those existing for (α, β) in an open neighborhood of $(\tilde{\alpha}, \tilde{\beta})$ intersected with Ω_-). Thus, the signs of the coefficients in K_3 (e.g., c_{d_3}) determine whether the corresponding term (e.g., $d_3 \beta L(y_t^3)$) tends to stabilize or destabilize $x = 0$ at criticality, as well as Hopf bifurcations.

The expression for $N_3(\omega)$ in Proposition 3.2 provides a direct means of computing these coefficients. We will illustrate with two examples. First, we present a theorem useful in reducing (or checking the validity of) the necessary computations.

THEOREM 3.3. *The ω-dependent coefficients in K_3 must satisfy the relations*

$$c_{a_2 c_2} - 2c_{a_2 b_2} = 2\tilde{\alpha} c_{a_2 a_2} \tag{3.4}$$

$$c_{a_3} + c_{b_2 c_2} - 4c_{b_2 a_2} - c_{b_3} = \tilde{\alpha} c_{a_2 b_2} \tag{3.5}$$

$$2(c_{d_3} + c_{c_2 c_2} - c_{c_3} - c_{b_2 c_2}) = \tilde{\alpha} c_{a_2 c_2} \tag{3.6}$$

$$c_{d_3} - 3c_{b_3} = 2\tilde{\alpha} c_{a_3}. \tag{3.7}$$

PROOF. Consider the change of variables of Lemma 3.1 and denote by $K_3(\omega; h_2, h_3)$ the corresponding coefficient in g. By the invariance of scale, $K_3(\omega; h_2, h_3)$ must be independent of h_2 and h_3. We compute $K_3(\omega; h_2, h_3)$ by substituting the identities of Lemma 3.1 into (3.2). The four conditions above are both necessary and sufficient for K_3 to be independent of h_2 and h_3. The details are omitted. □

EXAMPLE 3.4. Consider the equation

$$\begin{aligned} \dot{y} = \alpha y + \beta L y_t + a_2 y^2 + a_3 y^3 + b_2 y L y_t + b_3 y^2 L y_t \\ + c_2 L y_t^2 + c_3 y L y_t^2 + d_3 L y_t^3 + O(||y_t||^4), \end{aligned} \tag{3.8}$$

where

$$L(y_t) = \frac{1}{2} \int_{-2}^{0} y(t+s) ds.$$

The equation is derived from the example of Section 2 with $\varepsilon = 0$ by changing the time scale $t \to t/2$ (so that the resulting measure is normalized with $m_0 = 1$ and $m_1 = -1$) and by absorbing the factors of β in the higher order terms into their respective coefficients. Such scalings facilitate comparison with the nonlinear perturbation of the delay difference equation (2.7). See [**1**] and [**2**].

By direct (but symbolically assisted) computation, one obtains

$$\begin{aligned}
c_{a_2a_2}(\omega) = {}& -2\omega^2\sin(\omega)^2(4\omega\sin(\omega)^2\sin(2\omega) + 3\cos(\omega)\sin(\omega)\sin(2\omega) \\
&- 3\omega\sin(2\omega) + 16\sin(\omega)^4 - 16\omega\cos(\omega)\sin(\omega)^3 - 8\omega^2\sin(\omega)^2 \\
&- 6\omega\cos(\omega)\sin(\omega) + 6\omega^2)/D_1(\omega) \\
c_{a_2b_2}(\omega) = {}& \omega\sin(\omega)^3(-2\cos(\omega)\sin(\omega)^2(13\sin(\omega)^2 + 9) \\
&- 4\omega(4\cos(\omega)^4 - 14\cos(\omega)^2 + 1)\sin(\omega) \\
&- 2\omega^2\cos(\omega)(16\cos(\omega)^2 - 7))/D_1(\omega) \\
c_{b_2b_2}(\omega) = {}& \sin(\omega)^4(-2\omega^2(8\sin(\omega)^4 - 12\sin(\omega)^2 + 3) \\
&+ 4\omega\cos(\omega)(5\cos(\omega)^2 - 2)\sin(\omega) \\
&- 2(\cos(\omega)^2 - 1)\cos(\omega)^2(2\cos(\omega)^2 - 5))/D_1(\omega) \\
c_{a_2c_2}(\omega) = {}& 2\omega\sin(\omega)^3(-6\cos(\omega)\sin(\omega)^2(\sin(\omega)^2 + 1) \\
&+ 4\omega(4\cos(\omega)^2 - 1)\sin(\omega) \\
&- 2\omega^2\cos(\omega)(8\cos(\omega)^2 - 5))/D_1(\omega) \\
c_{c_2c_2}(\omega) = {}& 8\sin(\omega)^6(\cos(\omega)\sin(\omega) - 2\omega)(\cos(\omega)\sin(\omega) - \omega)/D_1(\omega)
\end{aligned}$$

$$\begin{aligned}
c_{b_2c_2}(\omega) = {}& -\sin(\omega)^4(-4\omega\cos(\omega)\sin(\omega)(2\sin(\omega)^4 + 3) \\
&+ 6\omega^2(\sin(\omega)^2 + 1) - 16\omega^3\cos(\omega)\sin(\omega) \\
&+ 2\sin(\omega)^2\cos(\omega)^2(4\cos(\omega)^4 \\
&- 13\cos(\omega)^2 + 12))/D_1(\omega) \\
c_{a_3}(\omega) = {}& -6\omega\sin(\omega)^3(\omega\cos(\omega) - \sin(\omega))/D_2(\omega) \\
c_{b_3}(\omega) = {}& \sin(\omega)^4(4\omega\sin(\omega)^2 + 5\cos(\omega)\sin(\omega) - 5\omega)/D_2(\omega) \\
c_{c_3}(\omega) = {}& -2\sin(\omega)^3(\cos(\omega)\sin(\omega)^4 - 2\omega\sin(\omega) \\
&+ 2\omega^2\cos(\omega))/D_2(\omega) \\
c_{d_3}(\omega) = {}& 3\sin(\omega)^4(\cos(\omega)\sin(\omega) - \omega)/D_2(\omega),
\end{aligned}$$

where

$$D_1(\omega) = 2\omega^3(3\sin(\omega)^4 - 4\omega\cos(\omega)\sin(\omega)^3 + \sin(\omega)^2 - 2\omega\cos(\omega)\sin(\omega) + \omega^2)(\sin(2\omega) - 2\omega)$$

$$D_2(\omega) = \omega(\sin(\omega)^2(3\sin(\omega)^2 + 1) + 2\omega\cos(\omega)(2\cos(\omega)^2 - 3)\sin(\omega) + \omega^2).$$

At $\omega = \pi/2$, one computes, with $(\tilde{\alpha}, \tilde{\beta}) = (0, -\pi^2/4)$,

$$K_3(\pi/2) = -4\frac{(16c_2^2 - 12b_2c_2 + 2b_2^2 - 8a_2c_2 - 4a_2b_2 + (\pi^2 - 32)a_2^2)}{\pi^2(\pi^2+16)} - 4\frac{(3d_3 - 4c_3 + b_3 - 6a_3)}{\pi^2 + 16}.$$

Since (for example) K_3 decreases with increasing d_3, we see that such an increase tends to stabilize $x = 0$, as well as small nearby periodic solutions. See [**5**] and [**8**] for specific applications related to this example.

EXAMPLE 3.5. Consider equation (1.2) where the measure is that of the example of Section 2 with $\varepsilon = 1$. The coefficients $c_{a_2a_2}, c_{a_2b_2}, \ldots$ can again be computed. As they are significantly more complicated than those of the previous example, we will state only the two required for consideration of the integrodifferential equation

$$\dot{x} = \alpha x + \beta \int_{-1}^{0} [c_2 x_t^2(s) + d_3 x_t^3(s) + \cdots](s+1)\, ds. \tag{3.9}$$

In particular, for $\tilde{\beta}(\omega)$ as defined in Section 2,

$$\begin{aligned}
c_{c_2c_2}(\omega) = \tilde{\beta}^2(\omega)[&-2\omega^2(\cos(\omega)-1)^3[12\cos(\omega)^3 - 107\cos(\omega)^2 \\
&\qquad - 374\cos(\omega) - 239]\sin(\omega) \\
&-2\omega^4(\cos(\omega)-1)^2[8\cos(\omega)^3 - 61\cos(\omega)^2 - 120\cos(\omega) \\
&\qquad - 97]\sin(\omega) \\
&+ 2\omega^6(\cos(\omega)-1)[28\cos(\omega)^2 - 120\cos(\omega) - 229]\sin(\omega) \\
&- 4(\cos(\omega)-1)^4(\cos(\omega)+1)[4\cos(\omega)^2 + 17\cos(\omega) \\
&\qquad + 21]\sin(\omega) \\
&+ 30\omega^8\sin(\omega) + 2\omega^3(\cos(\omega)-1)^3[40\cos(\omega)^3 - 59\cos(\omega)^2 \\
&\qquad - 258\cos(\omega) - 143] \\
&+4\omega^5(\cos(\omega)-1)^2[13\cos(\omega)^3 - 40\cos(\omega)^2 - 178\cos(\omega) \\
&\qquad - 134] \\
&+ 4\omega(\cos(\omega)-1)^4(\cos(\omega)+1)[24\cos(\omega)^2 + 95\cos(\omega) + 79] \\
&- 2\omega^7(\cos(\omega)-1)[10\cos(\omega)^2 - 70\cos(\omega) - 93]]/D_1(\omega) \\
c_{d_3}(\omega) = \tilde{\beta}(\omega)[&3(1-\cos(\omega))\sin(\omega)^2 + 3\omega^3\sin(\omega) \\
&+ 3\omega^2(\cos(\omega)-1)(\cos(\omega)+2)]/D_2(\omega),
\end{aligned}$$

where

$$\begin{aligned}
D_1(\omega) &= \omega^5[2\cos(\omega) + \omega^2 - 2][6\omega\sin(\omega)(\cos(\omega)-4) + 4\cos(\omega)^3 \\
&\qquad - 9\cos(\omega)^2 - 12\cos(\omega) + 9\omega^2 + 17] \\
D_2(\omega) &= \omega^2[-2\omega(\cos(\omega)+8)\sin(\omega) - (\cos(\omega)-1)(5\cos(\omega)+13) \\
&\qquad + \omega^2(4\cos(\omega)+5)].
\end{aligned}$$

At $\omega = 2\pi$, one has $(\tilde{\alpha}, \tilde{\beta}) = (0, -4\pi^2)$ and

$$K_3 = a_3 - 4\pi^2 c_3/3 + 2a_2^2/(3\pi^2) - [b_2 + 2c_2]a_2/3 + 4\pi^2 b_2 c_2/3.$$

Observe that K_3 is independent of the coefficients b_3 and d_3. Moreover, $K_3 \equiv 0$ if $a_3 = c_3 = a_2 = 0$ and either b_2 or c_2 are zero. Under such conditions, equation (1.2) is said to be *fully* 3^{rd} *order nongeneric*, and one must compute (at least) 5^{th} order terms in the expansion of g in order to resolve the stability at $x = 0$ at criticality. This generalizes the results of [**10**], where all but β, c_2, and d_3 in (1.2) are taken to be zero.

The following Lemma leads to a clearer understanding of when $K_3(\omega)$ can be independent of certain of the coefficients in (1.2).

LEMMA 3.6. *The following identities hold*:

$$\begin{aligned}\tilde\beta(\omega)L(e^{i\omega\cdot})/\Delta'(i\omega) &= [-\tilde\alpha'(\omega)\omega + i(-\tilde\alpha(\omega)\tilde\alpha'(\omega)\\ &\quad + \tilde\beta'(\omega)(\tilde\alpha(\omega)^2+\omega^2)/\tilde\beta(\omega))]/|\Delta'(i\omega)|^2 \qquad (3.10)\end{aligned}$$

(3.11)
$$\mathrm{Im}[\tilde\beta(\omega)L(e^{2i\omega\cdot})/\Delta(2i\omega)] = 2(\tilde\beta(\omega)/\tilde\beta(2\omega))[\tilde\alpha(2\omega)-\tilde\alpha(\omega)]/|\Delta(2i\omega)|^2$$

$$\begin{aligned}1/\Delta'(i\omega) &= [\omega\tilde\beta'(\omega)/\tilde\beta(\omega)\\ &\quad + i\tilde\beta(\omega)(\tilde\alpha(\omega)/\tilde\beta(\omega))']/|\Delta'(i\omega)|^2 \qquad (3.12)\end{aligned}$$

$$\mathrm{Im}[1/\Delta(2i\omega)] = 2\omega[\tilde\beta(\omega)-\tilde\beta(2\omega)]/(\tilde\beta(2\omega)|\Delta(2i\omega)|^2) \qquad (3.13)$$

PROOF. One differentiates the identity $i\omega = \tilde\alpha(\omega) + \tilde\beta(\omega)L(e^{i\omega\cdot})$ with respect to ω, then uses the fact that $\Delta'(i\omega) = 1 - \tilde\beta(\omega)L((\cdot)e^{i\omega\cdot})$, to obtain

$$\tilde\alpha'(\omega) + \tilde\beta'(\omega)L(e^{-i\omega\cdot}) = -i\Delta'(-i\omega). \qquad (3.14)$$

The first two identities follow by using $\tilde\beta(\omega)L(e^{i\omega\cdot}) = i\omega - \tilde\alpha(\omega)$. The last two are computed similarly, based on the fact that

$$\begin{aligned}\Delta(-2i\omega) &= -2i\omega - \tilde\alpha(\omega) - \tilde\beta(\omega)L(e^{-2i\omega\cdot})\\ &= -2i\omega - \tilde\alpha(\omega) + \tilde\beta(\omega)(2i\omega + \tilde\alpha(2\omega))/\tilde\beta(2\omega).\end{aligned}$$

The details are omitted. □

THEOREM 3.7. *Under hypotheses* H1 *and* H2*, a necessary and sufficient condition for*

$$\dot x(t) = \alpha x(t) + \beta\int_{-\infty}^0 h(x(t+s))\,d\eta(s),$$

$h(x) = x + c_2x^2 + d_3x^3 + \cdots$, *to be fully* 3rd *order nongeneric at* $(\alpha,\beta) = (\tilde\alpha(\omega),\tilde\beta(\omega))$ *is that* $\tilde\alpha'(\omega) = 0$ *and* $\tilde\alpha(2\omega) = \tilde\alpha(\omega)$, (*the first* *condition being both necessary and sufficient for* K_3 *for* (1.2) *to be independent of* d_3*).*

PROOF. By Proposition 3.2 and the first identity of the previous lemma, $c_{d_3}(\omega) = 0$ if and only if $\tilde{\alpha}'(\omega) = 0$. Continuing under this situation, $K_3(\omega)$ will be independent of c_2^2 if and only if $\mathrm{Re}[(\tilde{\beta}(\omega)L(e^{i\omega\cdot}))/(\Delta'(i\omega)\Delta(2i\omega))] = 0$. As $\tilde{\beta}(\omega)L(e^{i\omega\cdot})/\Delta'(i\omega)$ is purely imaginary by (3.10), Proposition 3.2 and (3.11) imply the result. □

We remark that the previous line of argument reveals that, at criticality, d_3 tends to stabilize (destabilize) $x = 0$ and Hopf bifurcations if and only if $\tilde{\alpha}'(\omega) > 0$ ($\tilde{\alpha}'(\omega) < 0$). Should $\tilde{\alpha}'(\omega) = 0$, (places of "infinite" slope for the imaginary root curve), $\mathrm{sign}(c_{c_2 c_2}(\omega)) = \mathrm{sign}(\tilde{\beta}'(\omega)(\tilde{\alpha}(2\omega) - \tilde{\alpha}(\omega)))$ since $\tilde{\beta}(\omega) < 0$. Note that, by equation (3.14), $\tilde{\beta}'(\omega)$ and $\tilde{\alpha}'(\omega)$ cannot vanish simultaneously by the assumption that $\lambda = i\omega$ is a simple characteristic root.

For the example of Section 2, the conditions of the previous Theorem apparently hold only at $\varepsilon = 1$ and when ω is an integer multiple of 2π (the equation of Levin and Nohel [**6, 10**]). The independence of K_3 on b_3 at that point is due to the fact (using Proposition 3.2) that $\tilde{\alpha}(2\pi) = 0$.

An analogue of the previous Theorem can be derived for the case where the nonlinear terms do not involve time delays.

THEOREM 3.8. *Under hypotheses* H1 *and* H2*, a necessary and sufficient condition for*

$$\dot{x}(t) = \beta \int_{-\infty}^{0} x(t+s)\, d\eta(s) + g(x(t))$$

$g(x) = \alpha x + a_2 x^2 + a_3 x^3 + \cdots$, *to be fully* 3^{rd} *order nongeneric at* $(\alpha, \beta) = (\tilde{\alpha}(\omega), \tilde{\beta}(\omega))$ *is that* $\tilde{\beta}'(\omega) = 0$ *and* $\tilde{\beta}(2\omega) = \tilde{\beta}(\omega)$ *(the first condition being both necessary and sufficient for* K_3 *for* (1.2) *to be independent of* a_3*).*

The proof of this result is entirely analogous to the previous theorem, using Proposition 3.2 and the last two identities of Lemma (3.6). Details are omitted. Since $\tilde{\beta}(\omega) < 0$, we have that $\text{sign}(c_{a_3}(\omega) = -\text{sign}(\tilde{\beta}'(\omega))$. The figures of Section 2 suggest that $\tilde{\beta}'(\omega) < 0$ on the border of Ω_-. Hence, the term a_3 for that example is expected to always have a destabilizing effect on $x = 0$ at criticality and on Hopf bifurcations. In general, if $\tilde{\beta}'(\omega) = 0$, then $\text{sign}(c_{a_2 a_2}(\omega)) = \text{sign}(\tilde{\alpha}'(\omega)(\tilde{\beta}(\omega) - \tilde{\beta}(2\omega)))$.

We remark that the expression for $N_3(\omega)$ as given in Proposition 3.2 shows that, unless $\tilde{\beta}'(\omega) = 0$ (points of zero slope for the imaginary root curve), $K_3(\omega)$ will always depend nontrivially on a_3. Finally, since $\tilde{\alpha}'(\omega)$ and $\tilde{\beta}'(\omega)$ cannot vanish simultaneously, $K_3(\omega)$ *must* depend nontrivially on both d_3 and b_3 whenever it is independent of a_3. This, of course, rules out the possibility of the general equation (1.2) ever being totally nongeneric.

Acknowledgments. The work of the author was supported by grants AFOSR 87-0268, NSF DMS-8701456, NSF DMS-8901893, and the Department of Commerce, National Bureau of Standards, through Georgia Tech. A grant of computing time from the Minnesota Supercomputer Institute and additional support from the Institute for Mathematics and Its Applications are gratefully acknowledged.

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